Gödel's
Second Incompleteness Theorem
(G2)

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Note: This is a version designed for those who have had at least one university-level course in formal logic with coverage through $\mathscr{L}_1$.

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Background Context …

Gödel's Great Theorems (OUP)

by Selmer Bringsjord

- Introduction ("The Wager")
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
- The Continuum-Hypothesis Theorem
- The Time-Travel Theorem
- Gödel's "God Theorem"
- Could a Finite Machine Match Gödel's Greatness?

STOP & REVIEW IF NEEDED!

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A corollary of the First Incompleteness Theorem: *We cannot prove (in "classical" mathematics) that mathematics is consistent.*

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By far the greatest of GGT; Selm's analysis based Sherlock Holmes' mystery "Silver Blaze."

The “Gödelian” Liar (from me)

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$\bar{P}$: This sentence is unprovable.

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Suppose that $\bar{P}$ is true. Then we can immediately deduce that $\bar{P}$ is provable, because here is a proof: $\bar{P} \rightarrow \bar{P}$ is an easy theorem, and from it and our supposition we deduce $\bar{P}$ by *modus ponens*. But since what $\bar{P}$ says is that it's unprovable, we have deduced that $\bar{P}$ is false under our initial supposition.

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Suppose on the other hand that $\bar{P}$ is false. Then we can immediately deduce that $\bar{P}$ is unprovable: Suppose for *reductio* that $\bar{P}$ *is* provable; then $\bar{P}$ holds as a result of some proof, but what $\bar{P}$ says is that it's unprovable; and so we have contradiction. But since what $\bar{P}$ says is that it's unprovable, and we have just proved that under our supposition, we arrive at the conclusion that $\bar{P}$ is true.

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T($\bar{P}$) iff (i.e., if & only if) $\neg$T($\bar{P}$) = F($\bar{P}$)

Contradiction!

The "Gödelian" Liar (from me)

$\bar{\pi}$ = "I'm unprovable."

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All of this is fishy; but Gödel, as we’ve seen, transformed it (by e.g. use of his encryption scheme) into utterly precise, impactful, indisputable reasoning …

PA (Peano Arithmetic):

$$
\begin{array}{ll}
\mathrm{A1} & \forall x(0 \neq s(x)) \\
\mathrm{A2} & \forall x \forall y(s(x) = s(y) \rightarrow x = y) \\
\mathrm{A3} & \forall x(x \neq 0 \rightarrow \exists y(x = s(y))) \\
\mathrm{A4} & \forall x(x + 0 = x) \\
\mathrm{A5} & \forall x \forall y(x + s(y) = s(x + y)) \\
\mathrm{A6} & \forall x(x \times 0 = 0) \\
\mathrm{A7} & \forall x \forall y(x \times s(y) = (x \times y) + x)
\end{array}
$$

And, every sentence that is the universal closure of an instance of

$$([\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x))] \rightarrow \forall x \phi(x))$$

where $\phi(x)$ is open wff with variable x, and perhaps others, free.

Is there buried inconsistency in here?!?

Courtesy of Gödel: Given certain limitative assumptions about "proof power," we can't prove that there isn't!

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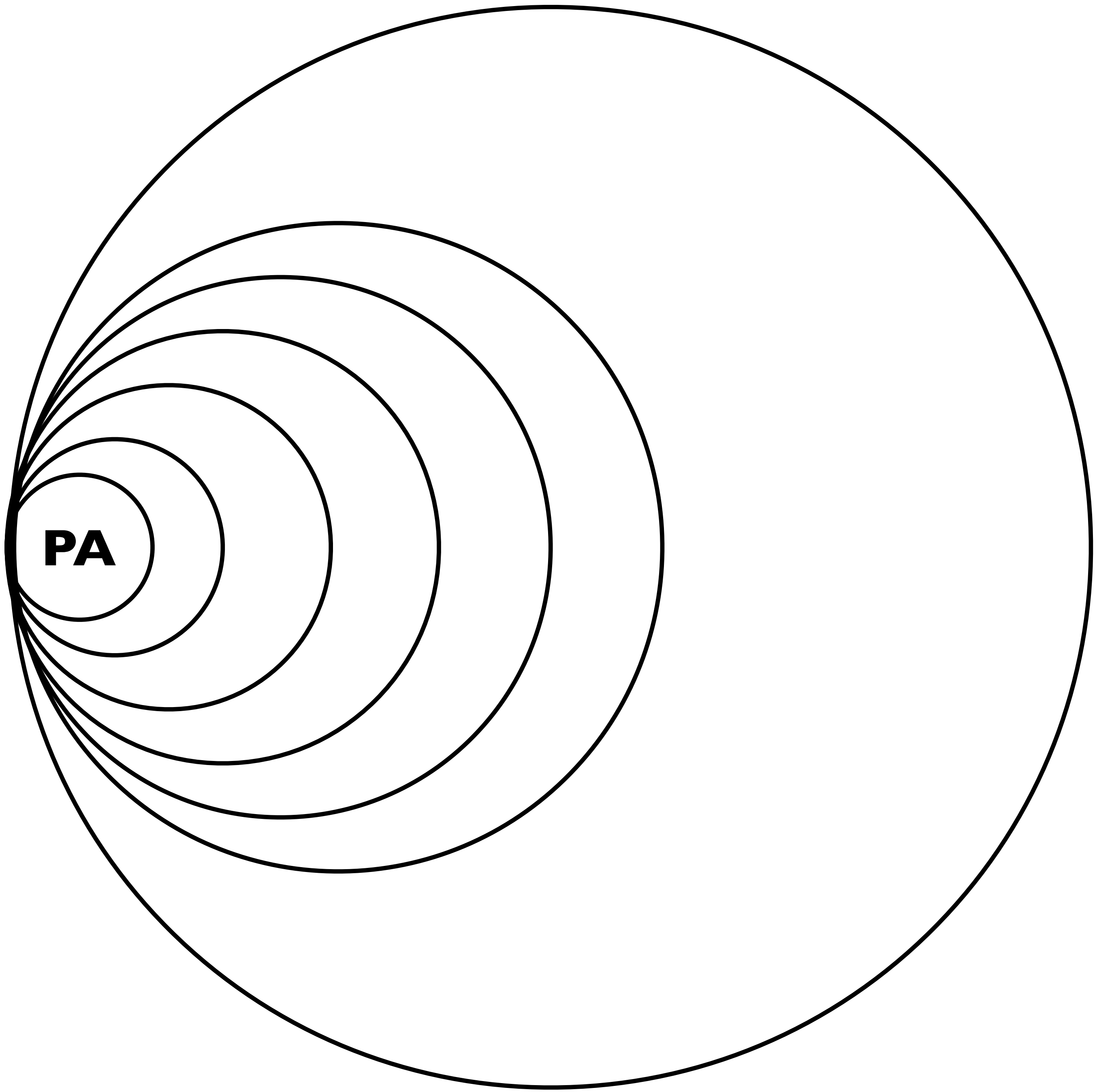

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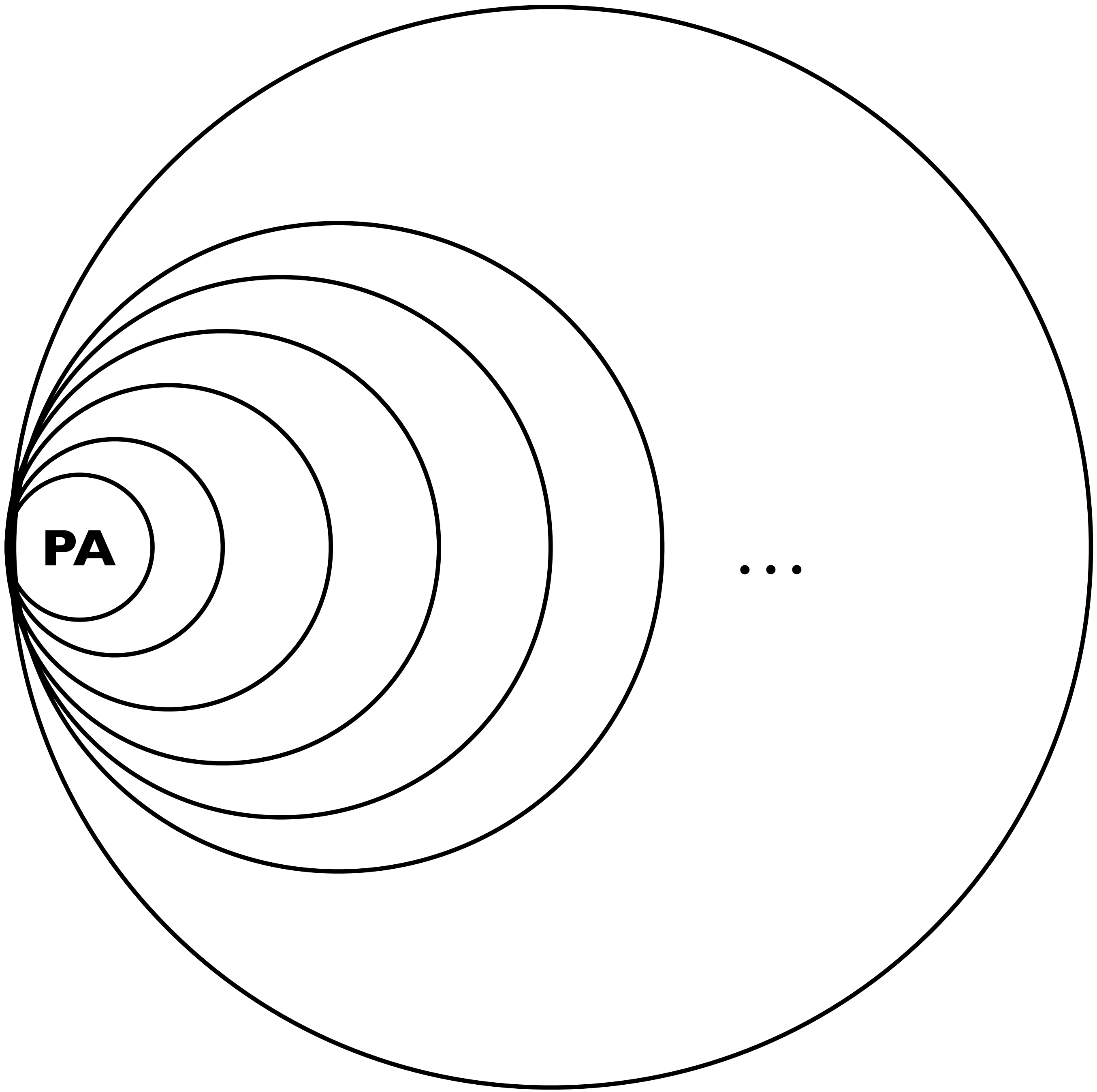

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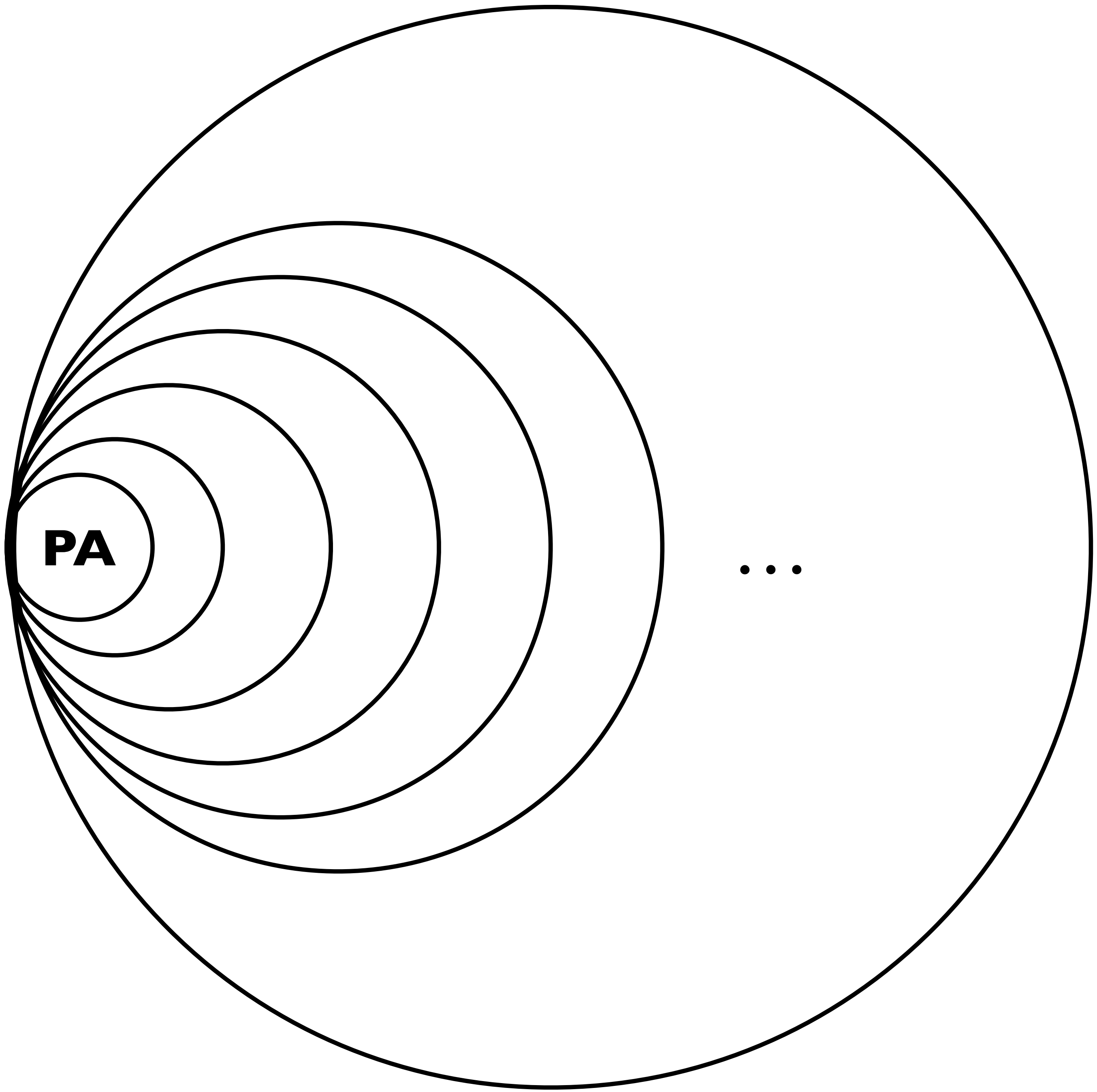

Each circle is a larger part of the formal sciences.

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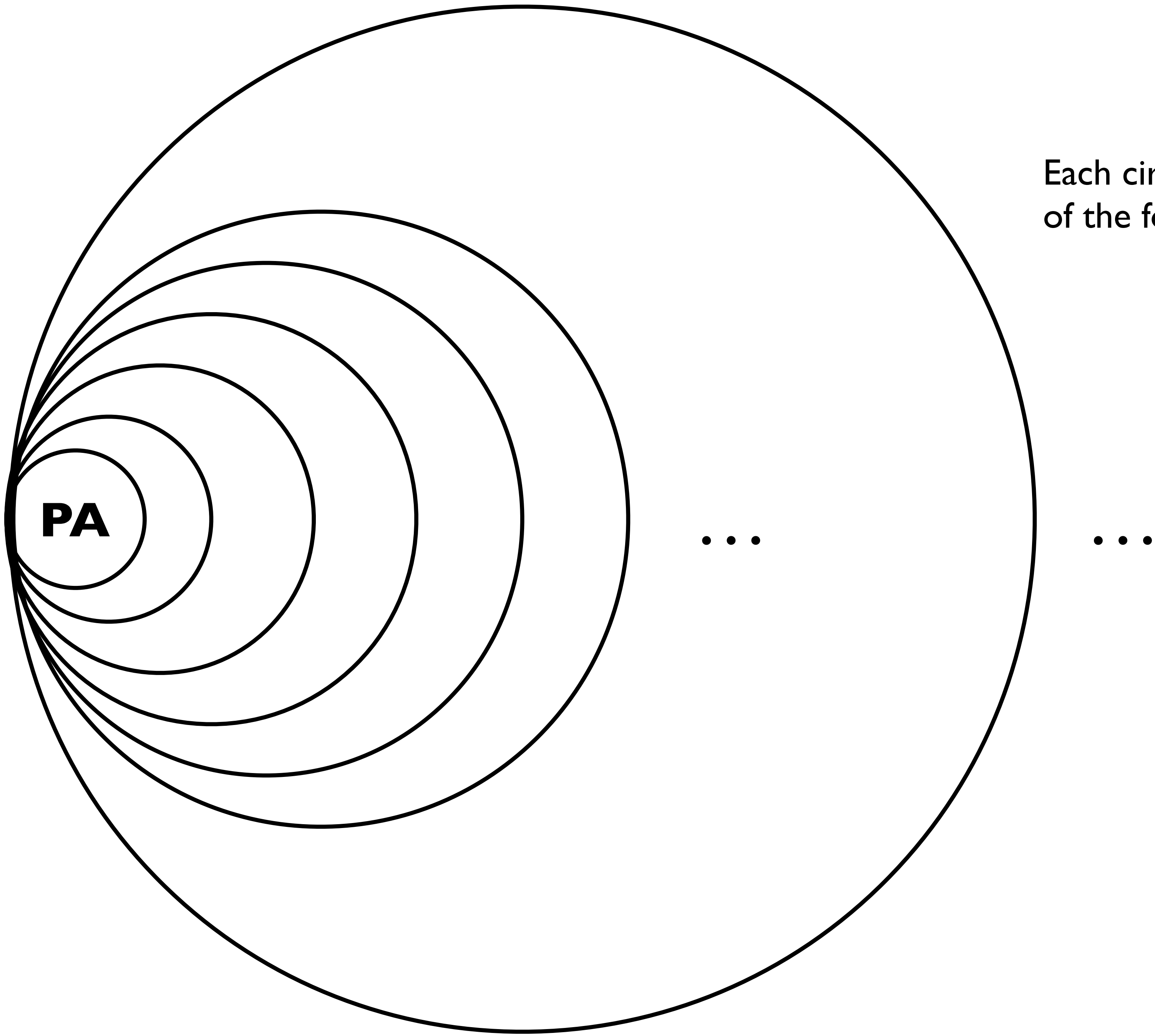

Each circle is a larger part of the formal sciences.

G2 as Slogan …

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"We can't use math to ascertain whether mathematics is consistent."

G2 as Slogan …

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"If we are restricted to certain kinds of formal reasoning, and feel we must have all of **PA** (math, engineering, etc.), we can't ascertain whether mathematics is consistent."

Gödel's <u>Second</u> Incompleteness Theorem

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Suppose $\Phi \supset \mathbf{PA}$ that is

(i) Con Φ;
(ii) Turing-decidable (i.e. membership in Φ is Turing-decidable); and
(iii) sufficiently expressive to capture all of the operations of a Turing machine (i.e. Repr Φ).

Then $\Phi \nvdash$ consis$_{\Phi}$.

Gödel's Second Incompleteness Theorem

Remember Church's Theorem!

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Then $\Phi \nvdash \text{consis}_{\Phi}$.

To prove G2, we shall once again allow ourselves …

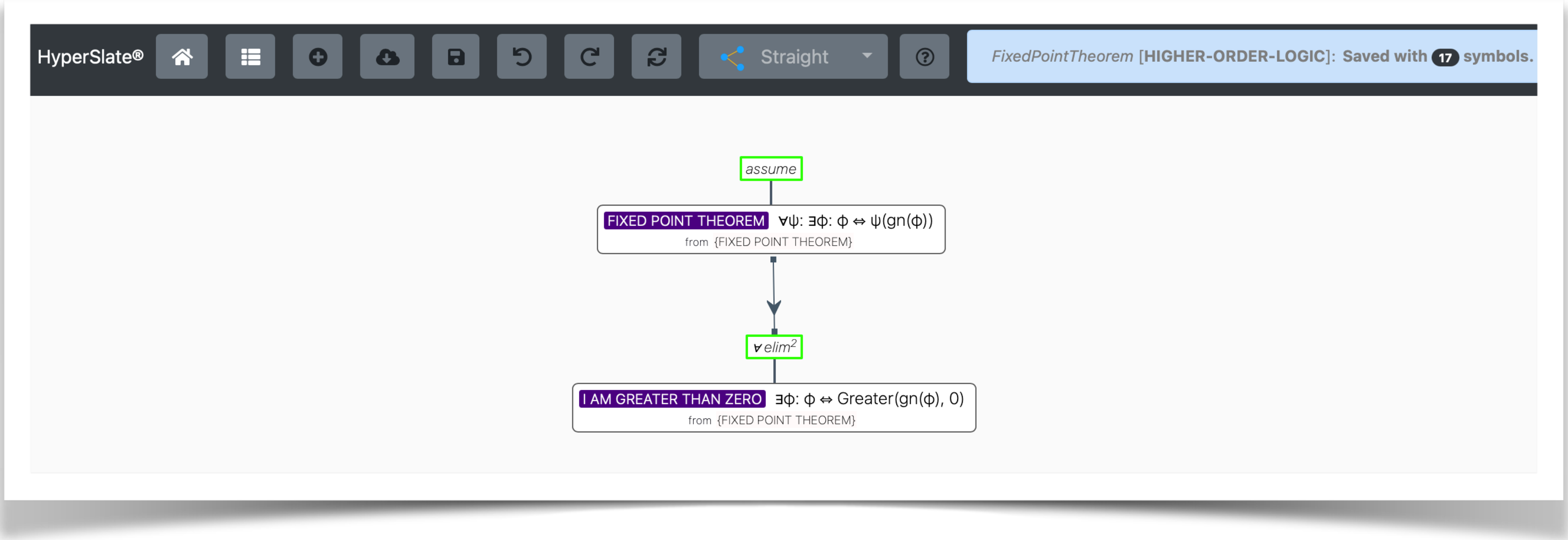
The Fixed Point Theorem (FPT)

Assume that Φ is a set of arithmetic sentences such that Repr Φ. There for every arithmetic formula $\psi(x)$ with one free variable x, there is an arithmetic sentence ϕ s.t.

$\Phi \vdash \phi \leftrightarrow \psi(\hat{n}^{\phi})$.

We can intuitively understand ϕ to be saying:
"I have the property ascribed to me by the formula ψ."

FPT in HyperSlate®!



Ok; so let's do it … and let's see if you can see why Gödel declared G2 to be a direct "corollary" of G1, and didn't bother to prove it in his original paper …

Proof: Suppose that the antecedent (i)–(iii) of **G2** holds. Suppose for reductio that

$\Phi \vdash$ consis$_\Phi$.

We need three ingredients, and we shall be done. First, from FPT we can again directly obtain:

(*) $\Phi \vdash \mathcal{G} \leftrightarrow \neg \mathscr{P}_\Phi(\hat{n}^{\mathcal{G}})$.

Next, we can prove (how? … from one half of **G1**!) that:

(7.9) If Con Φ, then $\Phi \nvdash \mathcal{G}$.

Thirdly, we can logicize the meta-logical proposition that Φ is consistent as an object-level conditional which can itself be proved formally from Φ:

(**) $\Phi \vdash$ consis$_\Phi \rightarrow \neg \mathscr{P}_\Phi(\hat{n}^{\mathcal{G}})$.

Contradiction! (Can you find it?) **QED**

Med nok penger, kan logikk løse alle problemer.