Gödel's
First Incompleteness Theorem
(G1)

Selmer Bringsjord*

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Department of Cognitive Science
Department of Computer Science
Lally School of Management & Technology
Rensselaer Polytechnic Institute (RPI)
Troy, New York 12180 USA

9/21/2020 (patched 10/5/20)

*With Naveen Sundar G's introduction to Gödel numbering, and his depiction of GT growth, etc.

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Note: This is a version designed for those who have had at least one robust, proof-intensive university-level course in formal logic.

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Background Context …

Gödel's Great Theorems (OUP)

by Selmer Bringsjord

- Introduction ("The Wager")
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
- The Continuum-Hypothesis Theorem
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- Gödel's "God Theorem"
- Could a Finite Machine Match Gödel's Greatness?

STOP & REVIEW IF NEEDED!

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A corollary of the First Incompleteness Theorem: *We cannot prove (in classical mathematics) that mathematics is consistent.*

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By far the greatest of GGT; Selm's analysis based Sherlock Holmes' mystery "Silver Blaze."

Deficient; Beware

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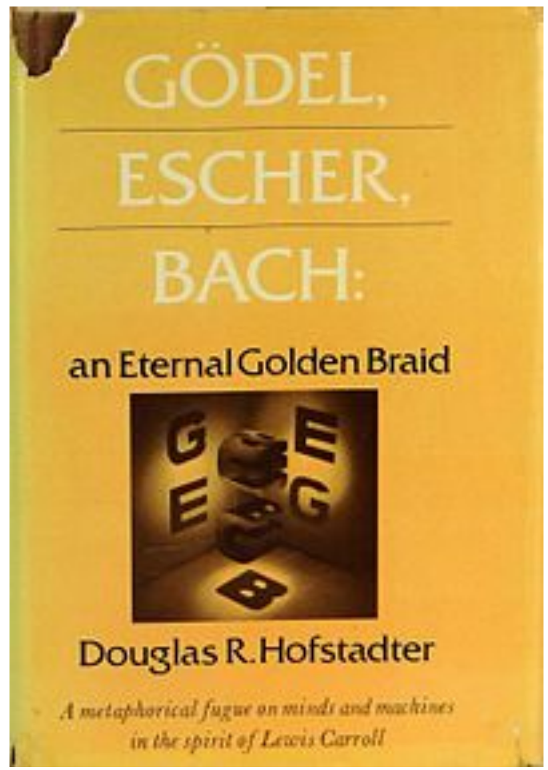

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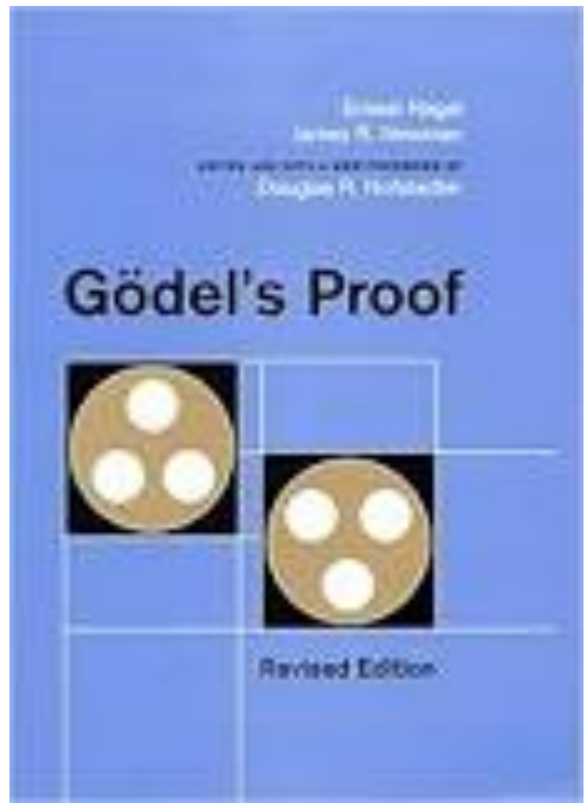

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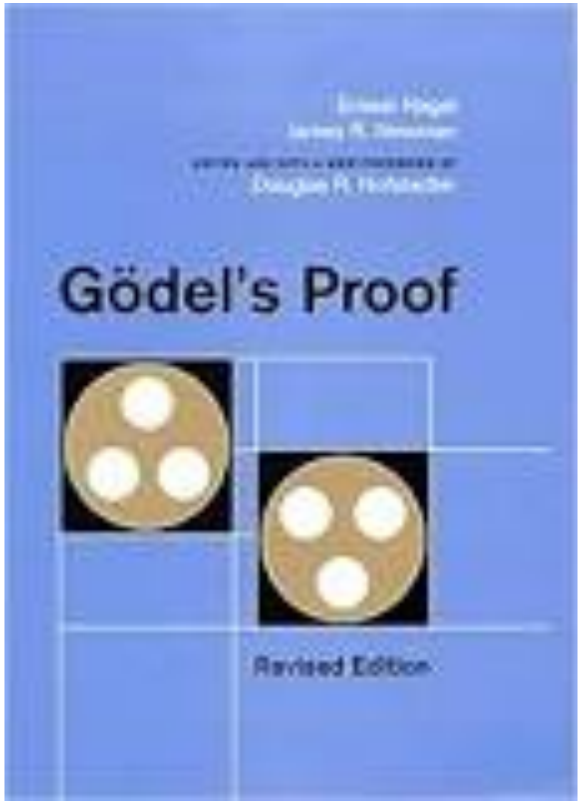

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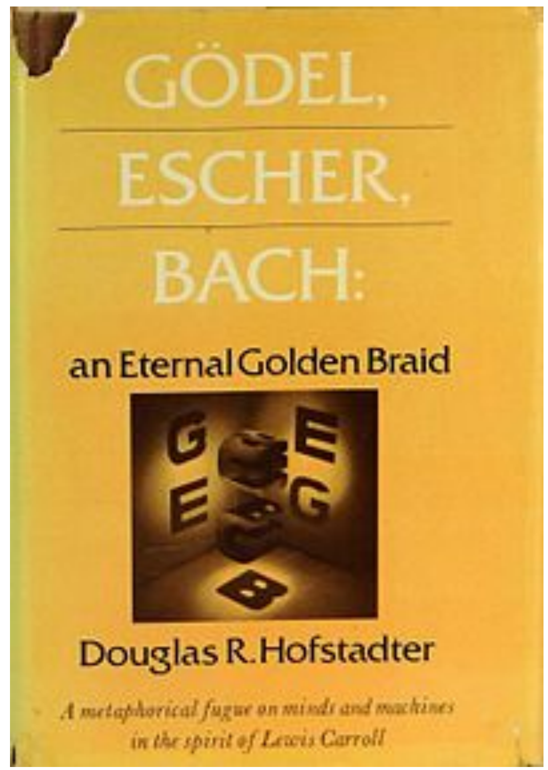


Some Timeline Points

1978 Princeton NJ USA.

1940 Back to USA, for good.

1936 Schlick murdered; Austria annexed

1933 Hitler comes to power.

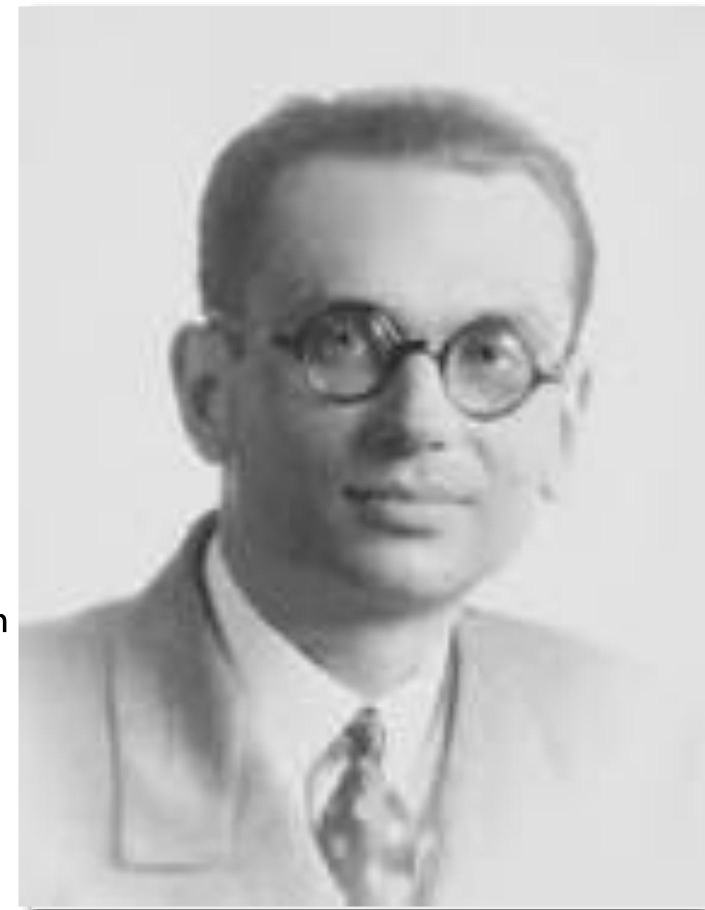
1930 Announces (First) *In*completeness Theorem

1929 Doctoral Dissertation: Proof of Completeness Theorem

Undergrad in seminar by Schlick

1923 Vienna

1906 Brünn, Austria-Hungary



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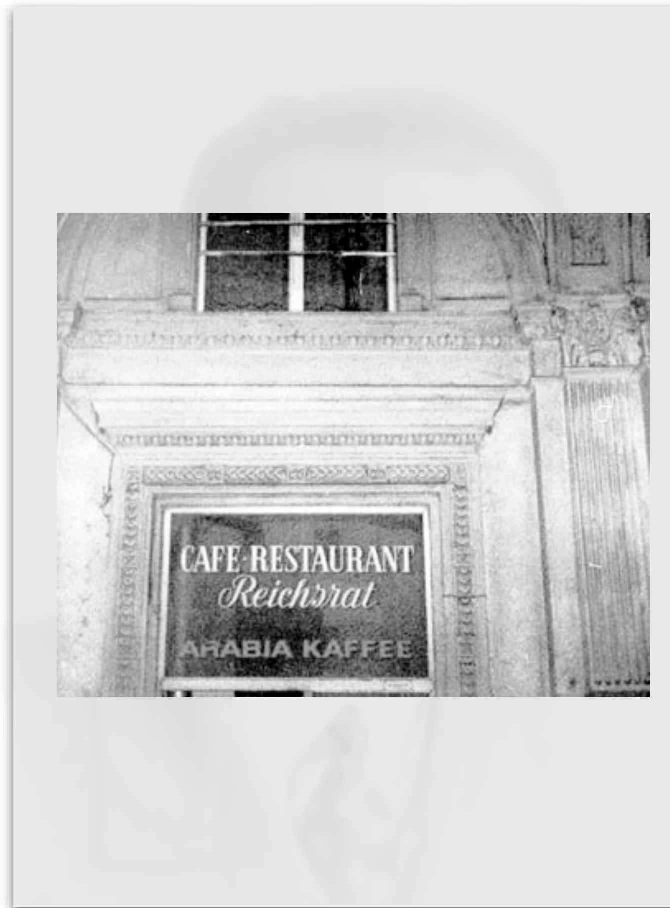

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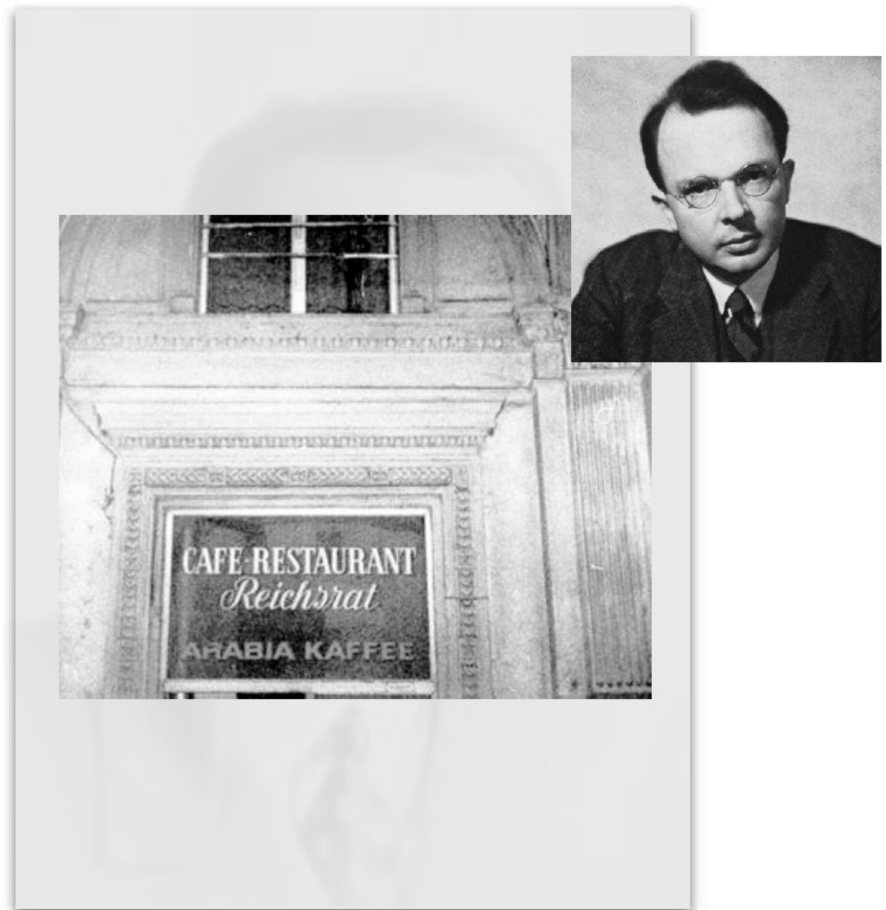

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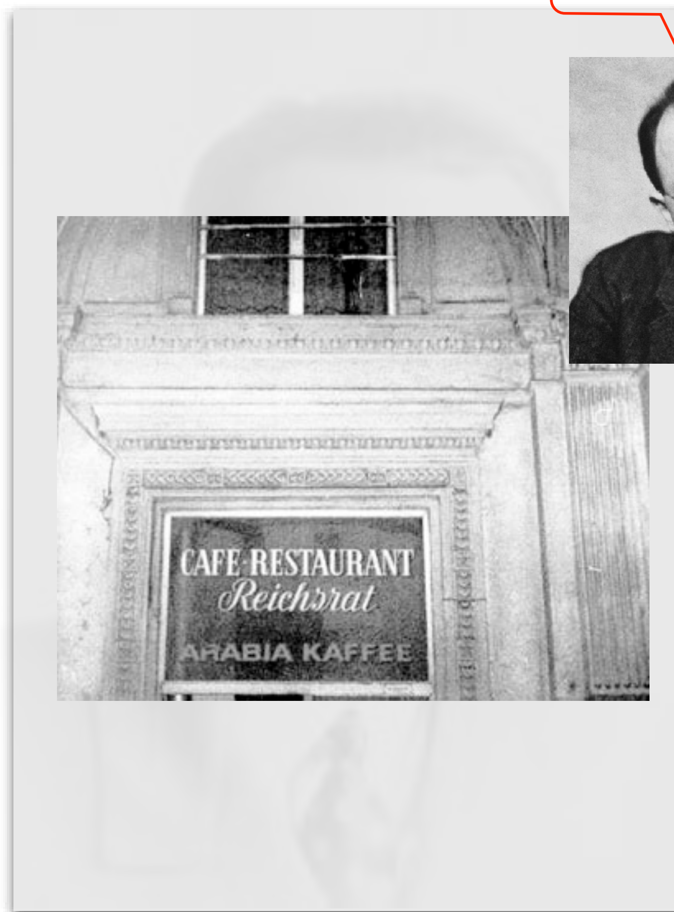
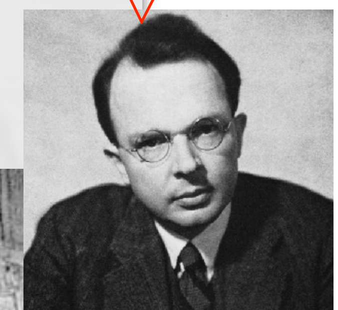


“Well, uh, hmm, …”



The “Liar Tree”

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The Liar Paradox

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Pure Proof-Theoretic Route

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Pure Proof-Theoretic Route

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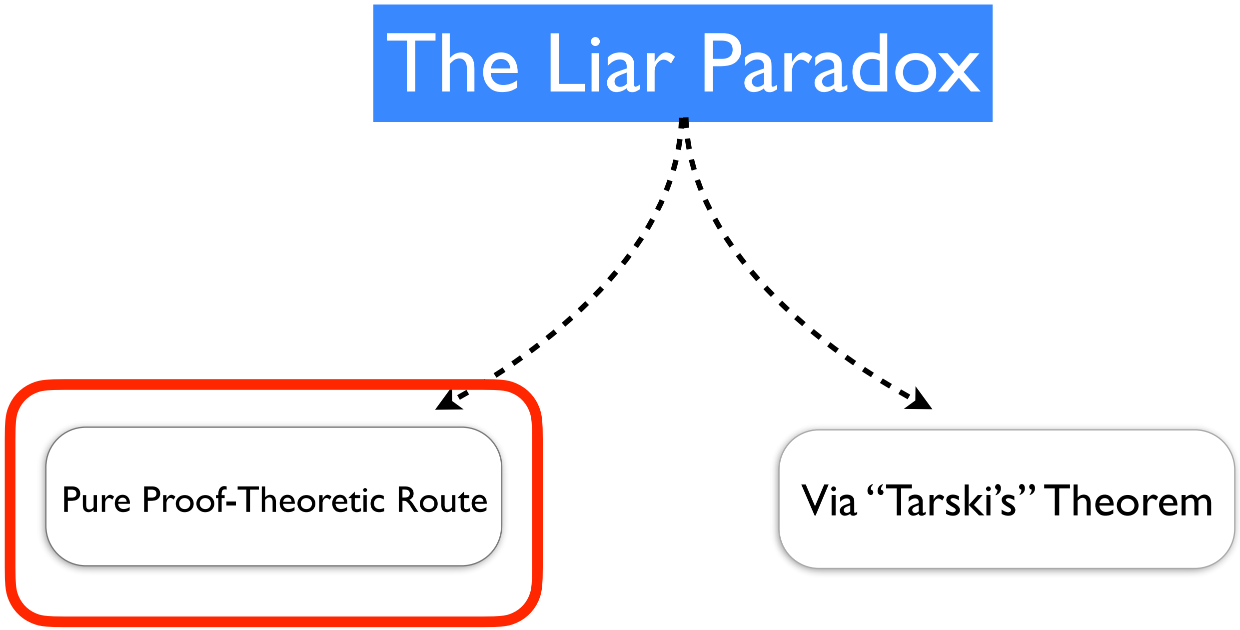
- Pure Proof-Theoretic Route
- Via “Tarski’s” Theorem

The "Liar Tree"



The “Liar Tree”

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Pure Proof-Theoretic Route

Via “Tarski’s” Theorem

Paul Erdős

“The Book”

The "Liar Tree"

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Via "Tarski's" Theorem

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"The Book"

Ergo, step one: What *is* LP?

“The (Economical) Liar”

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Suppose that T(**L);** then ¬T(**L**).

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Hence: T(**L**) iff (i.e., if & only if) ¬T(L).

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Contradiction!

The “Gödelian” Liar (from me)

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$\bar{P}$: This sentence is unprovable.

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Suppose that $\bar{P}$ is true. Then we can immediately deduce that $\bar{P}$ is provable, because here is a proof: $\bar{P} \rightarrow \bar{P}$ is an easy theorem, and from it and our supposition we deduce $\bar{P}$ by *modus ponens*. But since what $\bar{P}$ says is that it's unprovable, we have deduced that $\bar{P}$ is false under our initial supposition.

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Suppose on the other hand that $\bar{P}$ is false. Then we can immediately deduce that $\bar{P}$ is unprovable: Suppose for *reductio* that $\bar{P}$ *is* provable; then $\bar{P}$ holds as a result of some proof, but what $\bar{P}$ says is that it's unprovable; and so we have contradiction. But since what $\bar{P}$ says is that it's unprovable, and we have just proved that under our supposition, we arrive at the conclusion that $\bar{P}$ is true.

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All of this is fishy; but Gödel transformed it into utterly precise, impactful, indisputable reasoning …

PA (Peano Arithmetic):

$$
\begin{array}{ll}
\text{A1} & \forall x(0 \neq s(x)) \\
\text{A2} & \forall x \forall y(s(x) = s(y) \rightarrow x = y) \\
\text{A3} & \forall x(x \neq 0 \rightarrow \exists y(x = s(y)) \\
\text{A4} & \forall x(x + 0 = x) \\
\text{A5} & \forall x \forall y(x + s(y) = s(x + y)) \\
\text{A6} & \forall x(x \times 0 = 0) \\
\text{A7} & \forall x \forall y(x \times s(y) = (x \times y) + x)
\end{array}
$$

And, every sentence that is the universal closure of an instance of

$$([\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x))] \rightarrow \forall x \phi(x))$$

where $\phi(x)$ is open wff with variable x, and perhaps others, free.

Arithmetic Crucial Part of All Things Sci/Eng/Tech!

but alas, courtesy of Gödel: An infinite number of arithmetic propositions impossible to settle/decide.

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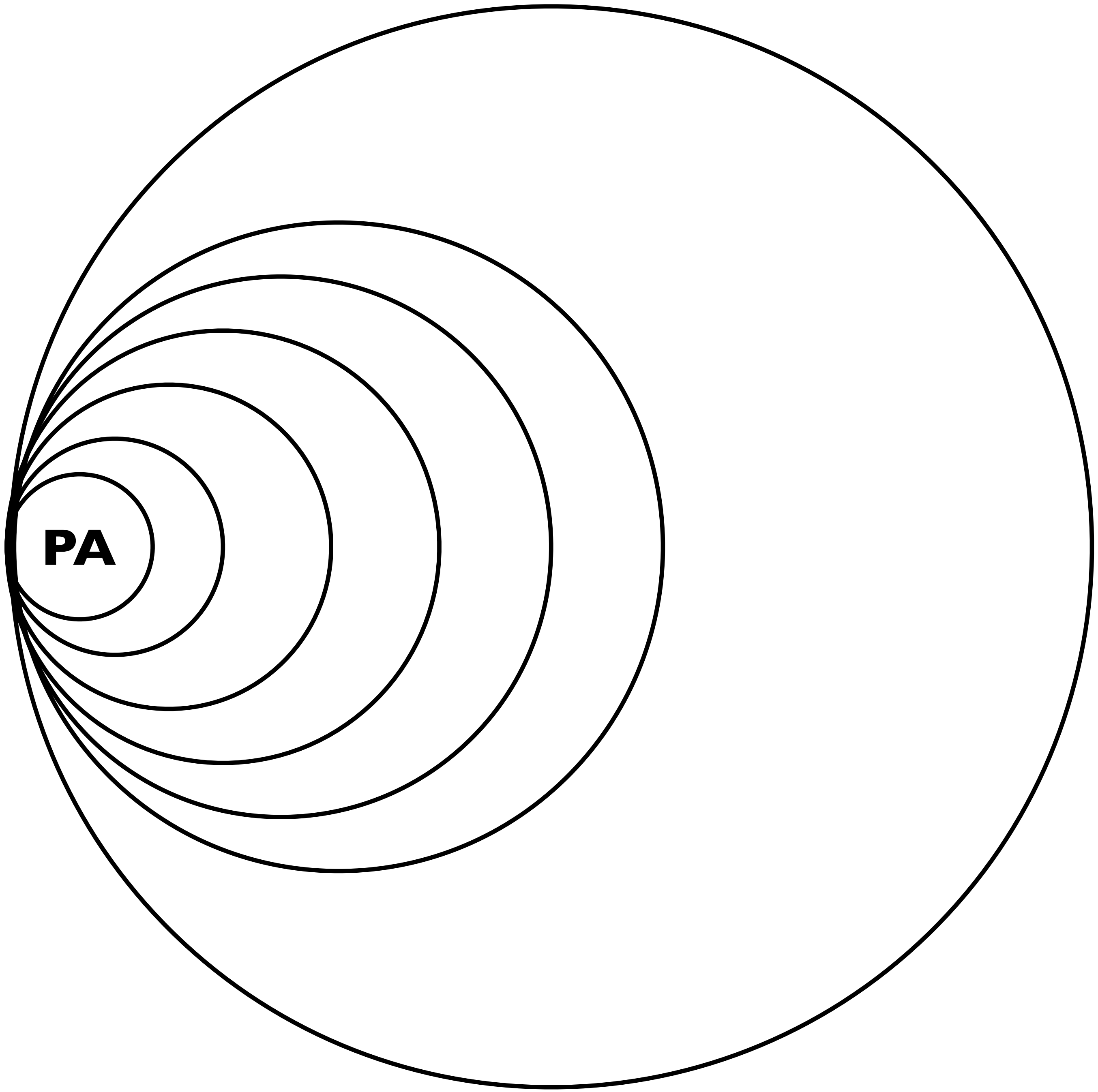

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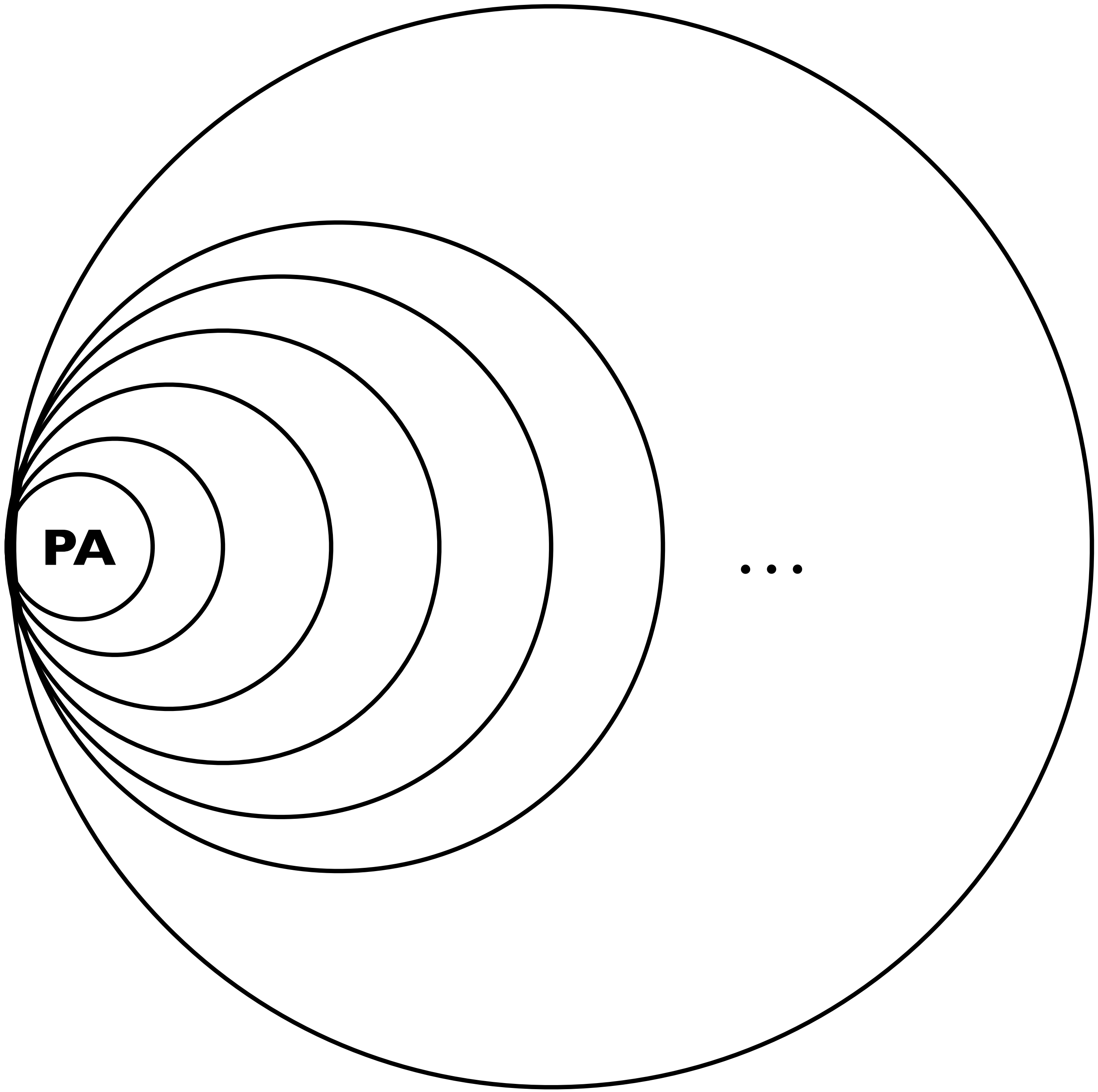

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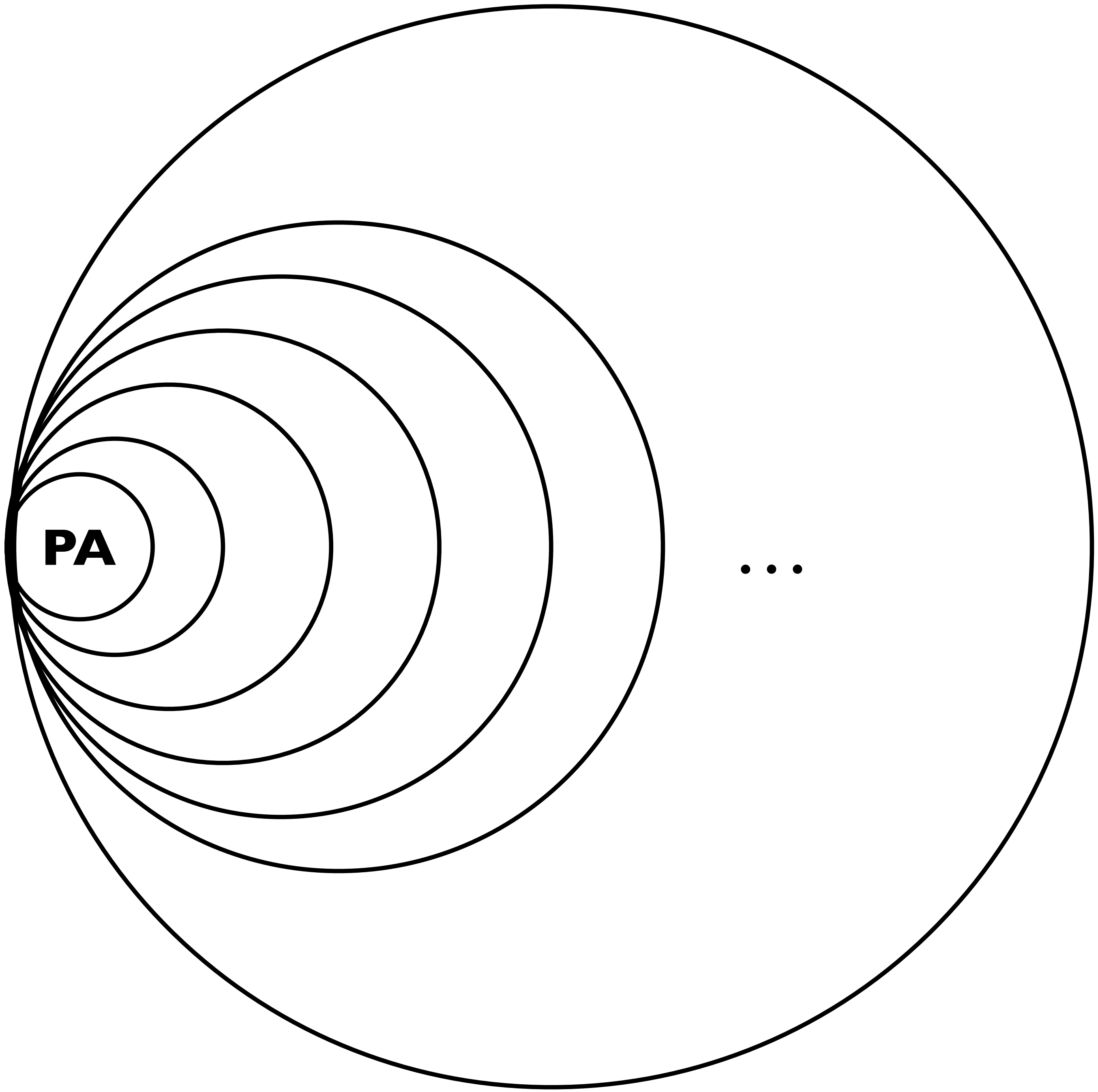

Each circle is a larger part of the formal sciences.

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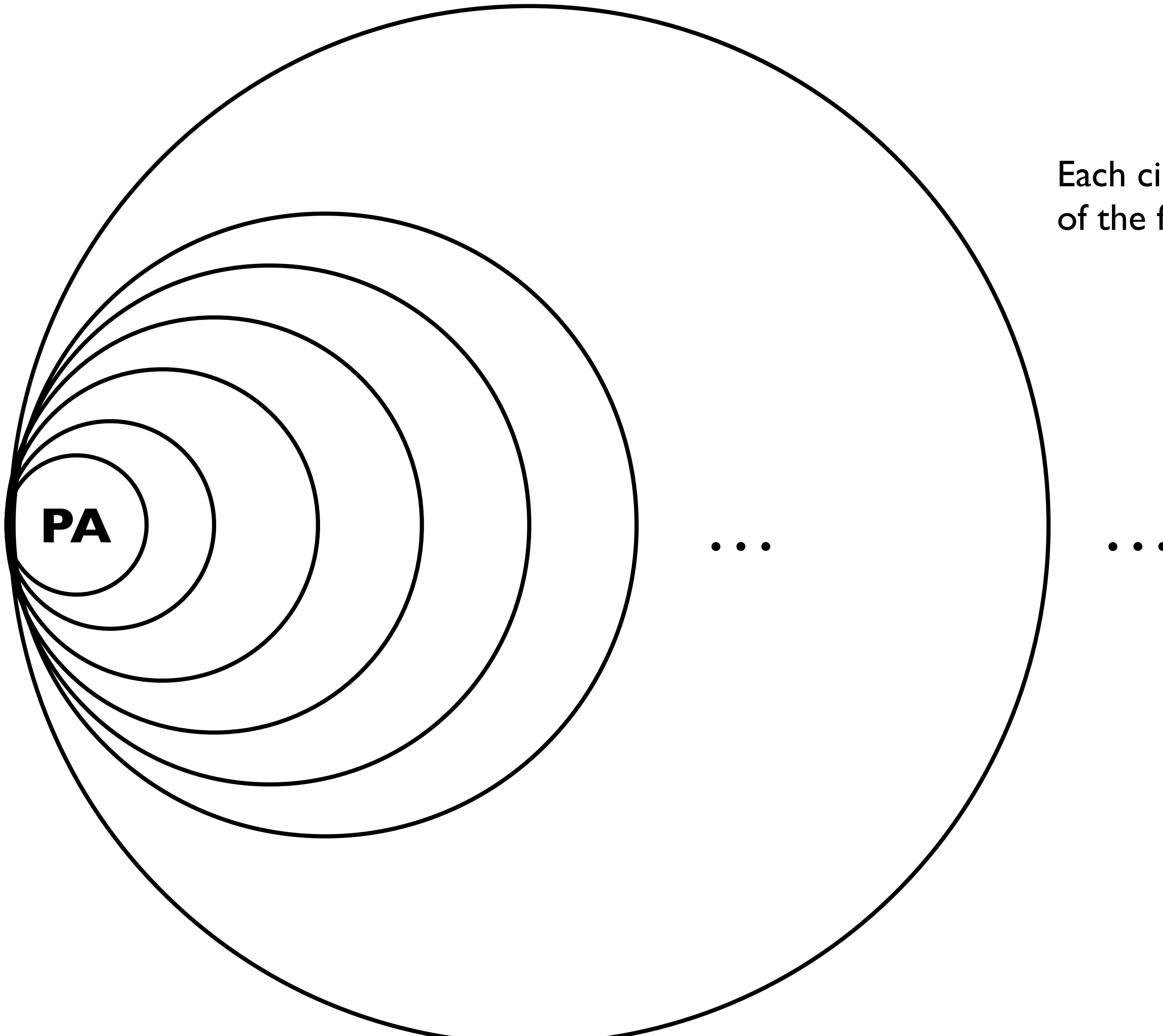

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Gödel Numbering

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Problem: How do we enable a formula to refer to other formulae and itself (and also other objects like proofs, terms etc.), in a perfectly consistent way?

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$$\boxed{\begin{array}{l} \phi \\ \phi \to \psi \\ f(x, a) \end{array}}$$

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Object-level objects in the language of $\mathscr{L}_1$

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(formulae, terms, proofs etc)

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$$f(x, a)$$

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ϕ	323
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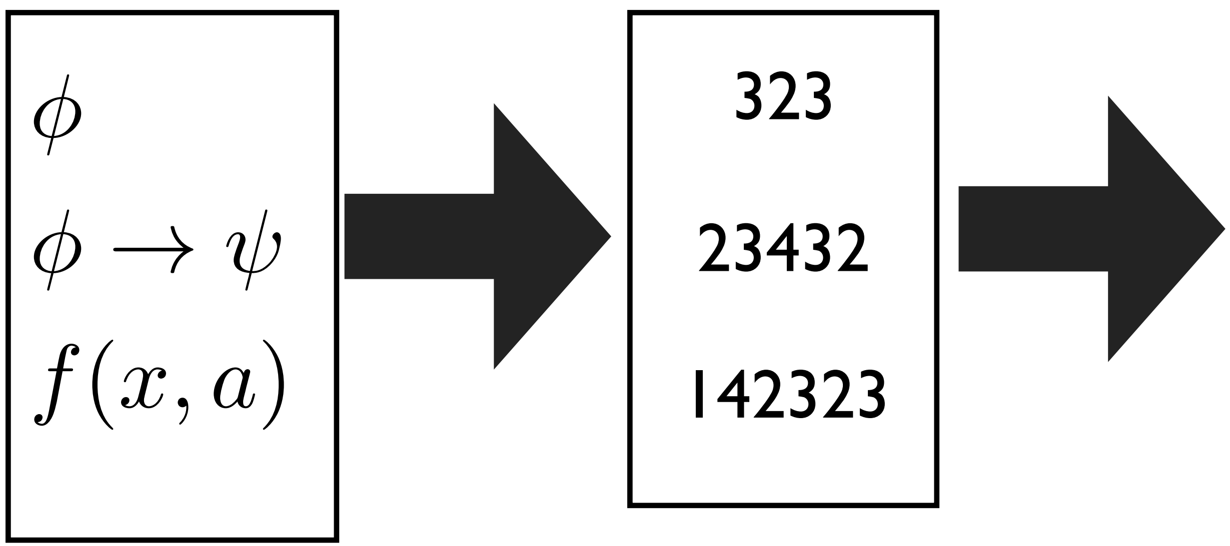

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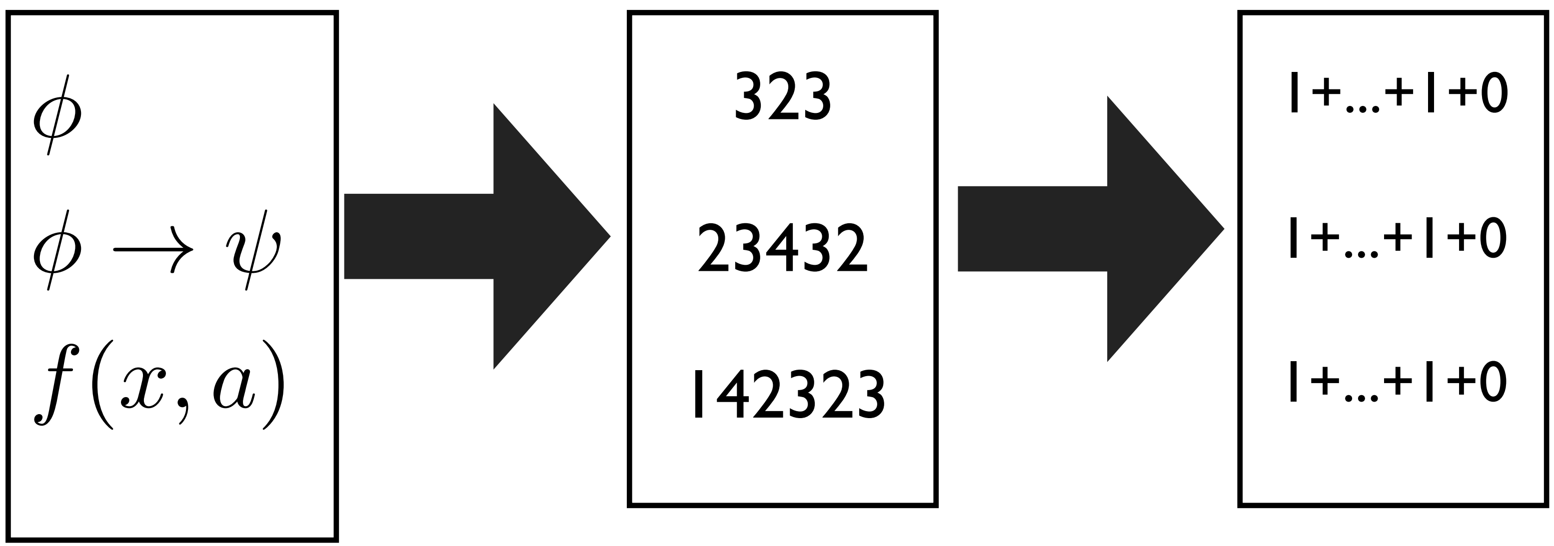

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Gödel number

n^{ϕ}

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Gödel Numbering, the Easy Way

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Just realize that every entry in a dictionary is named by a number *n*, and by the same basic lexicographic ordering, every computer program, formula, etc. is named by a number *m* in a lexicographic ordering going from 1, to 2, to …

Gödel Numbering, the Easy Way

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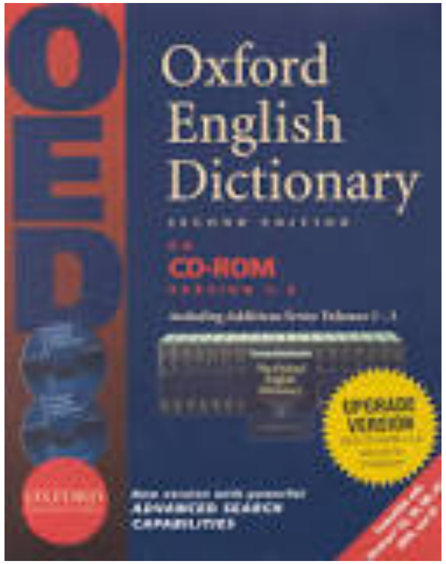

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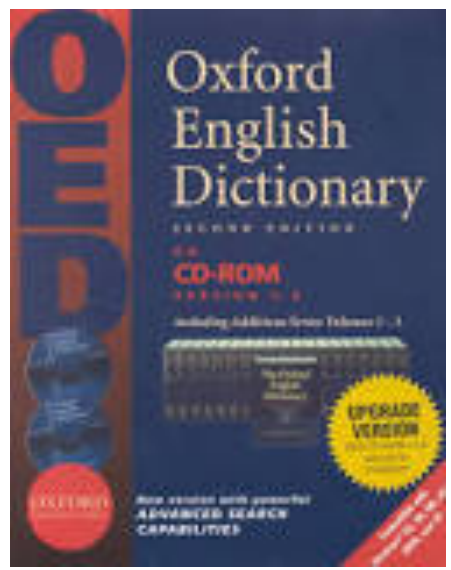

So, `gimcrack` is named by some positive integer *k*. Hence, I can just refer to this word as "*k*" Or in the notation I prefer: k^{gimcrack}.

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Or, every syntactically valid computer program in Clojure that you will ever write can be uniquely denoted by some number *m* in the lexicographic ordering of all syntactically valid such programs. So your program π can just be coded as a numeral m^{π} in a formal language that captures arithmetic (i.e., an *arithmetic language*).

Gödel's First Incompleteness Theorem

Let Φ be a set of arithmetic sentences that is

(i) consistent (i.e. no contradiction $\phi \wedge \neg\phi$ can be deduced from Φ);
(ii) s.t. an algorithm is available to decide whether or not a given string u is a member of Φ; and
(iii) sufficiently expressive to capture all of the operations of a standard computing machine (e.g. a Turing machine, register machine, KU machine, etc.).

Then there is an "undecidable" arithmetic sentence $\mathscr{G}$ from Gödel that can't be proved from Φ, nor can the negation of this sentence (i.e. $\neg\mathscr{G}$) be proved from Φ!

Alas, that's painfully verbose.

Gödel's First Incompleteness Theorem

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Suppose $\Phi \supset \mathbf{PA}$ that is

(i) Con Φ;
(ii) Turing-decidable, and
(iii) sufficiently expressive to capture all of the operations of a Turing machine (i.e. Repr Φ).

Then there is an arithmetic sentence $\mathscr{G}$ s.t.
$\Phi \nvdash \mathscr{G}$ and $\Phi \nvdash \neg\mathscr{G}$.

Gödel's First Incompleteness Theorem

Remember Church's Theorem!

Suppose $\mathbf{\Phi} \supset \mathbf{PA}$ that is

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To prove G1, we shall allow ourselves …

The Fixed Point Theorem (FPT)

Assume that Φ is a set of arithmetic sentences such that Repr Φ. There for every arithmetic formula $\psi(x)$ with one free variable x, there is an arithmetic sentence ϕ s.t.

$\Phi \vdash \phi \leftrightarrow \psi(n^{\phi})$.

We can intuitively understand ϕ to be saying:
"I have the property ascribed to me by the formula ψ."

Ok; so let’s do it …

Proof: Let $\mathbf{\Phi}$ be a set of arithmetic sentences, and suppose the antecedent of **G1** holds, i.e. (i)–(iii) hold. We must show that neither $\mathscr{G}$, nor the negation of this (Liar-Paradox-inspired) arithmetic sentence, can be proved from $\mathbf{\Phi}$. Let us instantiate Repr $\mathbf{\Phi}$ and FPT, respectively:

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(Repr*) = (1) Thm(n^{ϕ}) if and only if $\boldsymbol{\Phi} \vdash \phi$.

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(Repr*) = (1) Thm(n^{ϕ}) if and only if $\Phi \vdash \phi$.

(FPT*) = (2) $\Phi \vdash \mathscr{G}$ if and only if $\neg$Thm($n^{\mathscr{G}}$).

Proof: Let $\mathbf{\Phi}$ be a set of arithmetic sentences, and suppose the antecedent of **G I** holds, i.e. (i)–(iii) hold. We must show that neither $\mathscr{G}$, nor the negation of this (Liar-Paradox-inspired) arithmetic sentence, can be proved from $\mathbf{\Phi}$. Let us instantiate Repr $\mathbf{\Phi}$ and FPT, respectively:

(Repr*) = (1) Thm(n^{ϕ}) if and only if $\mathbf{\Phi} \vdash \phi$.

“I'm unprovable!”

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(FPT*) = (2) $\mathbf{\Phi} \vdash \mathscr{G}$ if and only if $\neg$Thm$(n^{\mathscr{G}})$.

Now suppose $\mathbf{\Phi} \vdash \mathscr{G}$. By right-to-left on (1) we deduce Thm$(n^{\mathscr{G}}) = \neg\neg$Thm$(n^{\mathscr{G}})$. Then $\mathbf{\Phi} \vdash \neg\mathscr{G}$, by right-to-left on (2). But therefore Inc $\mathbf{\Phi}$. Since by hypothesis we have Con $\mathbf{\Phi}$, contradiction!

Proof: Let $\mathbf{\Phi}$ be a set of arithmetic sentences, and suppose the antecedent of **G I** holds, i.e. (i)–(iii) hold. We must show that neither $\mathscr{G}$, nor the negation of this (Liar-Paradox-inspired) arithmetic sentence, can be proved from $\mathbf{\Phi}$. Let us instantiate Repr $\mathbf{\Phi}$ and FPT, respectively:

(Repr*) = (1) Thm(n^{ϕ}) if and only if $\mathbf{\Phi} \vdash \phi$.

(FPT*) = (2) $\mathbf{\Phi} \vdash \mathscr{G}$ if and only if $\neg$Thm($n^{\mathscr{G}}$).

"I'm unprovable!"

Now suppose $\mathbf{\Phi} \vdash \mathscr{G}$. By right-to-left on (1) we deduce Thm($n^{\mathscr{G}}$) $=$ $\neg\neg$Thm($n^{\mathscr{G}}$). Then $\mathbf{\Phi} \vdash \neg\mathscr{G}$, by right-to-left on (2). But therefore Inc $\mathbf{\Phi}$. Since by hypothesis we have Con $\mathbf{\Phi}$, contradiction!

Suppose on the other hand $\mathbf{\Phi} \vdash \neg\mathscr{G}$. Therefore by (2) we deduce $\mathbf{\Phi} \vdash \neg\neg$Thm($n^{\mathscr{G}}$), i.e. $\mathbf{\Phi} \vdash$ Thm($n^{\mathscr{G}}$). From this and an instantiation of (1) we have $\mathbf{\Phi} \vdash \mathscr{G}$. But this entails Inc $\mathbf{\Phi}$. Yet our original assumptions include Con $\mathbf{\Phi}$, so once again: contradiction! **QED**

"Silly abstract nonsense! There aren't any concrete examples of $\mathscr{G}$!"

Ah, but e.g.: Goodstein's Theorem!

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The Goodstein Sequence goes to zero!

Pure base *n* representation of a number *r*

- Represent *r* as only sum of powers of *n* in which the exponents are also powers of *n*, etc.

$$266 = 2^{2^{(2^{2^0}+2^0)}} + 2^{(2^{2^0}+2^0)} + 2^{2^0}$$

Grow Function

$Grow_k(n)$:

1. Take the pure base k representation of n
2. Replace all k by $k+1$. Compute the number obtained.
3. Subtract one from the number

Example of **Grow**

$$Grow_2(19)$$

$$19 = 2^{2^{2^{2^0}}} + 2^{2^0} + 2^0$$

$$3^{3^{3^{3^0}}} + 3^{3^0} + 3^0$$

$$3^{3^{3^{3^0}}} + 3^{3^0} + 3^0 - 1$$

$$7625597484990$$

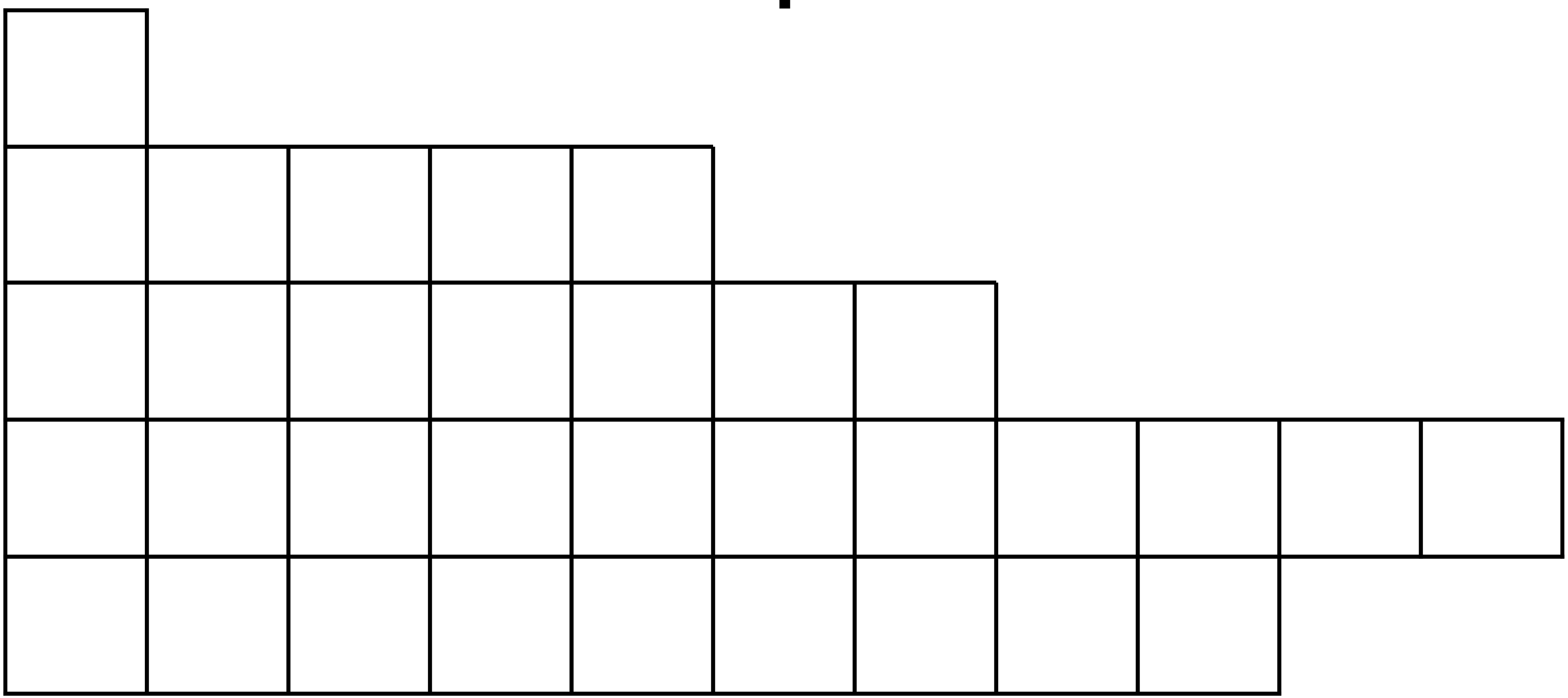
Goodstein Sequence

- For any natural number *m*

$$
\begin{aligned}
&m\\
&Grow_2(m)\\
&Grow_3(Grow_2(m))\\
&Grow_4(Grow_3(Grow_2(m))),\\
&\ldots
\end{aligned}
$$

Sample Values

Sample Values



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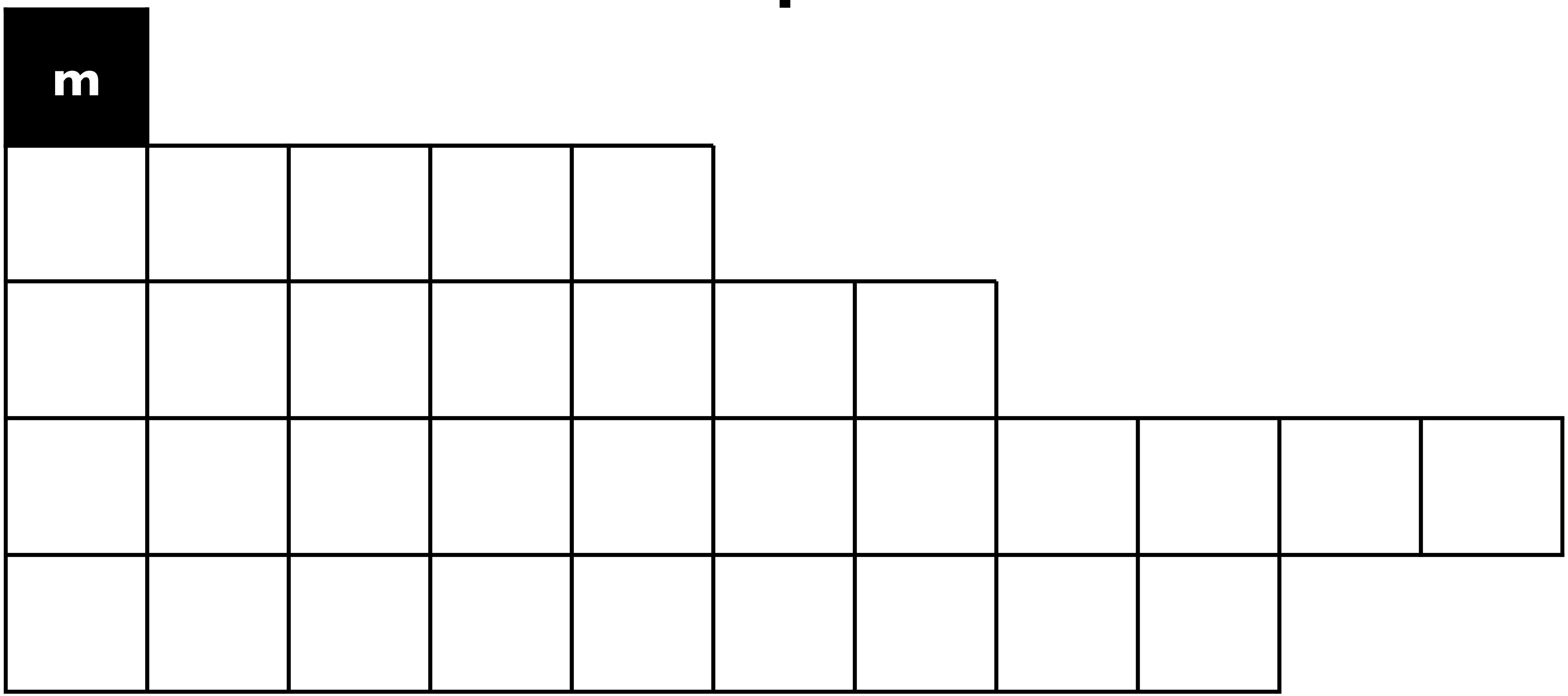

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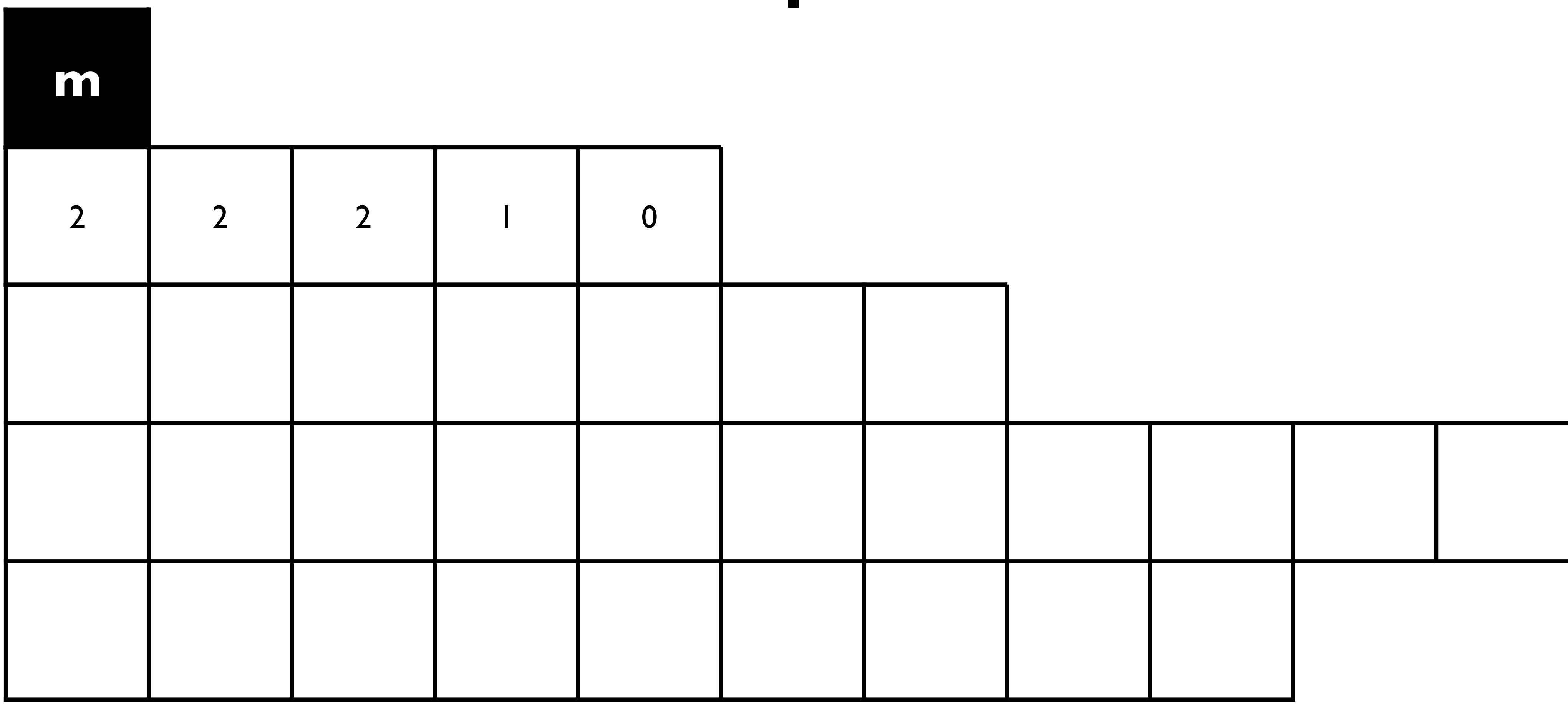


Sample Values

m										
2	2	2	1	0						
3	3	3	3	2	1	0				

Sample Values

m										
2	2	2	1	0						
3	3	3	3	2	1	0				
4	4	26	41	60	83	109	139	...	11327 (96th term)	...

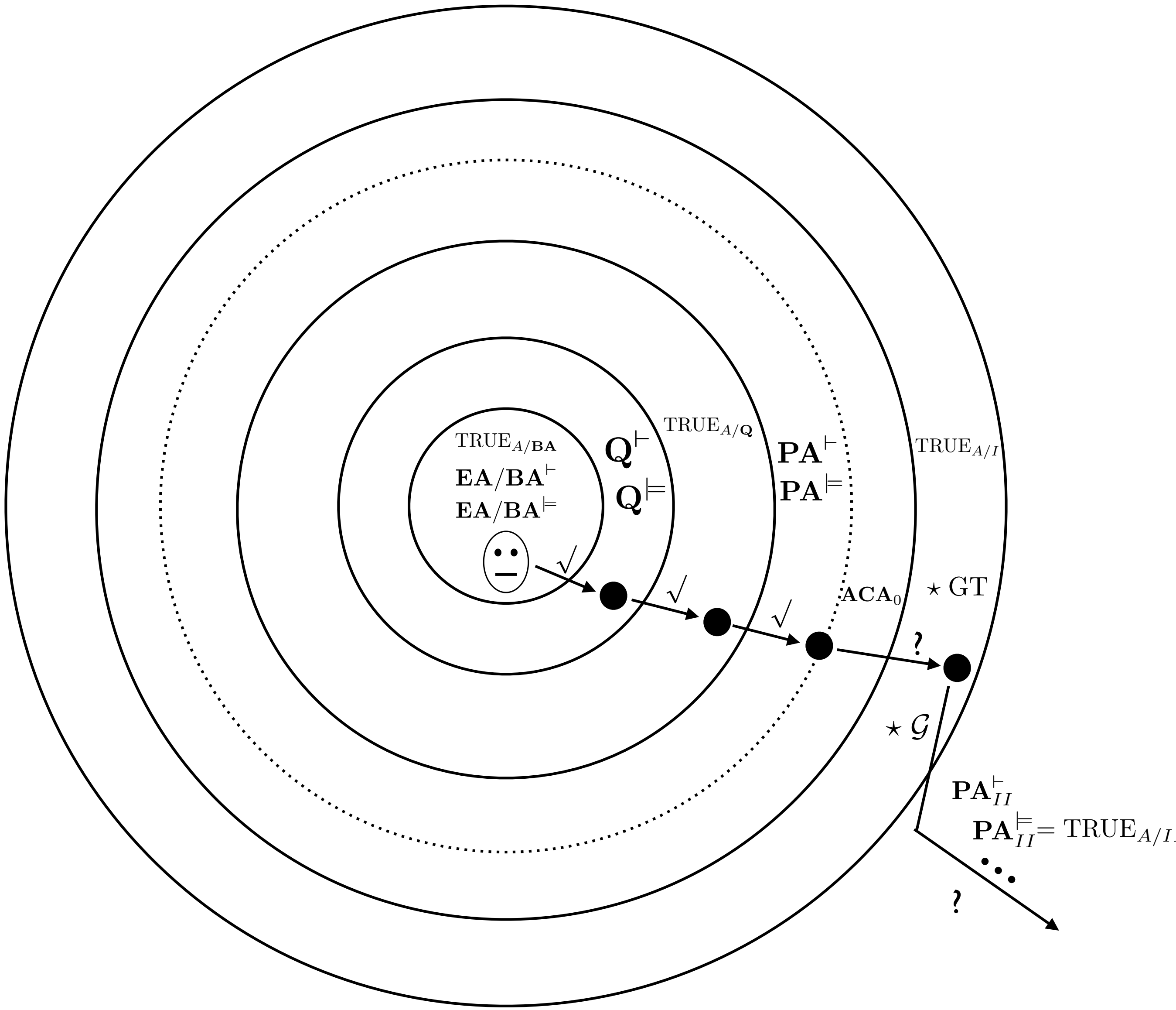
Sample Values

m										
2	2	2	1	0						
3	3	3	3	2	1	0				
4	4	26	41	60	83	109	139	...	11327 (96th term)	...
5	15	$\sim 10^{13}$	$\sim 10^{155}$	$\sim 10^{2185}$	$\sim 10^{36306}$	10^{695975}	$10^{15151337}$	...		

This sequence actually goes to zero!

Astrologic:

Rational Aliens Will be on the Same "Race Track"!



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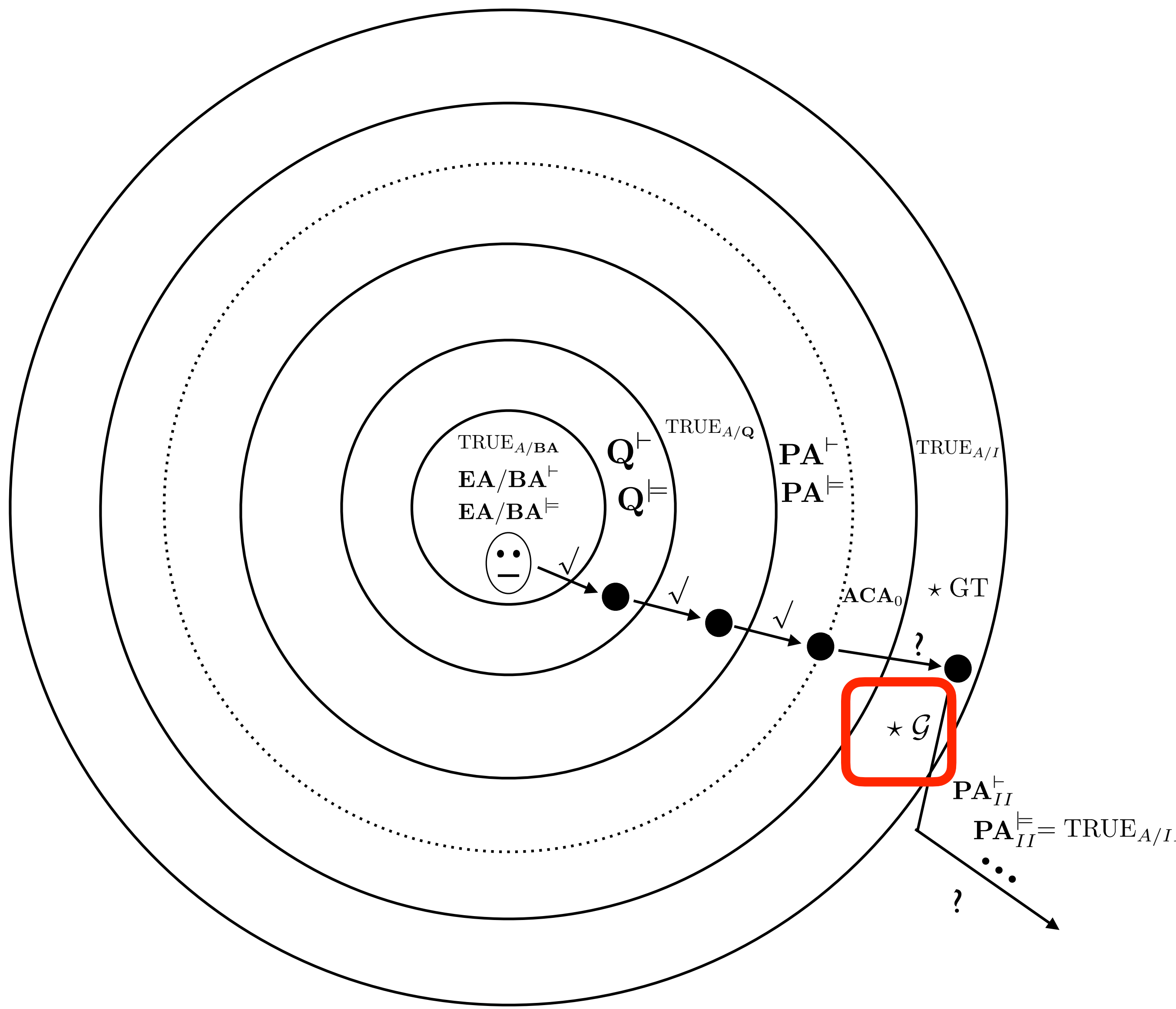

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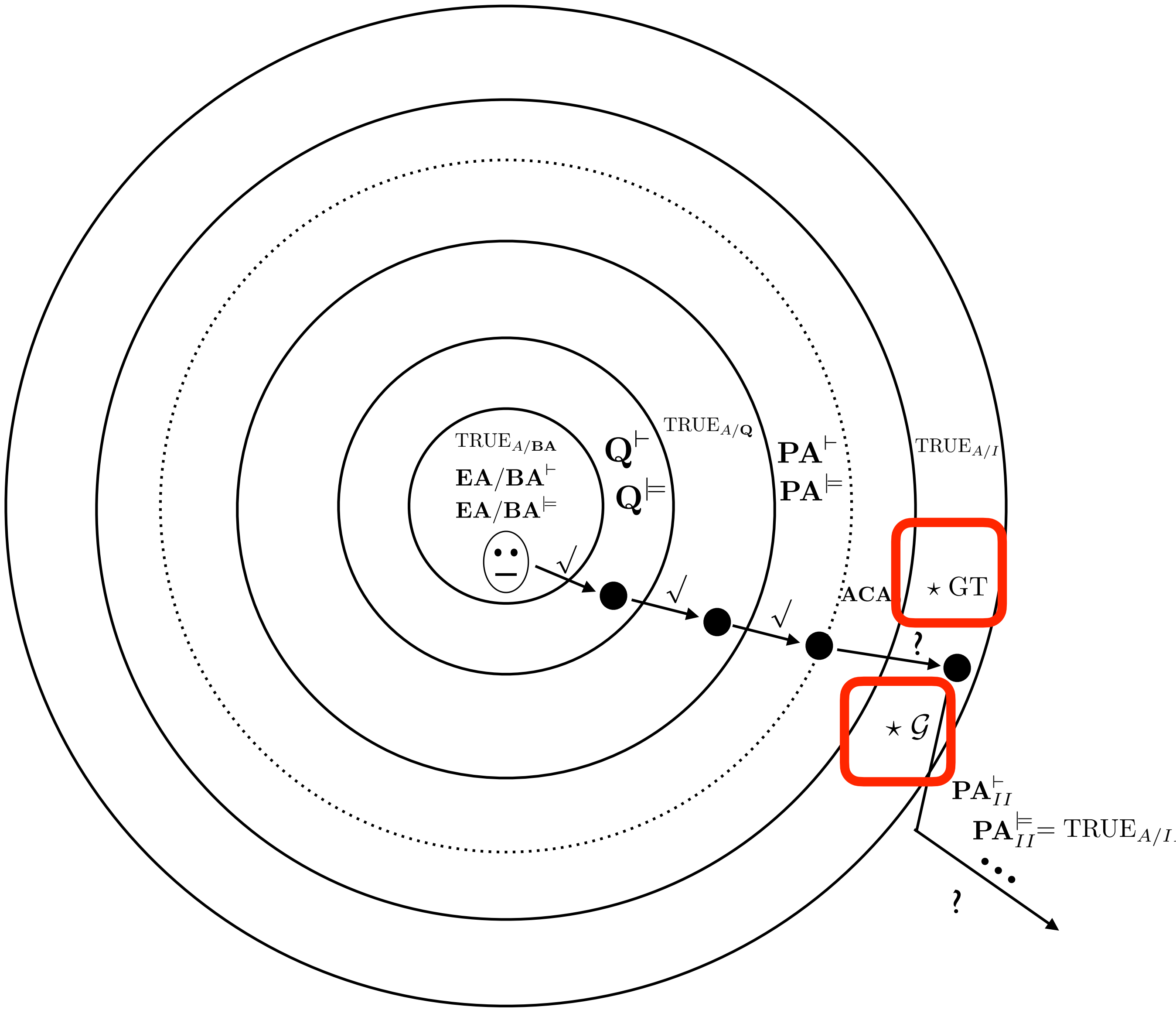

Could an AI Ever Match Gödel's G1 & G2?

Gödel's Great Theorems (OUP)

by Selmer Bringsjord

- Introduction ("The Wager")
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
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Med nok þenger, kan logikk løse alle þroblemer.