Church's Theorem*

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(w TM slides by Naveen Sundar G)

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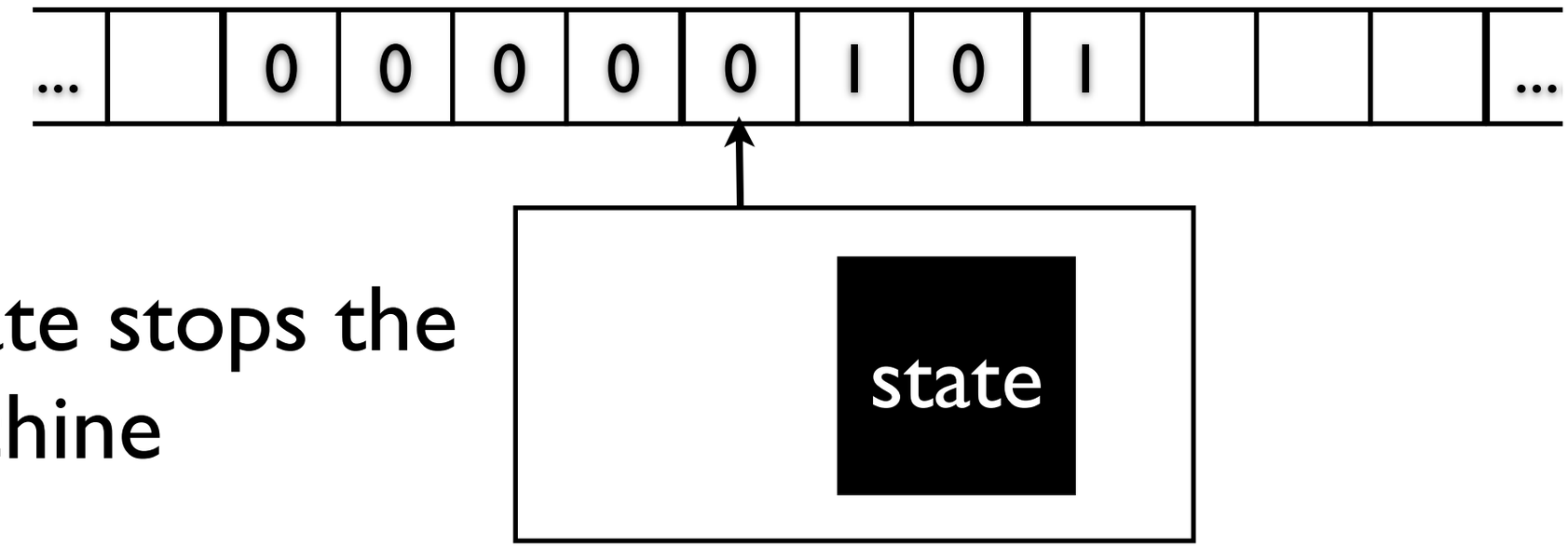
*Many thx to Can Mekik for corrections in the featured Turing machine.

Turing-decidability/computability

. . .

Turing Machines



a special state stops the machine

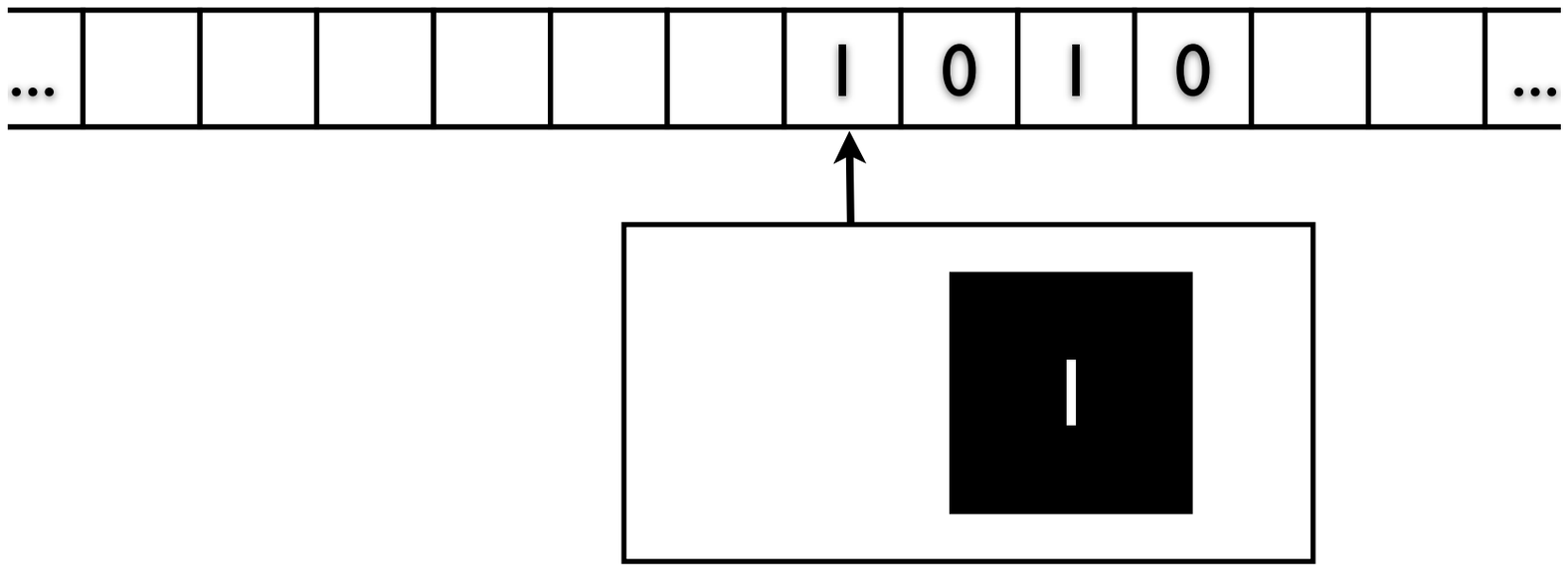
Program

current state	current symbol	next state	next symbol	direction

Even Number Function

- $f(n) = 1$ if n is even; else $f(n) = 0$

current state	current symbol	next state	next symbol	direction
3	0	3	blank	Left
3	1	3	blank	Left
3	blank	stop	0	Same
2	0	2	blank	Left
2	1	2	blank	Left
2	blank	stop	1	Same
1	1	1	1	Right
1	0	1	0	Right
1	blank	4	blank	Left
4	0	2	0	Same
4	1	3	1	Left



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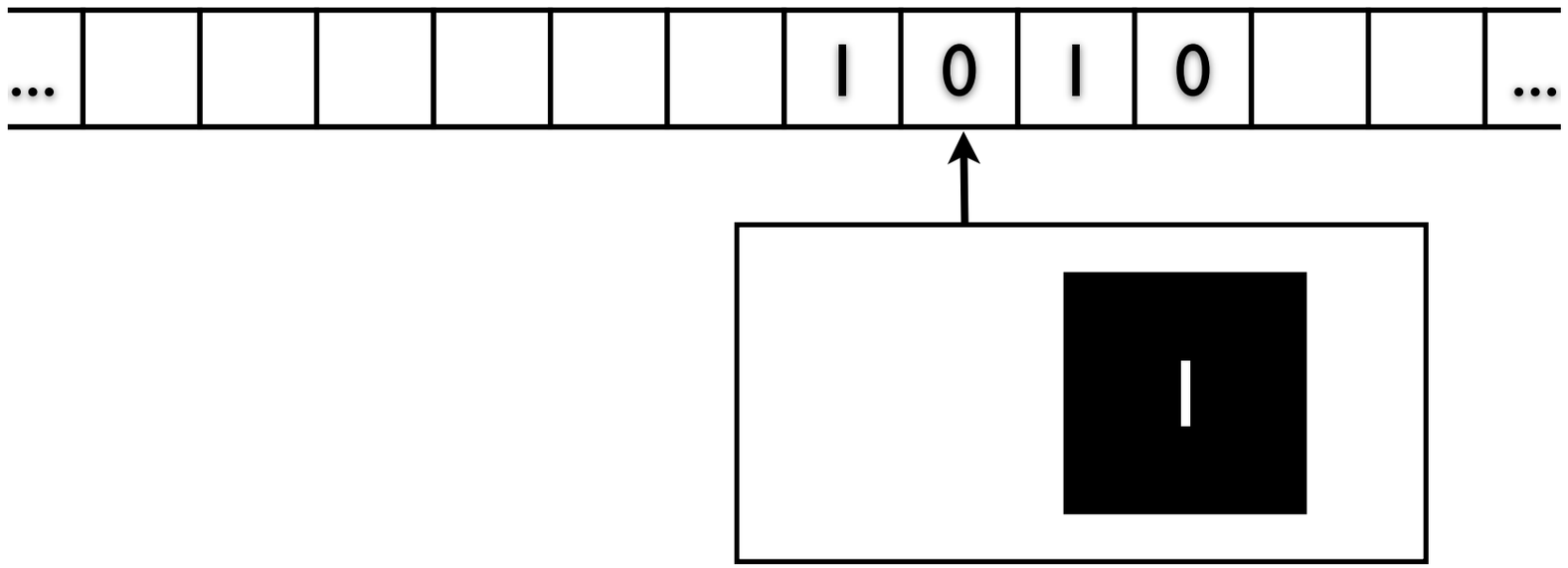

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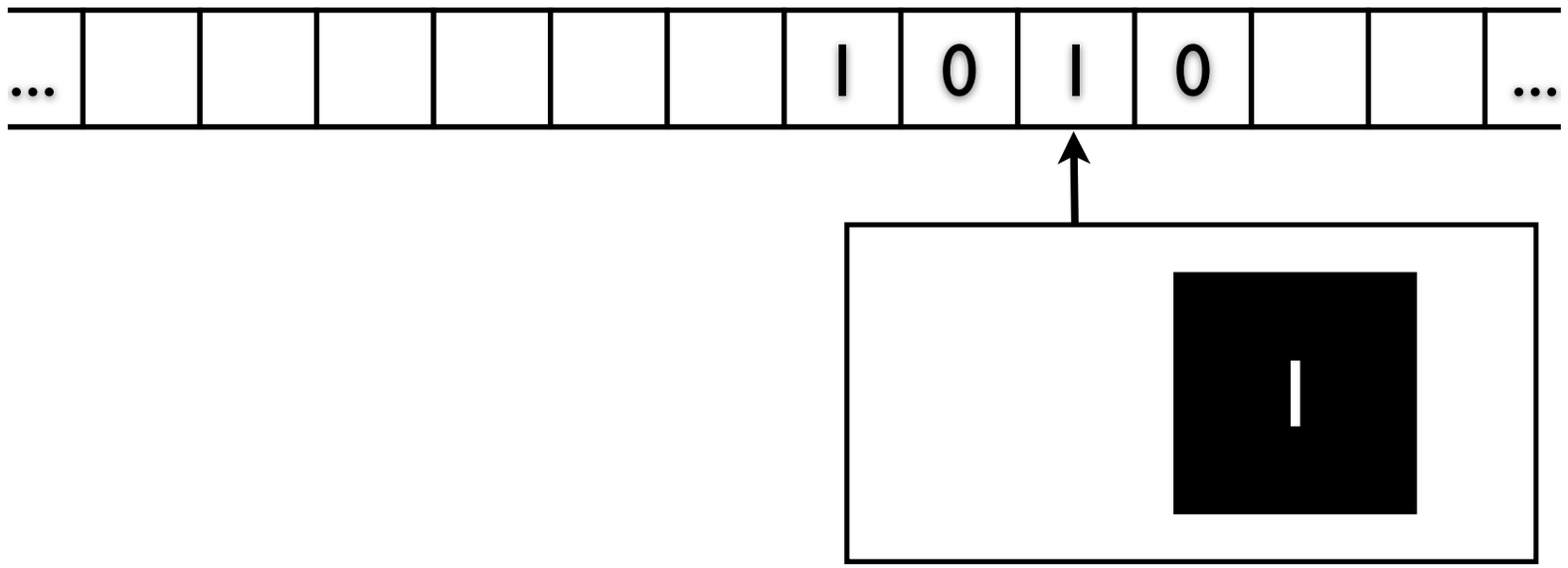

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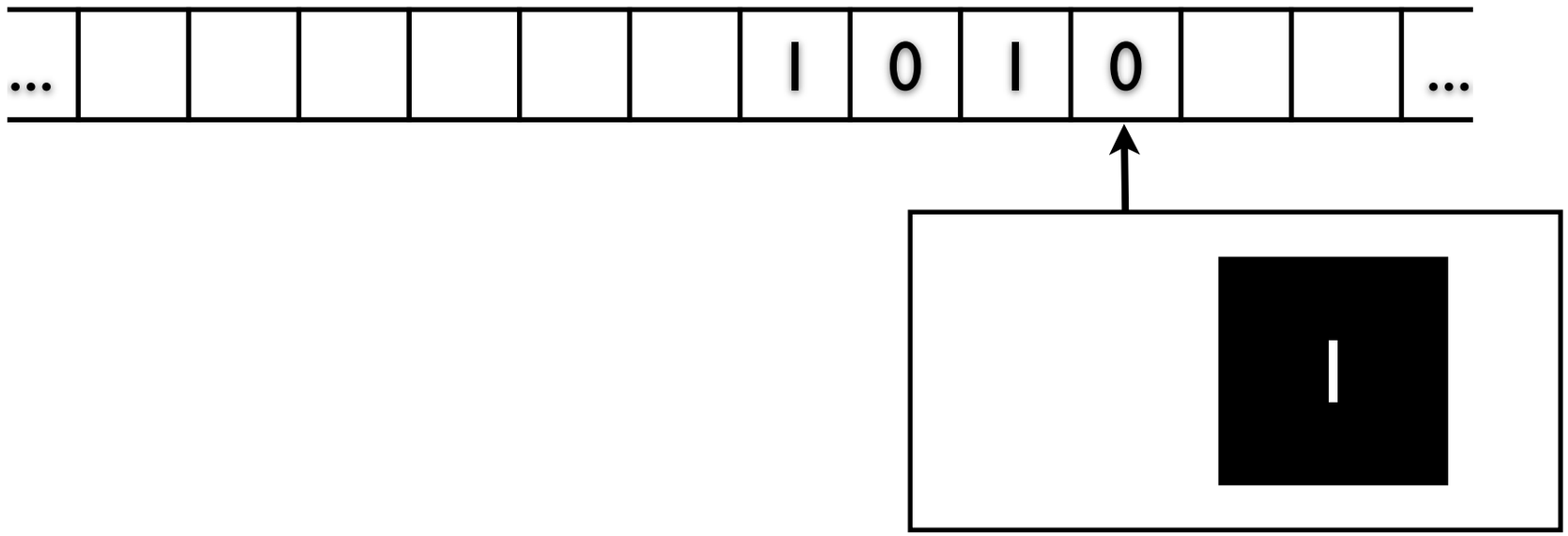

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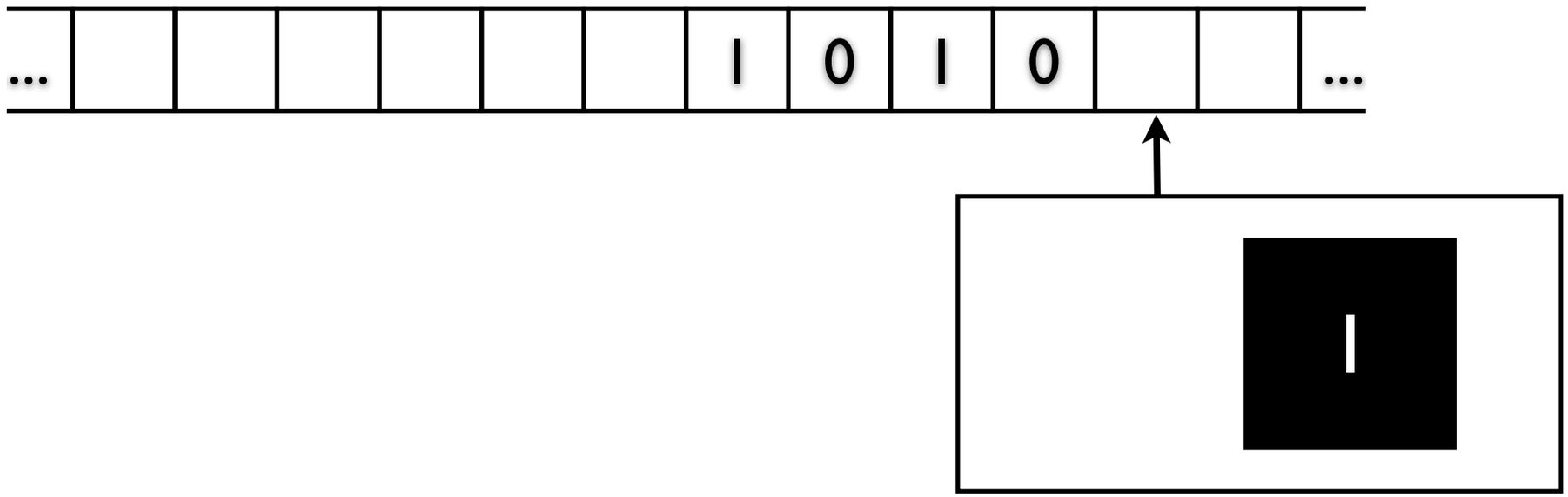

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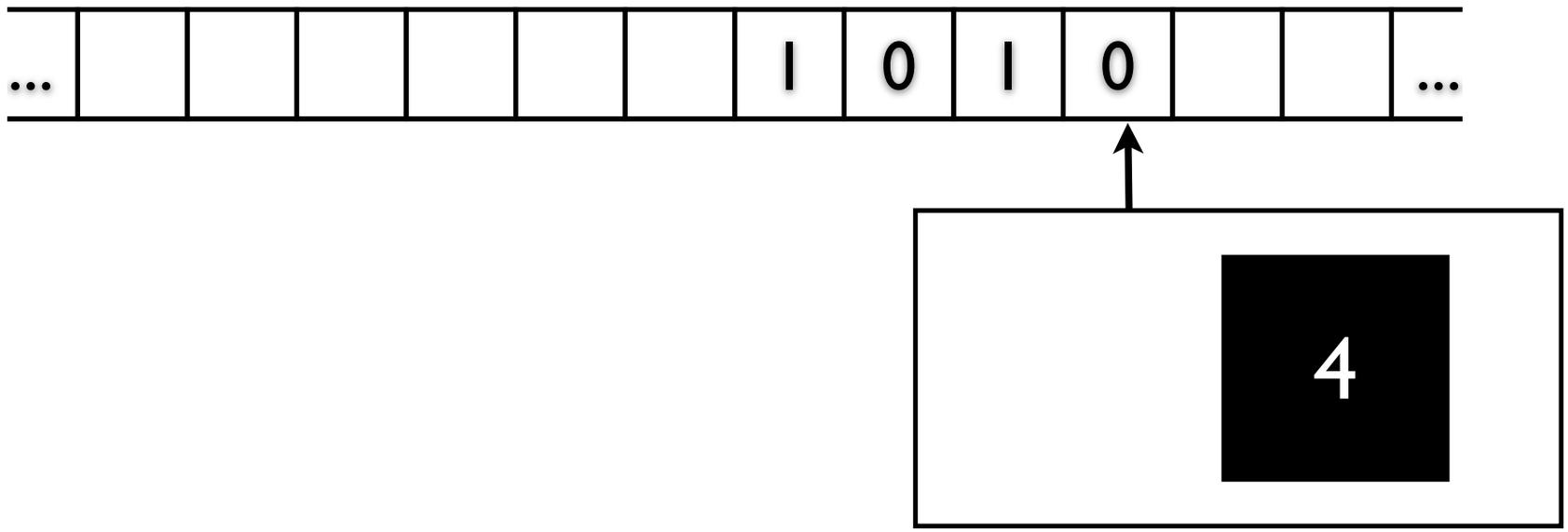

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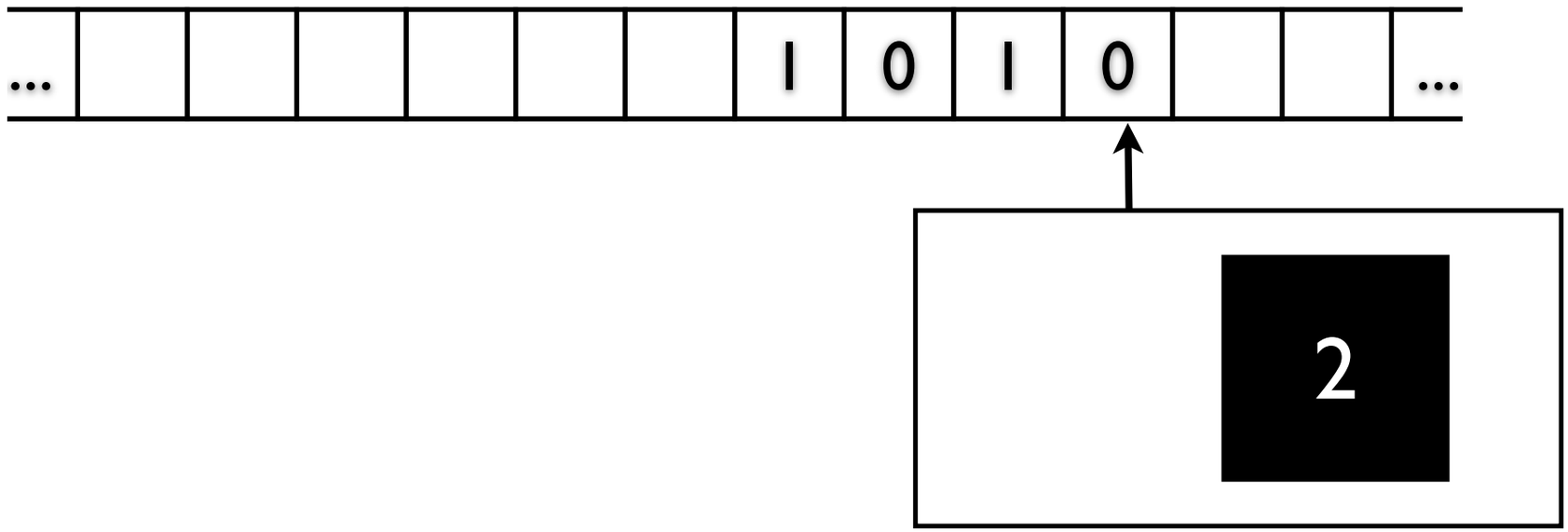

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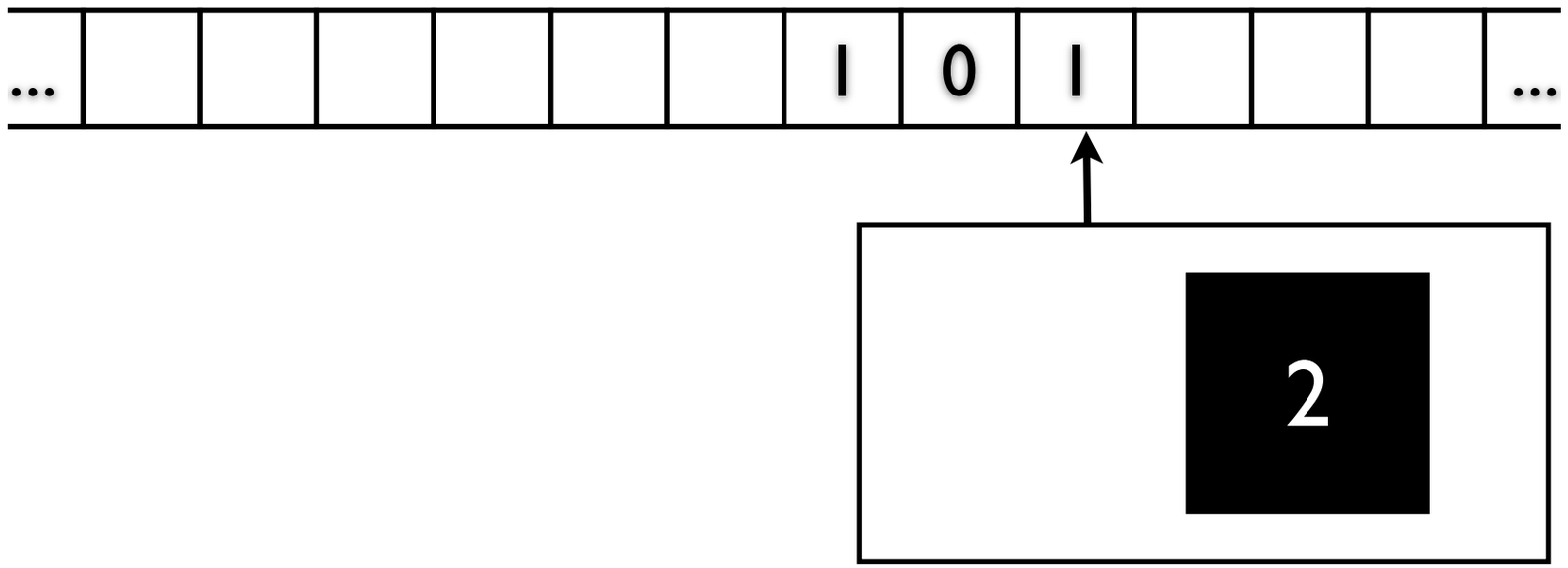

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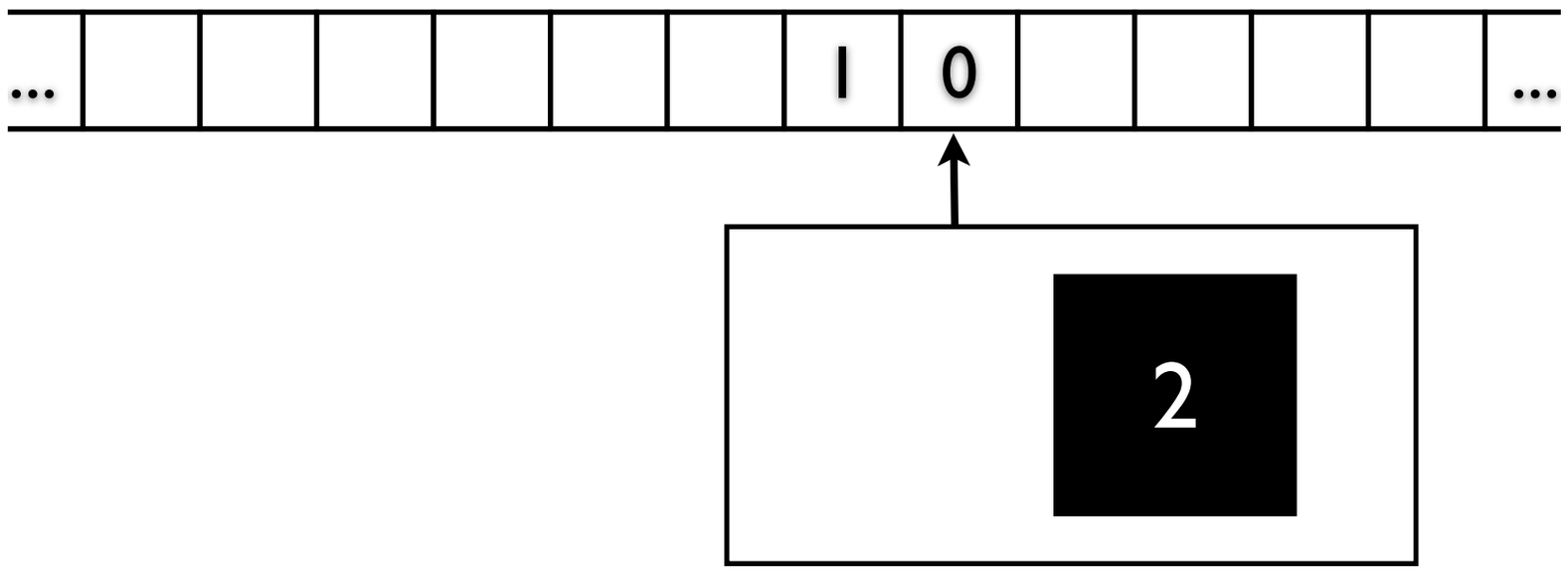

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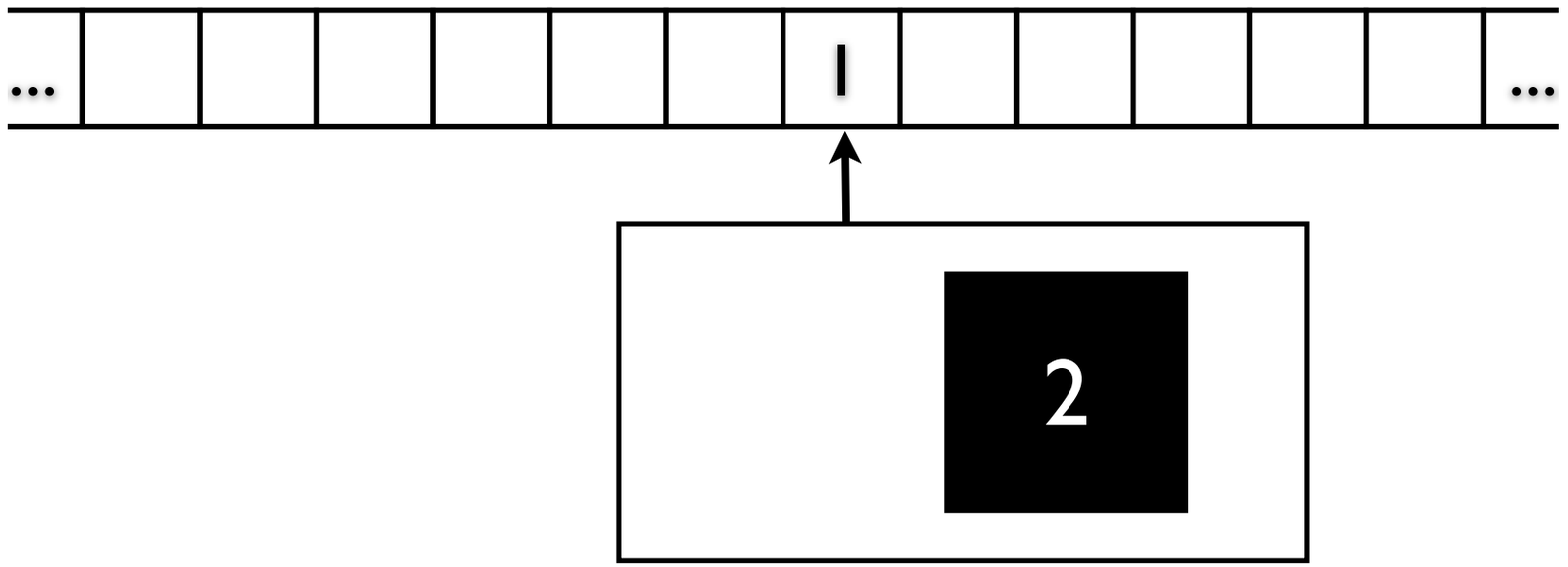

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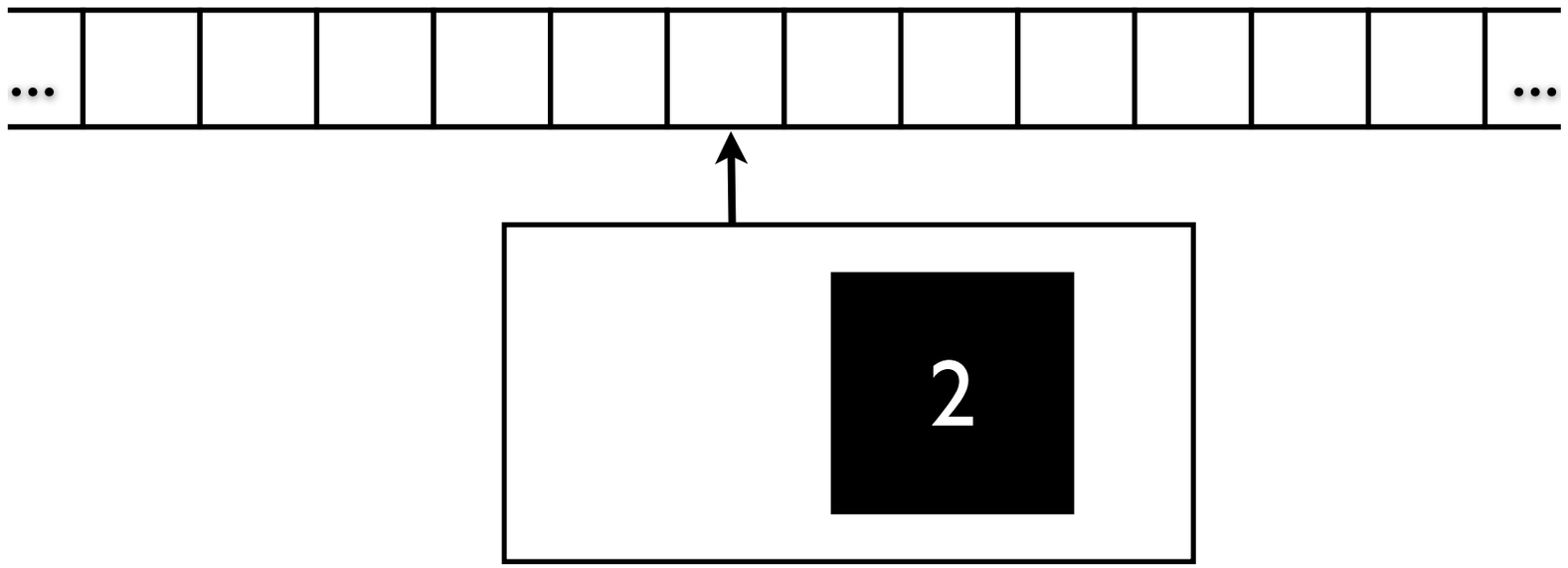

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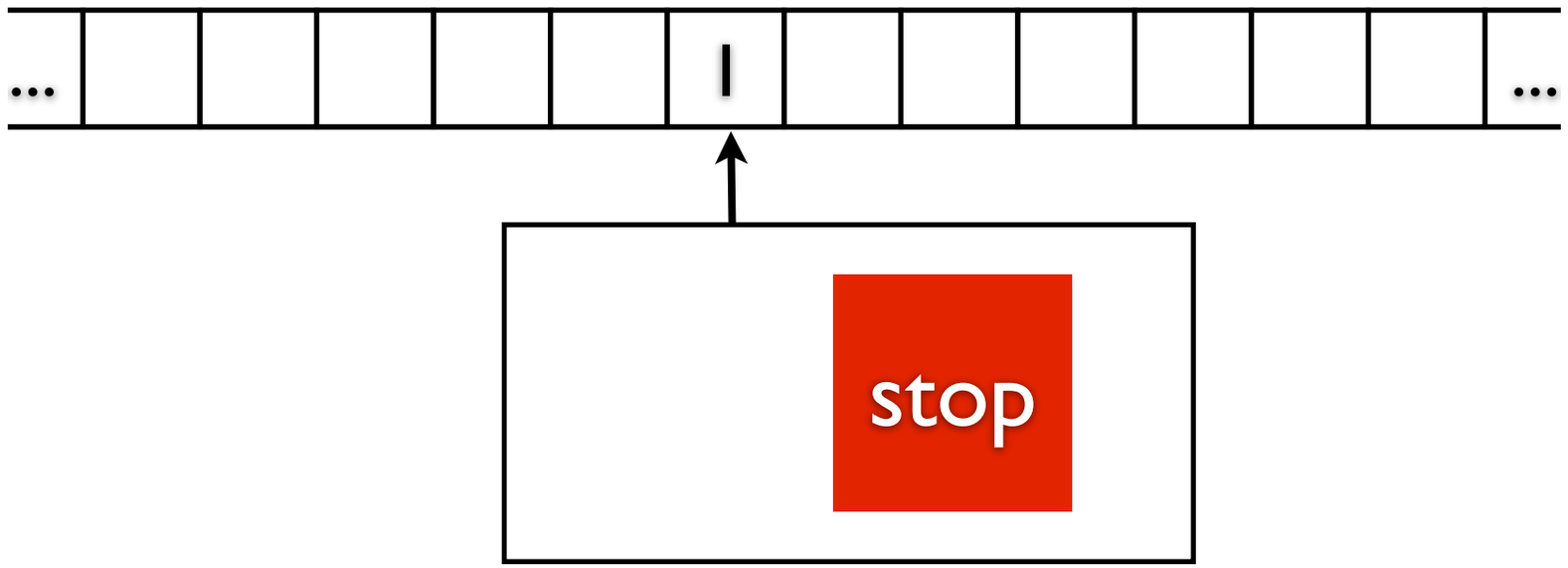

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- Functions that can be computed in this manner are *Turing-computable*.

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- Decision problems (Yes/No problems) that can answered in this manner are *Turing-decidable*. (Here, 1 can be used for **Y**; 2 for **N**.)

For more on TMs …

https://plato.stanford.edu/entries/turing-machine

Theorem: The Halting Problem is Turing-unsolvable.

. . .

We assume an encoding of TMs that permits identification of each with some $m \in \mathbb{Z}^+$, and say that the binary halt function h maps a machine and its input to 1 if that machine halts, and to 2 if it doesn't:

$h(m, n) = 1$ if $m : n \longrightarrow$ halt
$h(m, n) = 2$ if $m : n \longrightarrow \infty$

So, the theorem we need can be expressed this way:

$\neg \exists m_h$ [m_h computes h]

Next, let's contruct a TM m_c that copies a block of 1's (separated by a blank #), and (what BBJ call) a "dithering" TM:

$m_d : n \longrightarrow$ halt if $n > 1$; $m_d : n \longrightarrow \infty$ if $n = 1$

Proof: Suppose for *reductio* that m_h computes h. Then we can make a composite machine m^3 consisting of m_c connected to and feeding m_h and this machine connected to and feeding m_d. It's easy to see (use some paper and pencil/stylus and tablet!) that

(1) if $h(n, n) = 1$, then $m^3 : n \longrightarrow \infty$

and

(2) if $h(n, n) = 2$, then $m^3 : n \longrightarrow$ halt.

To reach our desired contradiction, we simply ask: What happens when we instantiate n to m^3 in (1) and (2)? The answer to this question, and its being just what the doctor ordered, is left to the reader. **QED**

Church's Theorem

& its proof …

Theorem: The *Entscheidungsproblem* is Turing-unsolvable.

Proof-sketch: We need to show that the question $\Phi \vdash \phi ?$ is not Turing-decidable. (Here we are working within the framework of $\mathscr{L}_1$.) To begin, note that competent users of HS® know that any Turing machine m can be formalized in a HS® workspace. (Explore! Prove it to yourself in hands-on fashion!) They will also then know that

$$(\dagger) \quad \forall m, n \; \exists \Phi, \phi \; [\Phi \vdash \phi \leftrightarrow \; m : n \longrightarrow \text{halt}]$$

where Φ and ϕ are built in HS®.

Now, let's assume for contradiction that theoremhood in first-order logic can be decided by a Turing machine m_t. But this is absurd. Why? Because imagine that someone now comes to us asking whether some arbitrary TM m halts. We can infallibly supply an answer, because we can formalize m in line with (†) and then employ m_t to given us the answer. **QED**