S5 Modal Logic And The Standard Translation

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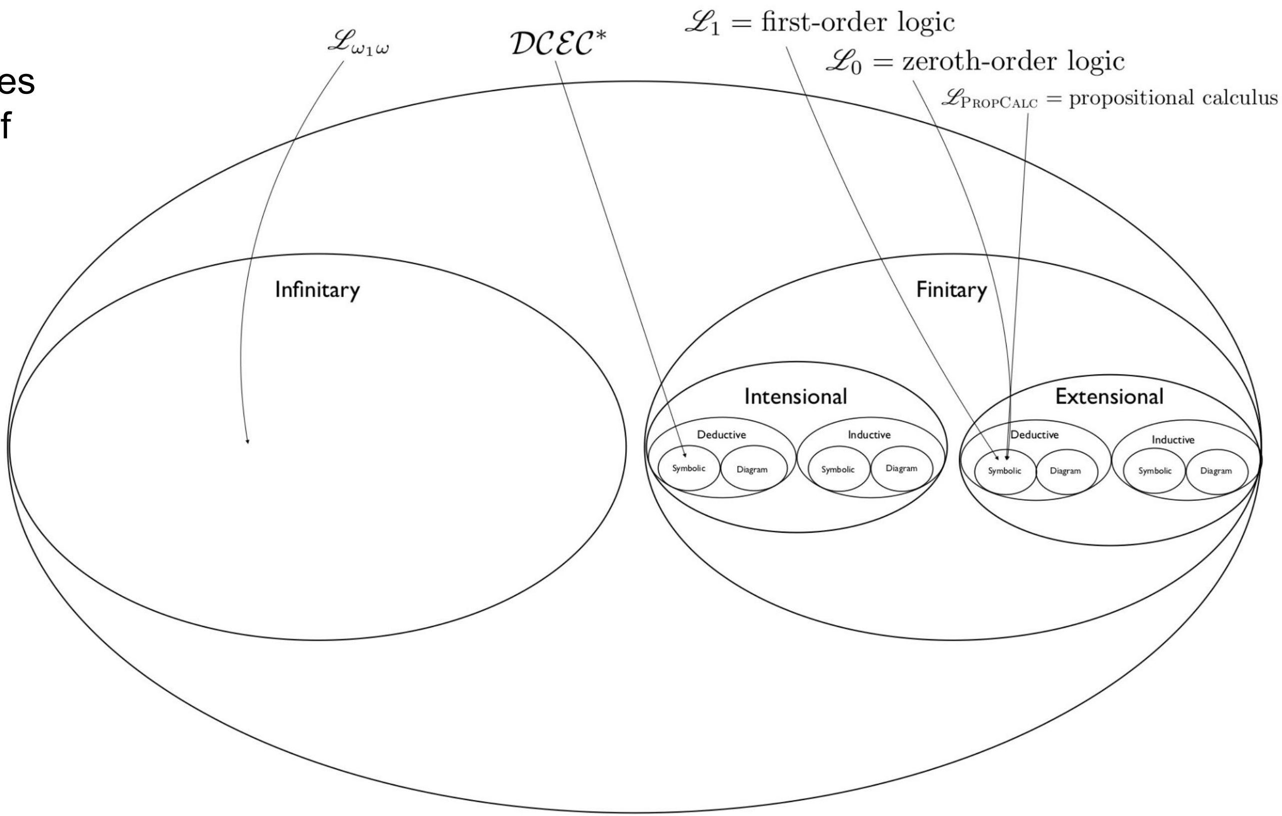
The Universe of Logics

Where are we today? Where does modal logic fall in the universe of logics?



Review of Modalities and Modal Logic

Modalities are the context we interpret the box and diamond operators in.

So far we have primarily looked at alethic and epistemic modalities.

	Box	Diamond
Alethic	“Necessary”	“Possible”
Epistemic (Knowledge)	“Knows”	(Dual Not Given Name)
Epistemic (Belief)	“Believes”	(Dual Not Given Name)
Deontic	“Obligatory” (Ought to)	“Permissible”
Temporal	“Always”	“Eventually”
Provability	“It is provable that”	(Dual Not Given Name)

An Axiomatic Look at A Universe of Modal Logics

K: $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

T: $\Box A \rightarrow A$

5: $\Diamond A \rightarrow \Box\Diamond A$

4: $\Box A \rightarrow \Box\Box A$

B: $A \rightarrow \Box\Diamond A$

5 is equivalent to having both 4 and B. We will show these are equivalent

Note that D = $\Box A \rightarrow \Diamond A$ and is just a weaker version of T.

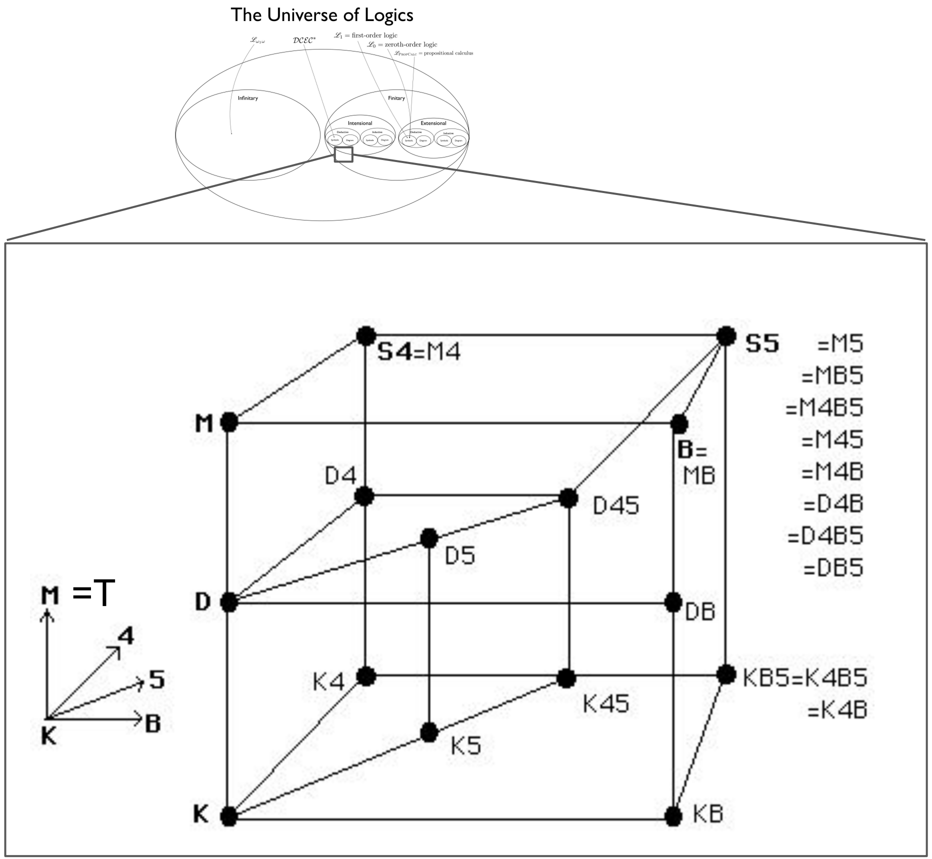


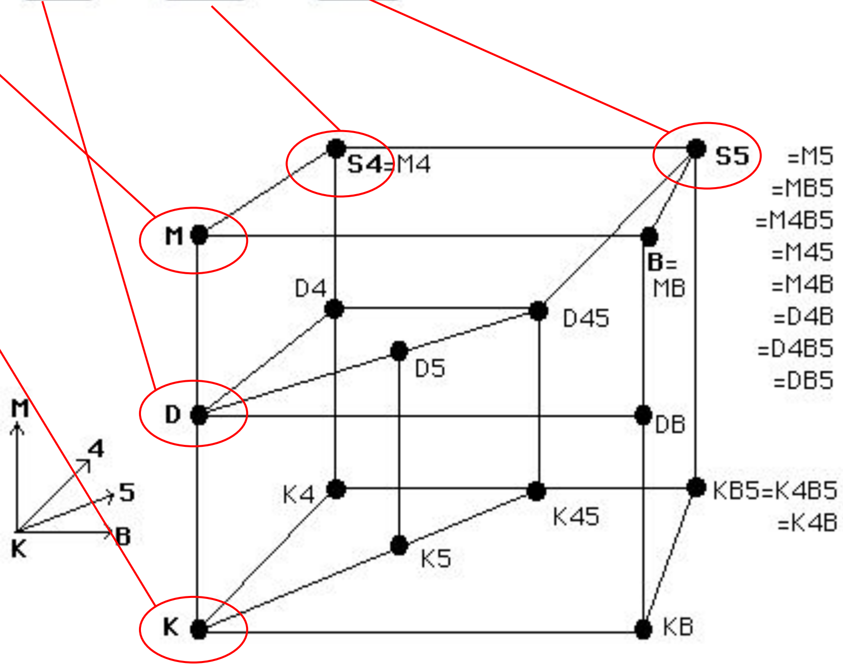
Figure From [Modal Logic (Stanford Encyclopedia of Philosophy)](https://plato.stanford.edu/entries/logic-modal/)

What do we have in Hyperslate?

Hyperslate offers K, D, T, S4, S5

Trace a path around edges of the modal subsumption cube leading to increasingly more powerful logics.

All Theorems in K are theorems in D, all theorems in D are theorems in T, and so on.



Other Logics In the Modal Subsumption Cube: KD45

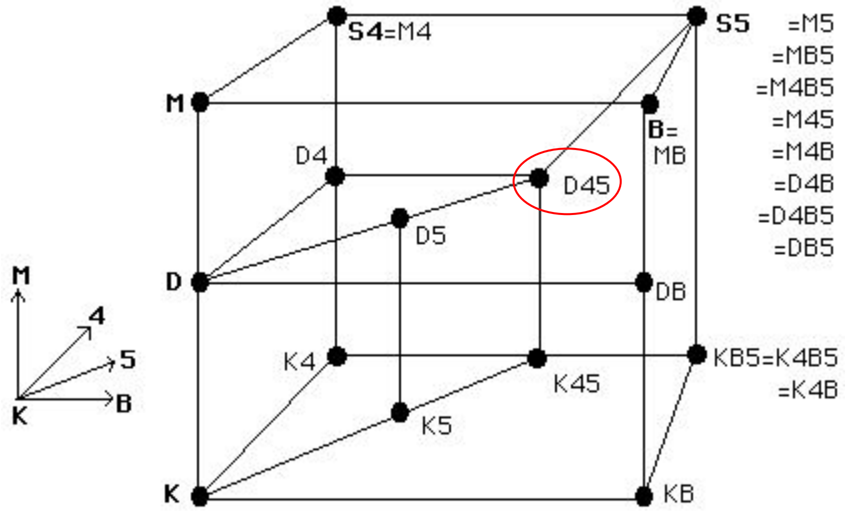
D45 (KD45) is the foundation of modal logic dealing with epistemic belief. (interpret the box as "believe")

Why throw out T? T: $\Box A \rightarrow A$

"If it is believed that A then A." NO!

D is still ok tho. $\Box A \rightarrow \Diamond A$

"If it is believed that A then it is not believed that not A"

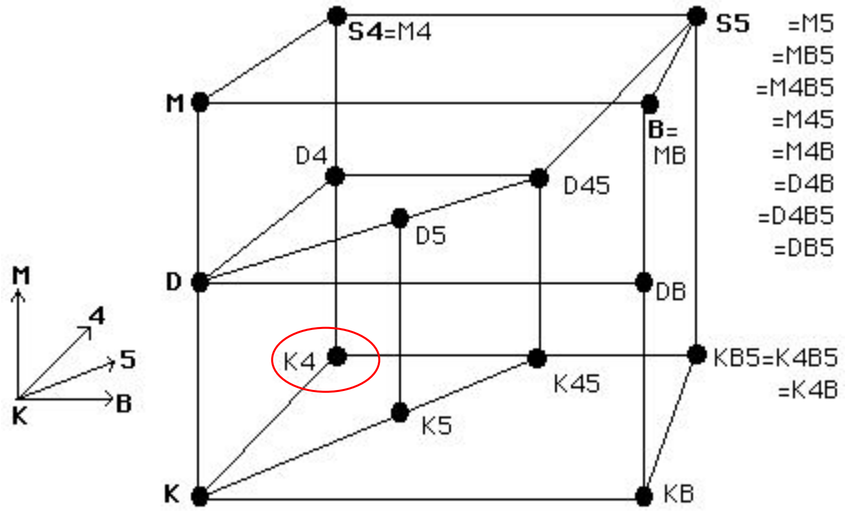


Other Logics In the Modal Subsumption Cube: K4

K4 is used as the base for many temporal logics. (where box is interpreted as “always in the future”)

Only admit 4. $\mathbf{4}: \Box A \rightarrow \Box\Box A$

“If always A, then always always A”

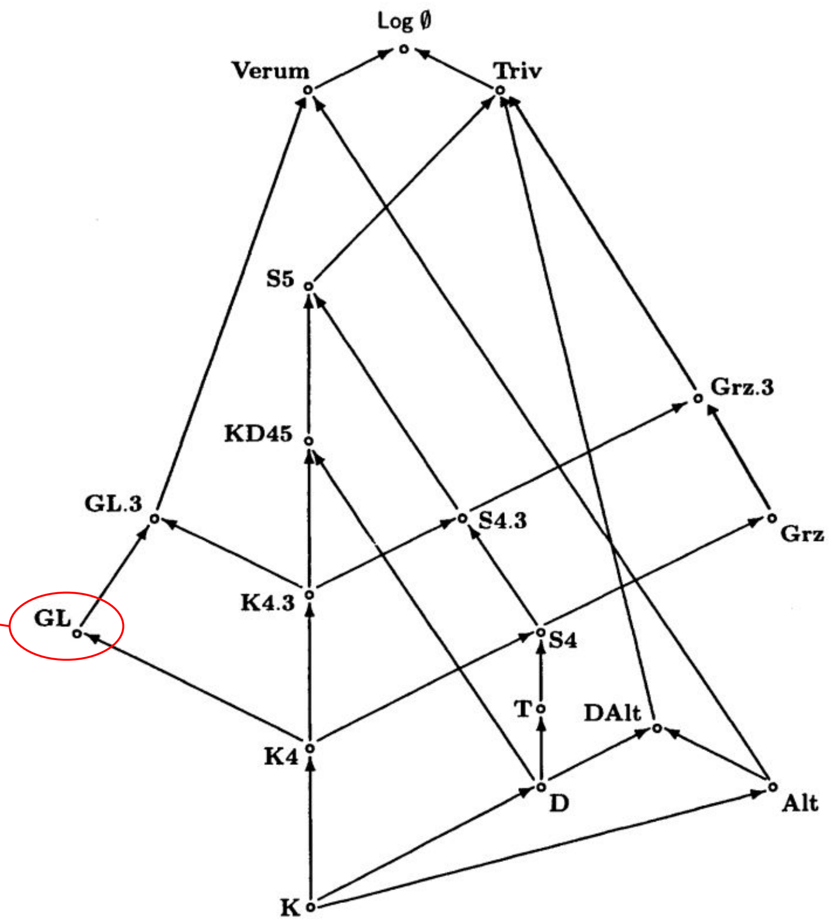


An even larger universe…

The modal cube is based off basic axioms from 1934, but this cube is just a small portion of an even larger modal logic lattice.

GL (Godel-Lob) Logic is used for reasoning about provability. Where box is read as “It is provable that”

Figure From “Many Dimensional Modal Logics: Theory and Applications” (2003) Page 14.



The Modal Logic S5

S5

S5 is the main modal logic we will look at in this class.

It is extremely versatile, and can be interpreted in the context of multiple modalities (we will look at alethic and epistemic).

Axiomatically, S5 is just like propositional calculus plus the following 4 axioms:

K: $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

T: $\Box A \rightarrow A$

5: $\Diamond A \rightarrow \Box\Diamond A$


C. I. Lewis

Along with the *necessitation rule* (Our box intro rule)

If A is a theorem (does not depend on any assumptions), then $\Box$A

Readings Of K

$$\textbf{K}: \Box(A \to B) \to (\Box A \to \Box B)$$

The K axiom merely states that box distributes over material implication.

Some readings in various modalities:

- Alethic: “If it is necessary that A implies B, then if A is necessary, B is necessary.”
- Epistemic: “If it is known that A implies B, then if A is known, B is known.”
- Deonic: “If it ought to be that A implies B, then if it ought to be that A, it ought to be that B.”
- Temporal: “If A always implies B, then if A is always true, B is always true.”
- Provability: “If it is provable that A implies B, then if A is provable, B is provable.”

Readings Of T

$$\textbf{T}: \Box A \rightarrow A$$

Some readings in various modalities for S5:

- Alethic: “If it is necessary that A then A is true.”
- Epistemic: “If A is known then A is true.” (JTB interpretation of knowledge)

Why can’t we use this for the following modalities?

- Deonic: “If it ought to be that A, then A.”
- Epistemic Belief. “If it is believed that A then A.”
- Temporal: “If always A, then A” (Future facing always operator, does not consider the past)
- Provability: “If it is provable that A then A” or “everything provable is true”. (Contradicts Godel's 2nd Incompleteness theorem, see: Many Dimensional Modal Logics: Theory and Applications, page 8)

Readings Of 5

$$\mathbf{5}: \Diamond A \to \Box\Diamond A \qquad\qquad \Diamond\Box A \to \Box A$$

Some readings in various modalities for S5:

- Alethic: “If it is possible that A then it necessary that A is possible.”
- Epistemic: (Expand Diamond and read as)

  ”If it is not known that A is not known, then it is known that A.”

  In epistemic logic this is called the the negative introspection axiom.

The Bringsjordian Inference System For S5

Inference Rules in S5 (in Hyperslate)

Easy Ones: Box Elim, Diamond Intro

$$\frac{\Gamma \vdash \Box\phi}{\Gamma \vdash \phi}\ \Box E$$

If something is necessary under assumptions (Gamma), it is true under those assumptions.

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \Diamond\phi}\ \Diamond I$$

If something is true under some assumptions (Gamma), it is possible under those assumptions.

Modal Dualities

Just a helper rule that rewrites box in terms of diamond and diamond in terms of box.

“Reversal” rules just shorthand for expanding and canceling the negations.

$$\Diamond\psi \Leftrightarrow \neg\Box\neg\psi$$

$$\Box\psi \Leftrightarrow \neg\Diamond\neg\psi$$

$$\neg\Diamond\psi \Leftrightarrow \Box\neg\psi$$

$$\neg\Box\psi \Leftrightarrow \Diamond\neg\psi$$

Diamond Elimination

$$\frac{\Gamma \cup \{\phi\} \vdash \psi \quad \Delta \vdash \Diamond\phi}{\Gamma \cup \Delta \vdash \Diamond\psi}\ \Diamond E$$

Given

1) A proof of psi from phi (and assumptions Gamma) and

2) The possibility of psi (from assumptions Delta),

We can derive that psi is possible (Assuming Gamma and Delta).

Box Introduction

$$\frac{\vdash \phi}{\vdash \Box\phi}\ \Box I_{theorem}$$

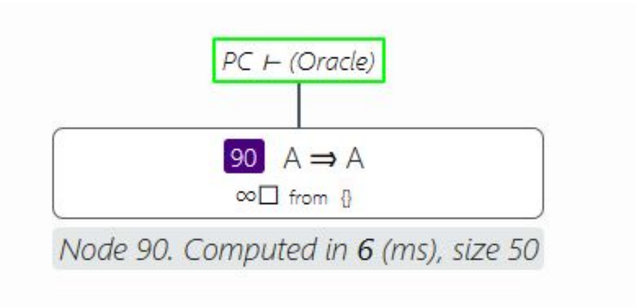
If something is a theorem then it is necessary.

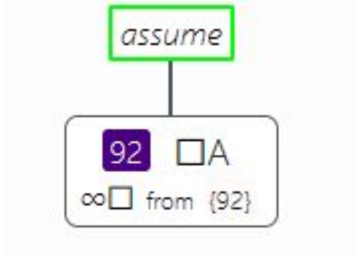
$$\frac{\Gamma \vdash \Box\phi}{\Gamma \vdash \Box\Box\phi}\ \Box I_4$$

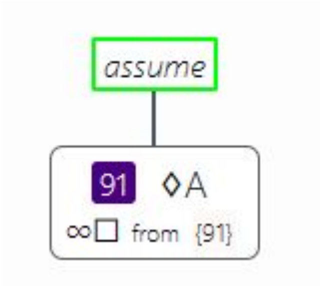
If something is necessary under some assumptions (Gamma), then it is necessarily necessary under those assumptions.

$$\frac{\Gamma \vdash \Diamond\phi}{\Gamma \vdash \Box\Diamond\phi}\ \Box I_5$$

If something is possible under some assumptions (Gamma), then it is necessarily possible under those assumptions.







Hyperslate Exercises

The inference rules provide a natural deduction style calculus for S5, while the axioms (along with box introduction) are a Hilbert style calculus for S5.

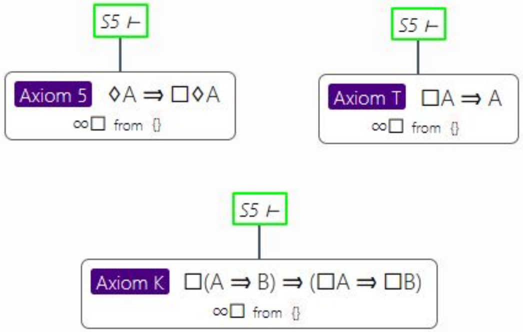
These systems are equivalent!

As an exercise we will prove all three axioms of S5 are theorems. (Start with T and 5 they are trivial, K is harder, you may use PC oracle)

K: $\Box(A \to B) \to (\Box A \to \Box B)$

T: $\Box A \to A$

5: $\Diamond A \to \Box\Diamond A$



The Standard Translation

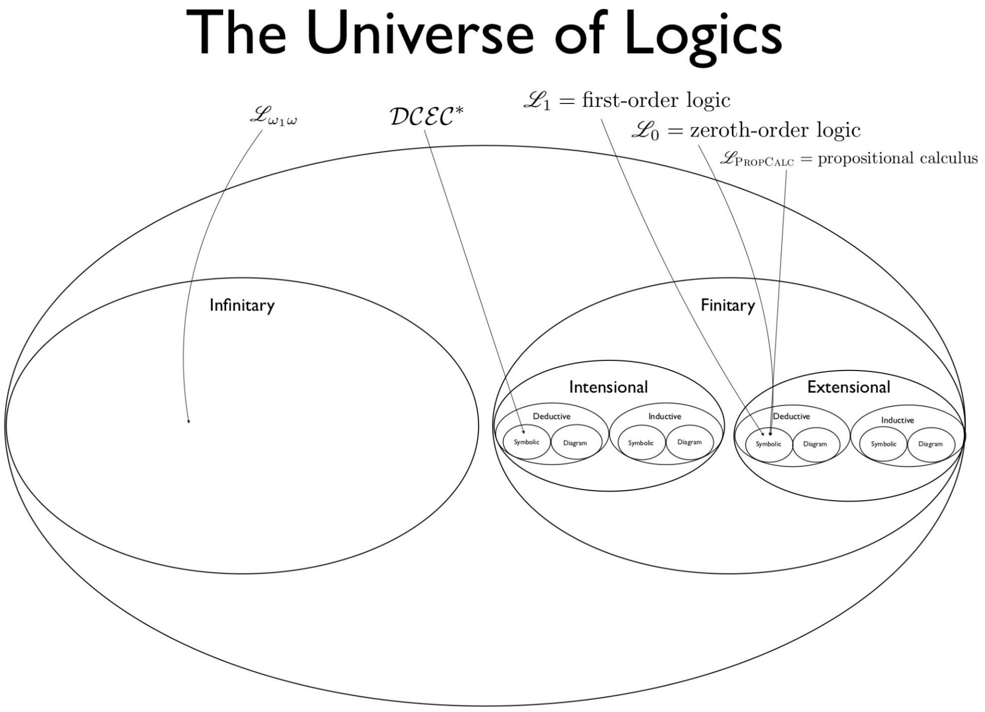
The Standard Translation

We're on the move!

Can translate any S5 formula into L1 and perform S5 reasoning in L1 with it.

Why do we care?

- S5 is decidable, what does this mean for L1?



Possible Worlds Interpretation

Imagine we have a multiverse of “worlds” in which each proposition symbol can be true or false.

We may move between worlds according to an “accessibility relation” R, which defines how possible worlds are connected.

R(x, y) is read “y is accessible from x”.

Can think of R as defining a directed graph of how we may move between worlds.

We can define possibility and necessity in terms of a world x and world accessibility relation R.

The standard Translation: *

$$p_i^\star = P_i(x)$$

$$\top^\star = \top \qquad \bot^\star = \bot$$

$$(\varphi \wedge \psi)^\star = \varphi^\star \wedge \psi^\star \qquad (\varphi \vee \psi)^\star = \varphi^\star \vee \psi^\star$$

$$(\varphi \rightarrow \psi)^\star = \varphi^\star \rightarrow \psi^\star \qquad (\neg\varphi)^\star = \neg\varphi^\star$$

$$(\Box\psi)^\star = \forall y\ (xRy \rightarrow \psi^\star\{y/x\}) \qquad (\Diamond\psi)^\star = \exists y\ (xRy \wedge \psi^\star\{y/x\}).$$

x is a constant representing the current world. y is a fresh variable not occurring in phi*

$\psi^\star\{y/x\}$ is the translated phi after substituting x with y.

Lets work out an example…

$$(\Box\Box p)^*$$
$$\forall y : R(x,y) \to (\Box p)^*\{y/x\}$$
$$\forall y : R(x,y) \to (\forall z : R(x,z) \to p^*\{z/x\})\{y/x\}$$
$$\forall y : R(x,y) \to (\forall z : R(x,z) \to P(x)\{z/x\})\{y/x\}$$
$$\forall y : R(x,y) \to (\forall z : R(x,z) \to P(z))\{y/x\}$$
$$\forall y : R(x,y) \to (\forall z : R(y,z) \to P(z))$$

$$(\Box\psi)^* = \forall y\ (xRy \to \psi^*\{y/x\})$$

$$(\Box\psi)^* = \forall y\ (xRy \to \psi^*\{y/x\})$$

$$p_i^* = P_i(x)$$

$\psi^*\{y/x\}$ is the translated phi after substituting x with y.

We’re in FOL, now what?

We still want to be able to reason with the modal formulae.

A very important result in modal logic is that we can get a different modal logic depending on the restrictions we place on the accessibility relation R.

We say a modal logic is determined by the properties of the R.

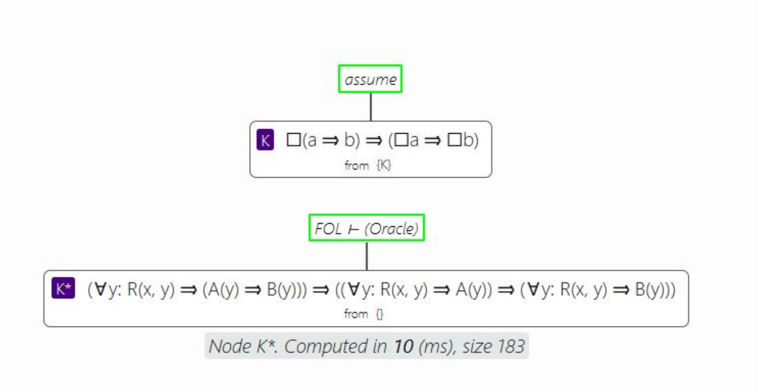
Without any restrictions, reasoning done on translated formulae is equivalent to K.

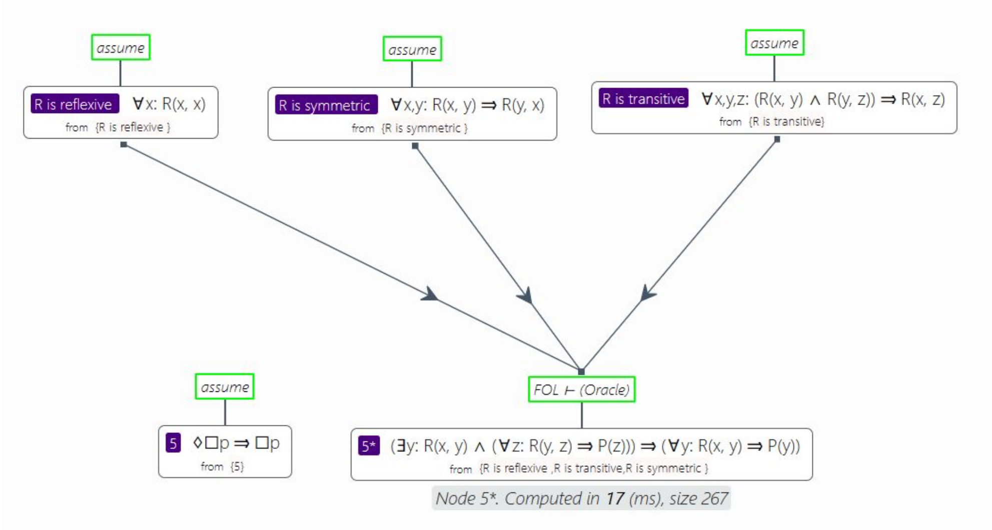
If we assume R is reflexive, we can reason like T.

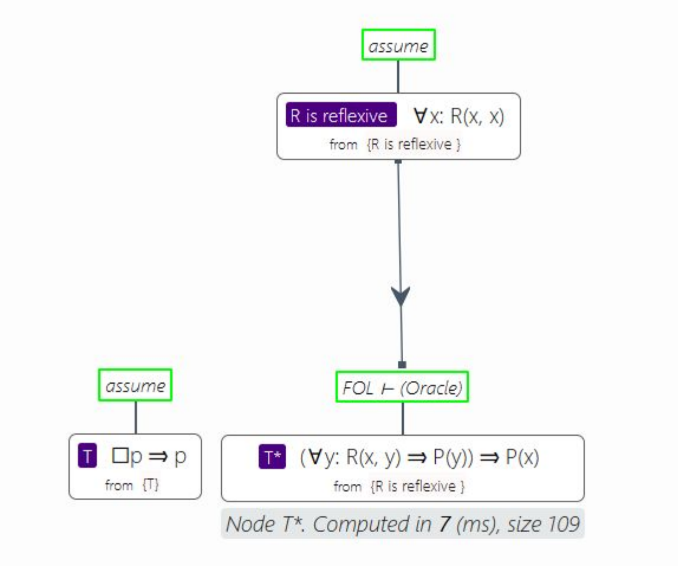
If we assume R is transitive, we can reason like K4

If we assume R is symmetric, reflexive, and transitive, we can reason like S5!

Translations of S5 Axioms Proved in FOL







Exercises!

$$p_i^* = P_i(x)$$
$$\top^* = \top \qquad \bot^* = \bot$$
$$(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^* \qquad (\varphi \vee \psi)^* = \varphi^* \vee \psi^*$$
$$(\varphi \to \psi)^* = \varphi^* \to \psi^* \qquad (\neg\varphi)^* = \neg\varphi^*$$
$$(\Box\psi)^* = \forall y\, (xRy \to \psi^*\{y/x\}) \qquad (\Diamond\psi)^* = \exists y\, (xRy \wedge \psi^*\{y/x\}).$$

4: $\Box A \to \Box\Box A$

B: $A \to \Box\Diamond A$

Translate B and 4 using the standard translation.

Prove that you're right since B and 4 are equivalent to 5. Hence the translation of 5 should be FOL Oracle provable from your translation of B and 4. And your translation of 5 should prove 4 and B.

Complete these proofs without the FOL Oracle