Gödel's SpeedupTheorem
(GST)

Selmer Bringsjord

Rensselaer AI & Reasoning (RAIR) Lab
Department of Cognitive Science
Department of Computer Science
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Ver 10/1/2020 (updated Oct 6 2020)

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Note: This is a version designed for those who have had at least one serious, proof-intensive university-level course in formal logic.

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Background Context …

Gödel's Great Theorems (OUP)

by Selmer Bringsjord

- Introduction ("The Wager")
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
- The Continuum-Hypothesis Theorem
- The Time-Travel Theorem
- Gödel's "God Theorem"
- Could a Finite Machine Match Gödel's Greatness?

STOP & REVIEW IF NEEDED!

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Switching to more expressive logics can produce a level of speedup beyond the reaching of standard computation.

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Switching to more expressive logics can produce a level of speedup beyond the reaching of standard computation.

By far the greatest of GGT; Selm's analysis based Sherlock Holmes' mystery "Silver Blaze."

Ascending Acceleration

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2 sec: 60 mph   5.5 sec: 100 mph   7.5 sec: 150 mph

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20 sec: 268 mph 520 sec: 17,000 mph

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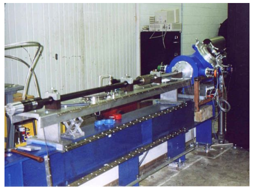

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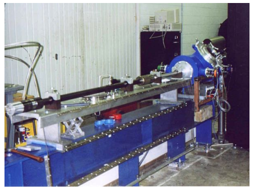

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PrRec: $h(x,0) = f(x); h(x, y') = g(x, y, h(x, y))$

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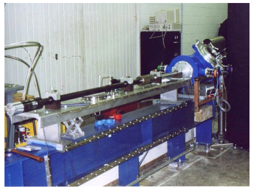

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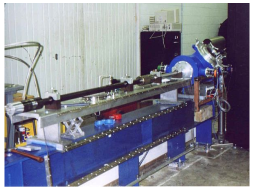

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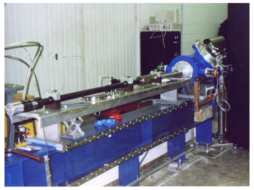

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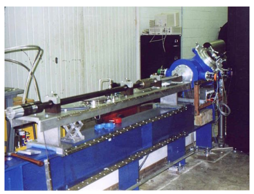

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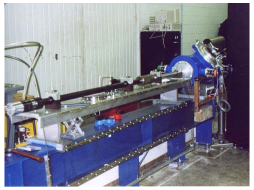

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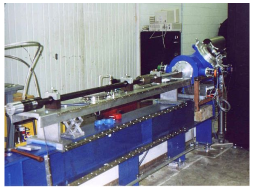

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Ackermann Function

$\alpha(x, y, z) = x\langle y\rangle z$ and $\gamma(x) = \alpha(x, x, x)$; then:

$\gamma(0) = 0 + 0 = 0$
$\gamma(1) = 1 \cdot 1 = 0$
$\gamma(2) = 2^2 = 4$
$\gamma(3) = 3^{3^3} = 3 \uparrow\uparrow 3 = 7,625,597,484,987$
$\gamma(4) = 4 \uparrow\uparrow 4 => 10^{1000}$(note: 10^{100} is googol)

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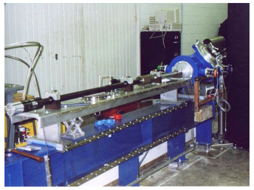

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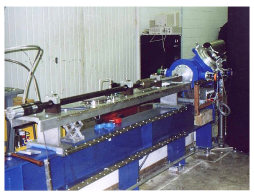

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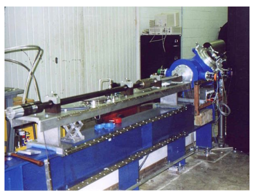

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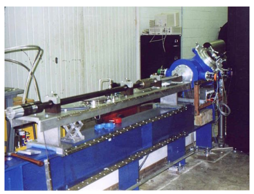

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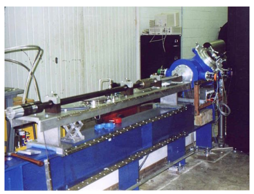

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Climbing the *k*-order Ladder

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a is a llama, as is *b*, *a* likes *b*, and the father of *a* is a llama as well.

Climbing the *k*-order Ladder

$$Llama(a) \wedge Llama(b) \wedge Likes(a,b) \wedge Llama(fatherOf(a))$$

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Climbing the *k*-order Ladder

ZOL $Llama(a) \wedge Llama(b) \wedge Likes(a, b) \wedge Llama(fatherOf(a))$

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Climbing the *k*-order Ladder

There's some thing which is a llama and likes *b* (which is also a llama), and whose father is a llama too.

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Climbing the *k*-order Ladder

$$\exists x \exists y \exists R[R(x) \wedge R(y) \wedge Likes(x,y) \wedge R(fatherOf(x))]$$

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Things *x* and *y*, along with the father of *x*, share a certain property; and, *x* R^2s *y*, where R^2 is a positive property.

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$$\exists x, y\ \exists R, R^2[R(x) \land R(y) \land R^2(x,y) \land Positive(R^2) \land R(fatherOf(x))]$$

Things x and y, along with the father of x, share a certain property; and, x R^2s y, where R^2 is a positive property.

SOL $$\exists x \exists y \exists R[R(x) \land R(y) \land Likes(x,y) \land R(fatherOf(x))]$$

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FOL $\exists x [Llama(x) \land Llama(b) \land Likes(x,b) \land Llama(fatherOf(x))]$

There's some thing which is a llama and likes *b* (which is also a llama), and whose father is a llama too.

ZOL $Llama(a) \land Llama(b) \land Likes(a,b) \land Llama(fatherOf(a))$

a is a llama, as is *b*, *a* likes *b*, and the father of *a* is a llama as well.

Climbing the *k*-order Ladder

TOL $\mathscr{L}_3$

$$\exists x, y\ \exists R, R^2[R(x) \land R(y) \land R^2(x,y) \land Positive(R^2) \land R(fatherOf(x))]$$

Things *x* and *y*, along with the father of *x*, share a certain property; and, *x* R^2s *y*, where R^2 is a positive property.

SOL $\mathscr{L}_2$

$$\exists x \exists y \exists R[R(x) \land R(y) \land Likes(x,y) \land R(fatherOf(x))]$$

Things *x* and *y*, along with the father of *x*, share a certain property (and *x* likes *y*).

FOL $\mathscr{L}_1$

$$\exists x[Llama(x) \land Llama(b) \land Likes(x,b) \land Llama(fatherOf(x))]$$

There's some thing which is a llama and likes *b* (which is also a llama), and whose father is a llama too.

ZOL $\mathscr{L}_0$

$$Llama(a) \land Llama(b) \land Likes(a,b) \land Llama(fatherOf(a))$$

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Climbing the *k*-order Ladder

$\vdots$

TOL $\mathscr{L}_3$

$$\exists x, y\ \exists R, R^2[R(x) \land R(y) \land R^2(x, y) \land Positive(R^2) \land R(fatherOf(x))]$$

Things x and y, along with the father of x, share a certain property; and, x R^2s y, where R^2 is a positive property.

SOL $\mathscr{L}_2$

$$\exists x \exists y \exists R[R(x) \land R(y) \land Likes(x, y) \land R(fatherOf(x))]$$

Things x and y, along with the father of x, share a certain property (and x likes y).

FOL $\mathscr{L}_1$

$$\exists x[Llama(x) \land Llama(b) \land Likes(x, b) \land Llama(fatherOf(x))]$$

There's some thing which is a llama and likes b (which is also a llama), and whose father is a llama too.

ZOL $\mathscr{L}_0$

$$Llama(a) \land Llama(b) \land Likes(a, b) \land Llama(fatherOf(a))$$

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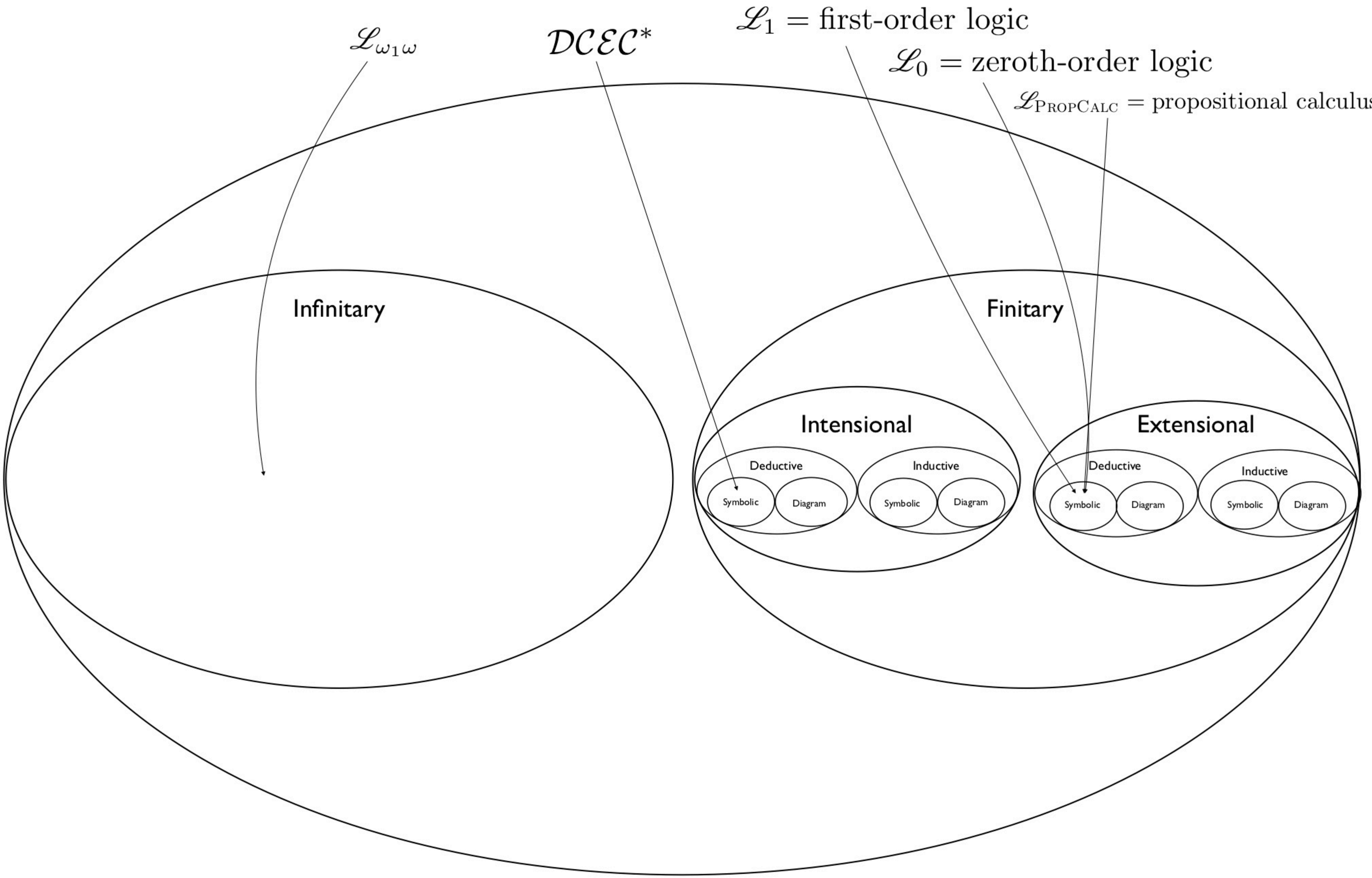
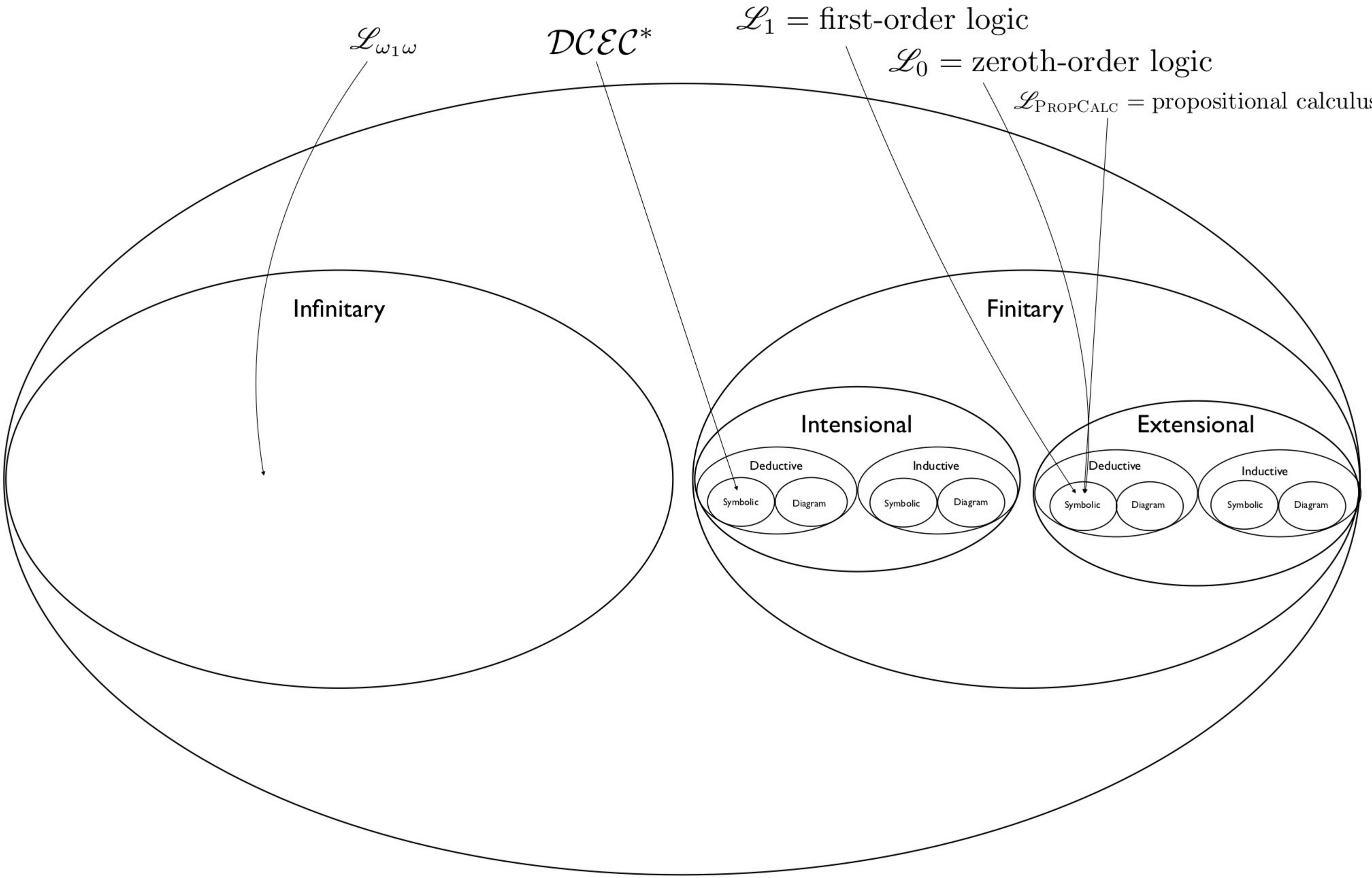
The Universe of Logics



$\mathscr{L}_3$
$\mathscr{L}_2$

The Universe of Logics



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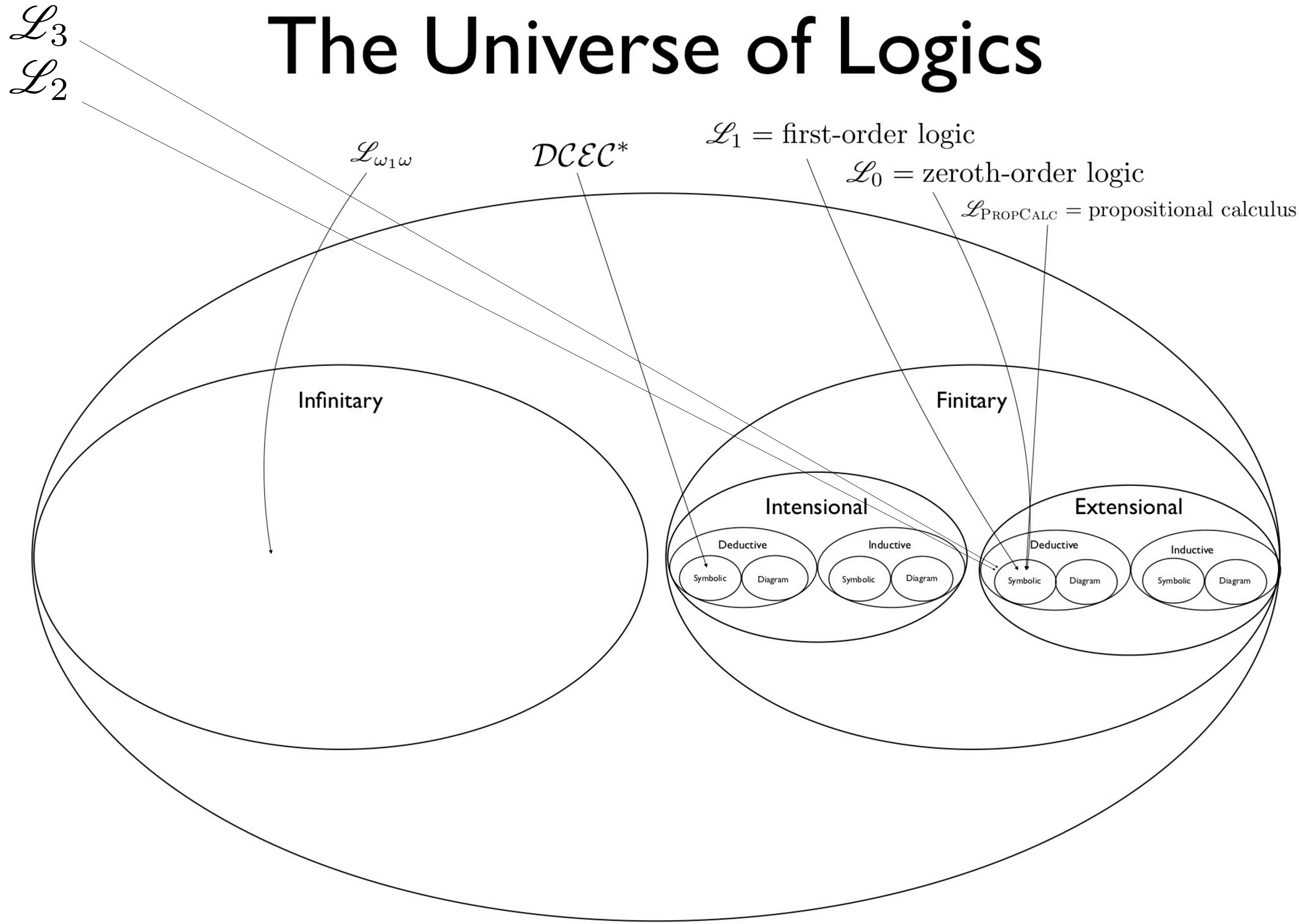

Climbing the *k*-order Ladder

⋮

TOL $\mathscr{L}_3$

$$\exists x, y\ \exists R, R^2[R(x) \wedge R(y) \wedge R^2(x,y) \wedge Positive(R^2) \wedge R(fatherOf(x))]$$

Things *x* and *y*, along with the father of *x*, share a certain property; and, *x* R^2s *y*, where R^2 is a positive property.

SOL $\mathscr{L}_2$

$$\exists x \exists y \exists R[R(x) \wedge R(y) \wedge Likes(x,y) \wedge R(fatherOf(x))]$$

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Gödel's Speedup Theorem

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1. $\forall \phi \in \mathcal{F}, Z_i \vdash \phi$; and
2. $\forall \phi \in \mathcal{F}$, if k is the least integer s.t. $Z_{i+1} \vdash^{k \text{ symbols}} \phi$, then $Z_i \nvdash^{f(k) \text{ symbols}} \phi$.

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!

A Simpler Speedup Theorem

A Simpler Speedup Theorem

Let f be any recursive function, and again let us refer to $\Phi \supset \mathbf{PA}$. Then there are arithmetic $\mathscr{L}_1$ sentences ϕ s.t. $\Phi \vdash \phi$, where the shortest proof P confirming this has more more than $f(n^{\phi})$ symbols.

To prove GST, we shall once again allow ourselves …

The Fixed Point Theorem (FPT)

Assume that Φ is a set of arithmetic sentences such that Repr Φ. There for every arithmetic formula $\psi(x)$ with one free variable x, there is an arithmetic sentence ϕ s.t.

$$\Phi \vdash \phi \leftrightarrow \psi(n^{\phi}).$$

We can intuitively understand ϕ to be saying:
"I have the property ascribed to me by the formula ψ."

Ok; so let's do it …

Proof: Let f^* be an arbitrary (total) recursive function. We can clearly write a formula that expresses the property of having a proof in **PA** shorter, symbol-wise, than $f(n^{\phi})$, for the Gödel number of any formula ϕ. Let us do it like this: Prov-sh$_{\Phi}(n^{\phi})$. By Repr Φ, since a Turing machine can compute this relation, we then have:

(Rep*) = (1) Prov-sh$_{\Phi}(n^{\phi})$ iff $\Phi \vdash \phi$

Next, we can instantiate the Fixed Point Theorem to yield a formula that declares "There's no proof of me shorter than what f^* applied to me returns!" More formally, the instantiation will be:

(FPT*) = (2) $\Phi \vdash \bar{\pi}_{sh} \leftrightarrow \neg$Prov-sh$_{\Phi}(n^{\bar{\pi}_{sh}})$

Now what about this self-referential sentence? Can it have a proof shorter than f^* applied to its Gödel number? Suppose it does. Then by left-to-right on (1) it's provable in Φ. But given this, combined with (2), this self-referential sentence is *not* provable by a derivation shorter than f^* applied to it — contradiction! **QED**

Proof (short!): Let f^* be a (total) recursive function. Write Prov-sh$_\Phi(n^\phi)$ to express having a proof in **PA** shorter, symbol-wise, than $f(n^\phi)$. Since Repr Φ, and this relation is Turing-computable:

(Rep*) = (1) Prov-sh$_\Phi(n^\phi)$ iff $\Phi \vdash \phi$

Next, instantiate the Fixed Point Theorem to yield:

(FPT*) = (2) $\Phi \vdash \bar{\pi}_{sh} \leftrightarrow \neg$Prov-sh$_\Phi(n^{\bar{\pi}_{sh}})$

Suppose this self-referential formula has a short proof. Then by left-to-right on (1) it's provable in Φ. But given this, combined with (2), this self-referential sentence is *not* provable by a derivation shorter than f^* applied to it — contradiction! **QED**

Med nok penger, kan logikk løse alle problemer.