

Gödel: **ZFC**  $\not\vdash$   $\neg$ **CH**

(Gödel's "Silver Blaze" Theorem)

# Part II: Further Context & Proof Sketch

Selmer Bringsjord

IFLAI2

Nov 30 2020

RPI

Troy NY USA

version 1130201500NY



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**Context ...**

# *Gödel's Great Theorems* (OUP)

by Selmer Bringsjord

- Introduction (“The Wager”)
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
- The Continuum-Hypothesis Theorem
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- Could a Finite Machine Match Gödel’s Greatness?





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# Hilbert's problems

From Wikipedia, the free encyclopedia

**Hilbert's problems** are twenty-three problems in [mathematics](#) published by German mathematician [David Hilbert](#) in 1900. The problems were all unsolved at the time, and several of them were very influential for 20th-century mathematics. Hilbert presented ten of the problems (1, 2, 6, 7, 8, 13, 16, 19, 21, and 22) at the [Paris](#) conference of the [International Congress of Mathematicians](#), speaking on August 8 in the [Sorbonne](#). The complete list of 23 problems was published later, most notably in English translation in 1902 by [Mary Frances Winston Newson](#) in the *[Bulletin of the American Mathematical Society](#)*.<sup>[1]</sup>

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- [Nature and influence of the problems](#)
- [Ignorabimus](#)
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David Hilbert



# Gödel as logician/mathematician; Gödel as prophet.

## Table of problems [\[ edit \]](#)

Hilbert's twenty-three problems are (for details on the solutions and references, see the detailed articles that are linked to in the first column):

Problem ↕	Brief explanation	Status ↕	Year Solved ↕
1st	The <a href="#">continuum hypothesis</a> (that is, there is no <a href="#">set</a> whose <a href="#">cardinality</a> is strictly between that of the <a href="#">integers</a> and that of the <a href="#">real numbers</a> )	Proven to be impossible to prove or disprove within <a href="#">Zermelo–Fraenkel set theory</a> with or without the <a href="#">Axiom of Choice</a> (provided <a href="#">Zermelo–Fraenkel set theory</a> is <a href="#">consistent</a> , i.e., it does not contain a contradiction). There is no consensus on whether this is a solution to the problem.	1940, 1963
2nd	Prove that the <a href="#">axioms</a> of <a href="#">arithmetic</a> are <a href="#">consistent</a> .	There is no consensus on whether results of <a href="#">Gödel</a> and <a href="#">Gentzen</a> give a solution to the problem as stated by Hilbert. Gödel's <a href="#">second incompleteness theorem</a> , proved in 1931, shows that no proof of its consistency can be carried out within arithmetic itself. Gentzen proved in 1936 that the consistency of arithmetic follows from the <a href="#">well-foundedness</a> of the <a href="#">ordinal</a> $\epsilon_0$ .	1931, 1936
3rd	Given any two <a href="#">polyhedra</a> of equal volume, is it always possible to cut the first into finitely many polyhedral pieces that can be reassembled to yield the second?	Resolved. Result: No, proved using <a href="#">Dehn invariants</a> .	1900
4th	Construct all <a href="#">metrics</a> where lines are <a href="#">geodesics</a> .	Too vague to be stated resolved or not. <sup>[h]</sup>	—
5th	Are continuous <a href="#">groups</a> automatically <a href="#">differential groups</a> ?	Resolved by <a href="#">Andrew Gleason</a> , assuming one interpretation of the original statement. If, however, it is understood as an equivalent of the <a href="#">Hilbert–Smith conjecture</a> , it is still unsolved.	1953?
6th	Mathematical treatment of the <a href="#">axioms</a> of <a href="#">physics</a>  (a) axiomatic treatment of probability with limit theorems for foundation of <a href="#">statistical physics</a>  (b) the rigorous theory of limiting processes "which lead from the atomistic view to the laws of motion of continua"	Partially resolved depending on how the original statement is interpreted. <sup>[9]</sup> Items (a) and (b) were two specific problems given by Hilbert in a later explanation. <sup>[1]</sup> <a href="#">Kolmogorov's axiomatics</a> (1933) is now accepted as standard. There is some success on the way from the "atomistic view to the laws of motion of continua." <sup>[10]</sup>	1933– 2002?
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# Hilbert's First Problem

Let  $A$  be an infinite set of real numbers. Then we need to prove that  $A$  is in one-to-one correspondence either with the set of natural numbers, or with the set of *all* real numbers (i.e. with *the continuum*).

Alternatively (with transfinite numbers allowed):

$$\aleph_1 = \mathcal{P}(\mathbb{N}) = 2^{\aleph_0}$$

**Admission:** We “AI-got” (ie obtained automated proofs of) G1, G-RT, G2, GST, but getting this quite another story!

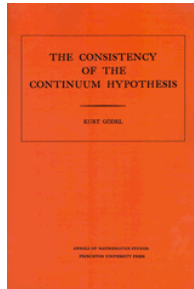
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Hilbert’s #1 (1900): “very plausible theorem”: **CH**



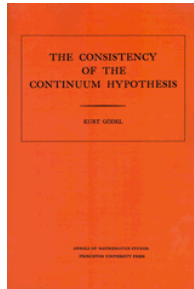
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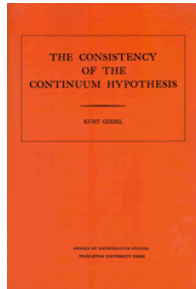
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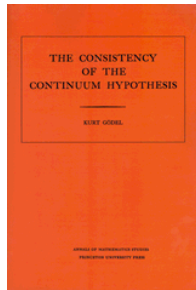
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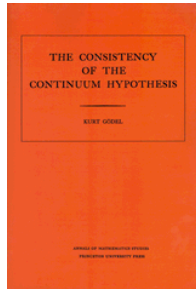


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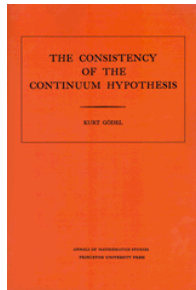
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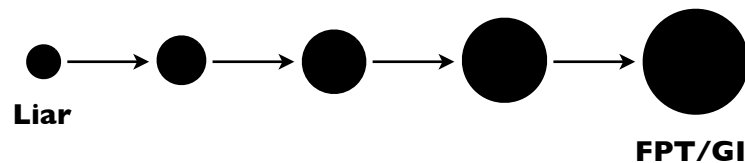
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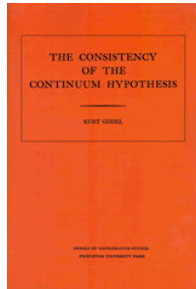
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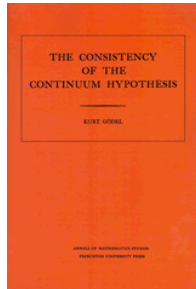
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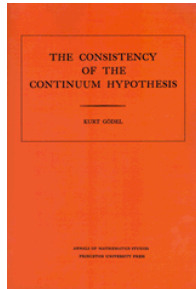
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**Note:** Gödel (1940): Not like G2!

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**ZFC to the Rescue ...**

Friday's Hill, Haslemere, 16 June 1902

Dear colleague,

For a year and a half I have been acquainted with your *Grundgesetze der Arithmetik*, but it is only now that I have been able to find the time for the thorough study I intended to make of your work. I find myself in complete agreement with you in all essentials, particularly when you reject any psychological element [[Moment]] in logic and when you place a high value upon an ideography [[Begriffsschrift]] for the foundations of mathematics and of formal logic, which, incidentally, can hardly be distinguished. With regard to many particular questions, I find in your work discussions, distinctions, and definitions that one seeks in vain in the works of other logicians. Especially so far as function is concerned (§ 9 of your *Begriffsschrift*), I have been led on my own to views that are the same even in the details. There is just one point where I have encountered a difficulty. You state (p. 17 [[p. 23 above]]) that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let  $w$  be the predicate : to be a predicate that cannot be predicated of itself. Can  $w$  be predicated of itself? From each answer its opposite follows. Therefore we must conclude that  $w$  is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [[Menge]] does not form a totality.

I am on the point of finishing a book on the principles of mathematics and in it I should like to discuss your work very thoroughly.<sup>1</sup> I already have your books or shall buy them soon, but I would be very grateful to you if you could send me reprints of your articles in various periodicals. In case this should be impossible, however, I will obtain them from a library.

The exact treatment of logic in fundamental questions, where symbols fail, has remained very much behind; in your works I find the best I know of our time, and therefore I have permitted myself to express my deep respect to you. It is very regrettable that you have not come to publish the second volume of your *Grundgesetze*; I hope that this will still be done.

Very respectfully yours,

BERTRAND RUSSELL

The above contradiction, when expressed in Peano's ideography, reads as follows :

$$w = \text{cls } \cap x \varepsilon (x \sim \varepsilon x). \supset : w \varepsilon w . = . w \sim \varepsilon w.$$

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[FregTHEN2](#)[KnightKnave\\_SmullyanKKProblem1.1](#)[AthenCfromAthenBandBthenC](#)[BiconditionalIntroByChaining](#)[BogusBiconditional](#)[CheatersNeverPropser](#)[Contrapositive\\_NYS\\_2](#)[Disj\\_Syll](#)[GreenCheeseMoon2](#)[HypSyll](#)[LarryIsSomehowSmart](#)[Modus\\_Tollens](#)[RussellsLetter2Frege](#)[ThxForThePCOracle](#)[Explosion](#)[OnlyMediumOrLargeLlamas](#)[GreenCheeseMoon1](#)[Disj\\_Elim](#)[kok13\\_28](#)[KingAce2](#)[kok\\_13\\_31](#) RussellsLetter2Frege

The challenge here is to prove that from Russell's instantiation of Frege's doomed Axiom V a contradiction can be promptly derived. The letter has of course been examined in some detail by S Bringsjord (in the Mar 16 2020 lecture in [the 2020 lecture lineup](#)); it, along with an astoundingly soft-spoken reply from Frege, can be found [here](#). Put meta-logically, your task in the present problem is to build a proof that confirms this:

$$\{\exists x \forall y ((y \in x) \rightarrow (y \notin y))\} \vdash \zeta \wedge \neg \zeta.$$

Make sure you understand that the given here is an instantiation of Frege's Axiom V; i.e. it's an instantiation of

$$\exists x \forall y ((y \in x) \rightarrow \phi(y)).$$

(The notation  $\phi(y)$ , recall, is the standard way in mathematical logic to say that  $y$  is free in  $\phi$ .) **Note:** Your finished proof is allowed to make use the PC-provability oracle (but *only* that oracle).

(Now a brief remark on matters covered by in class by Bringsjord when second-order logic =  $\mathcal{L}_2$  arrives on the scene: Longer term, and certainly constituting evidence of Frege's capacity for ingenious, intricate deduction, it has recently been realized that while Frege himself relied on Axiom V to obtain what is known as **Hume's Principle** (= HP), this reliance is avoidable. That from just HP we can deduce all of Peano Arithmetic (**PA**) (!) is a result Frege can be credited with showing; the result is known today as [Frege's Theorem](#) (= FT). Following the link just given will reward the reader with an understanding of HP, and how how to obtain **PA** from it.)

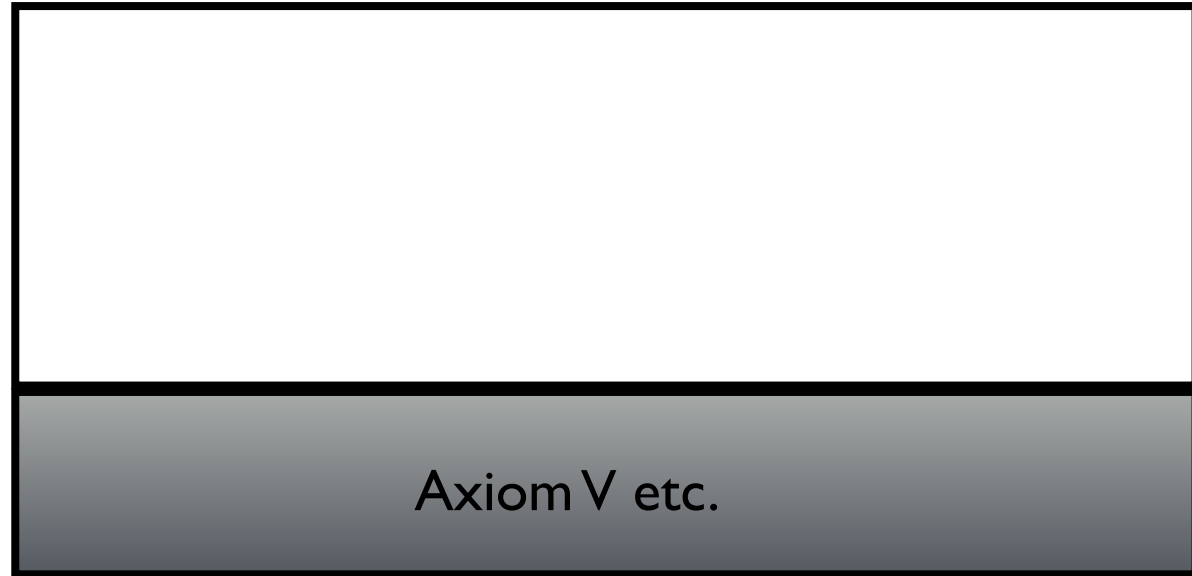
**Deadline** 22 Apr 2020 23:59:00 EST

[Solve](#)

# The Foundation Crumbles

The Rest of Math,  
Engineering, etc.

Foundation





# The Foundation Crumbles

The Rest of Math,  
Engineering, etc.

Foundation



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Foundation

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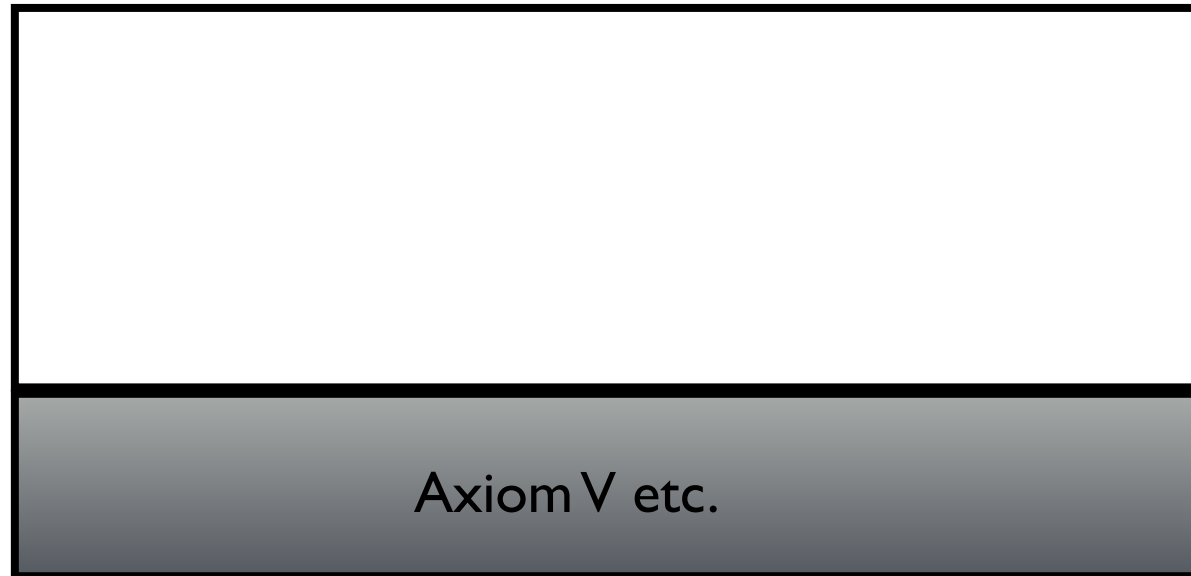
The Rest of Math,  
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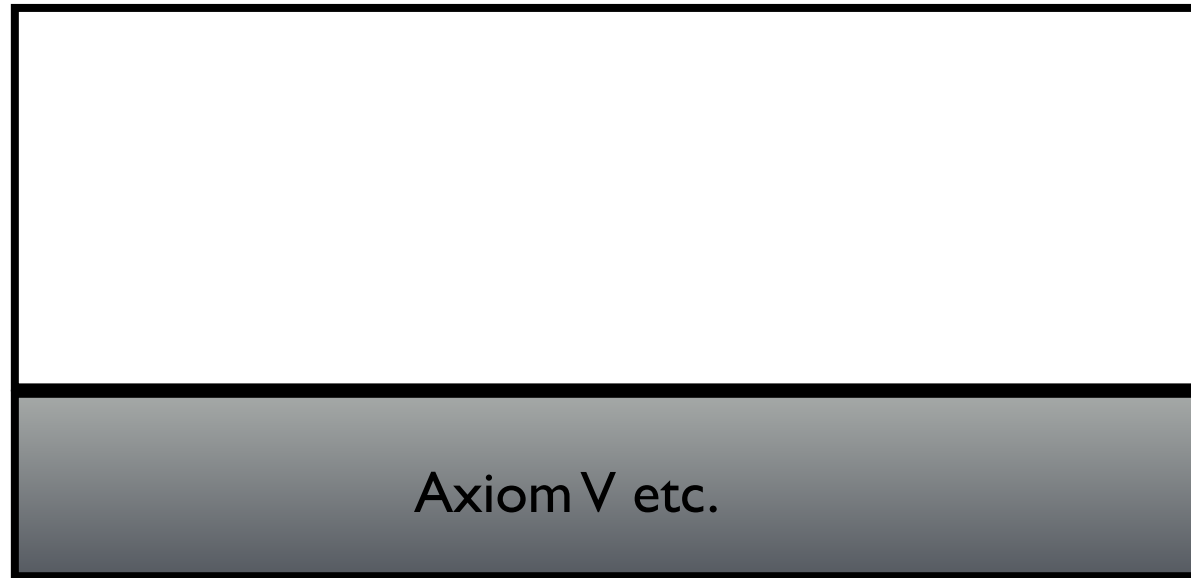


Axiom V etc.

# The Foundation Crumbles

The Rest of Math,  
Engineering, etc.

Foundation



$$\text{Axiom V} \quad \exists x \forall y [y \in x \leftrightarrow \phi(y)]$$

# The Foundation Crumbles

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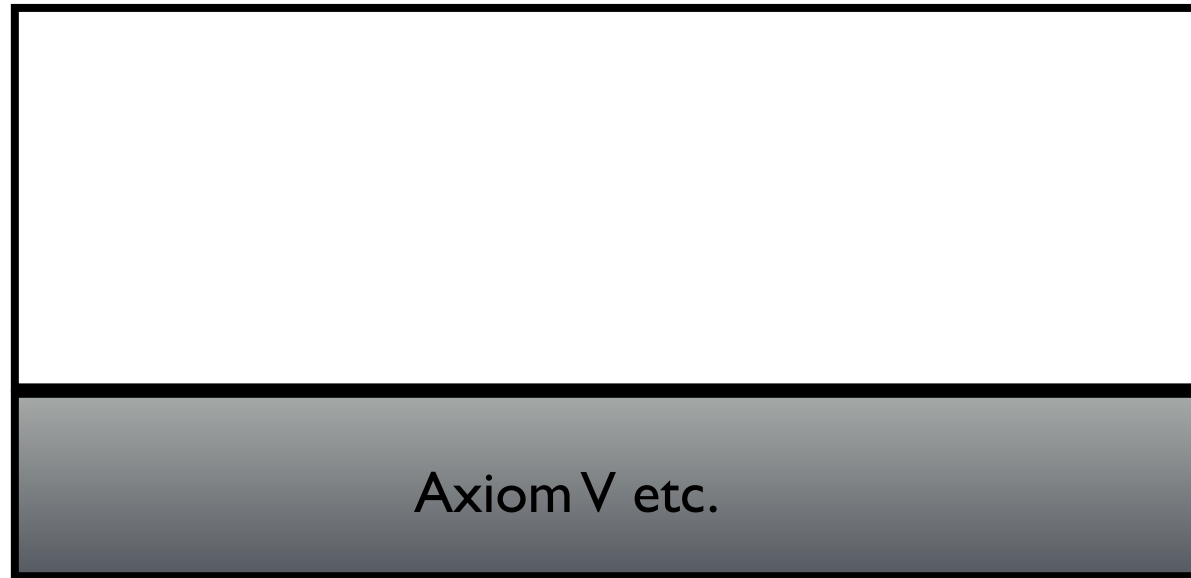
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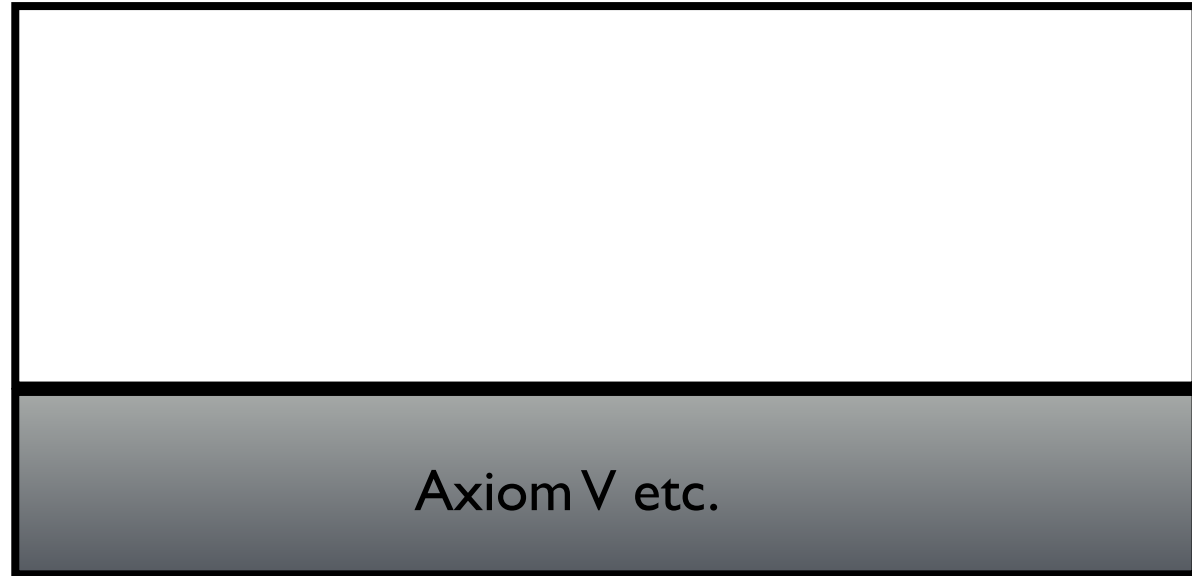
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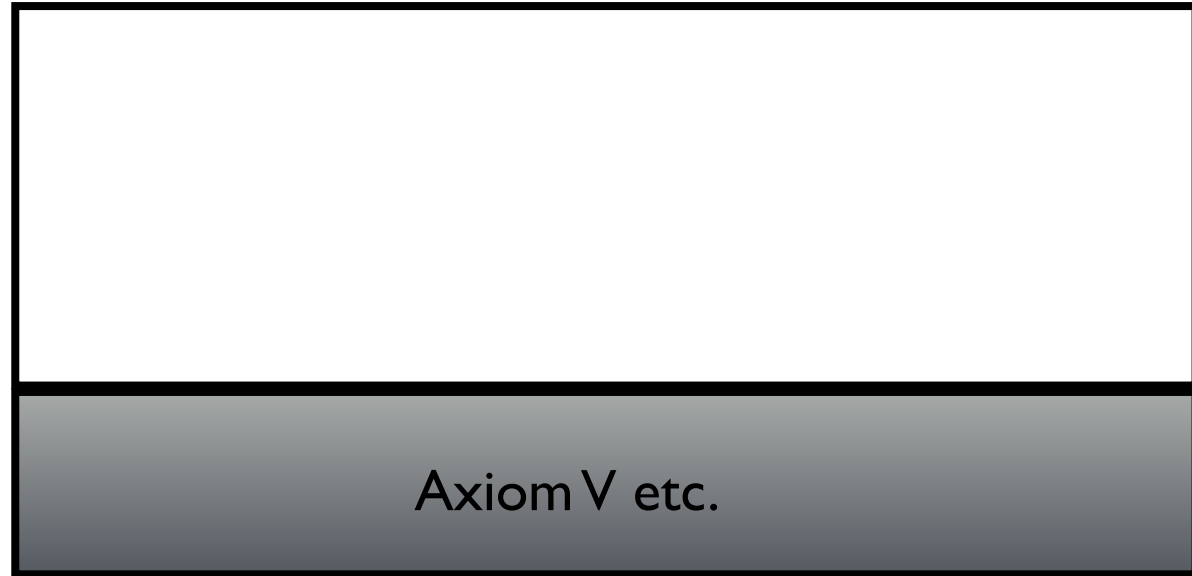
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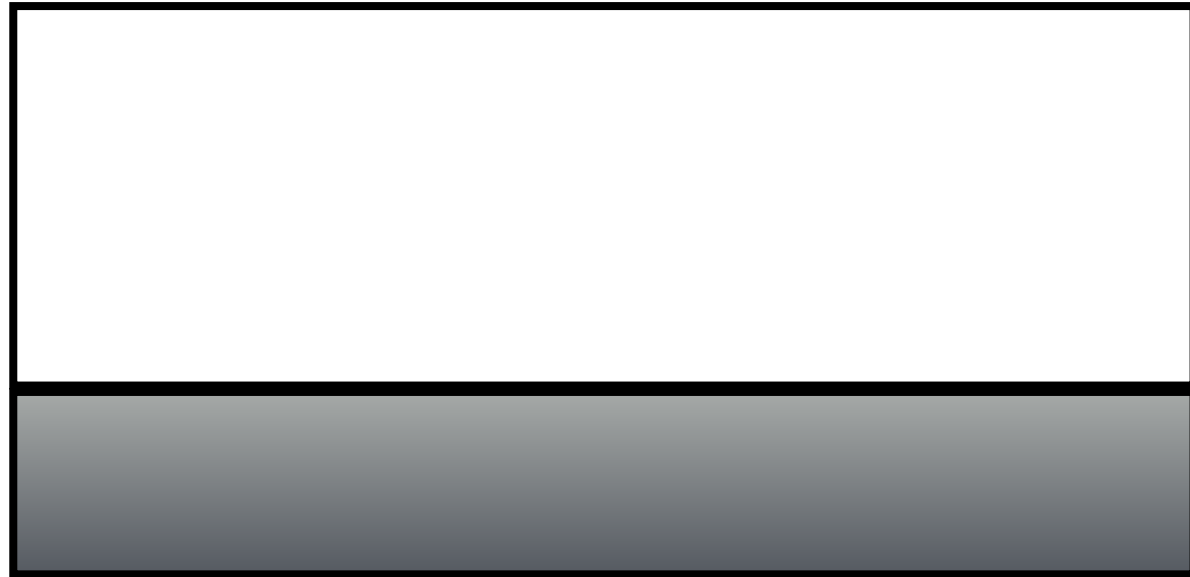
Foundation



# The Foundation Crumbles

The Rest of Math,  
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Foundation



# The Foundation Rebuilt

The Rest of Math,  
Engineering, etc.

New Foundation

# The Foundation Rebuilt

The Rest of Math,  
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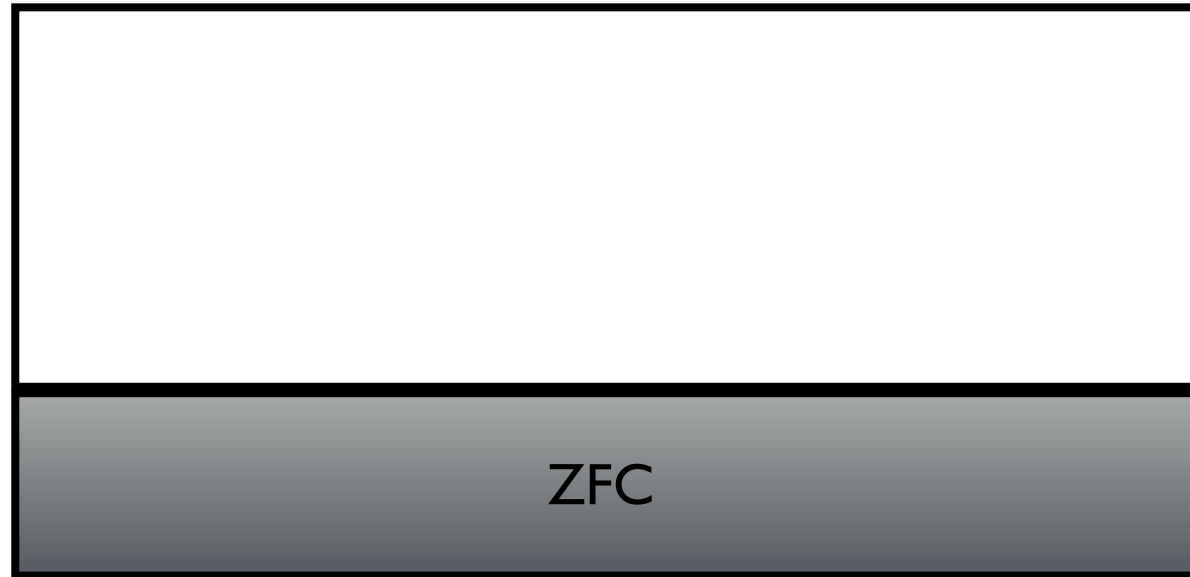
New Foundation



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The Rest of Math,  
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New Foundation





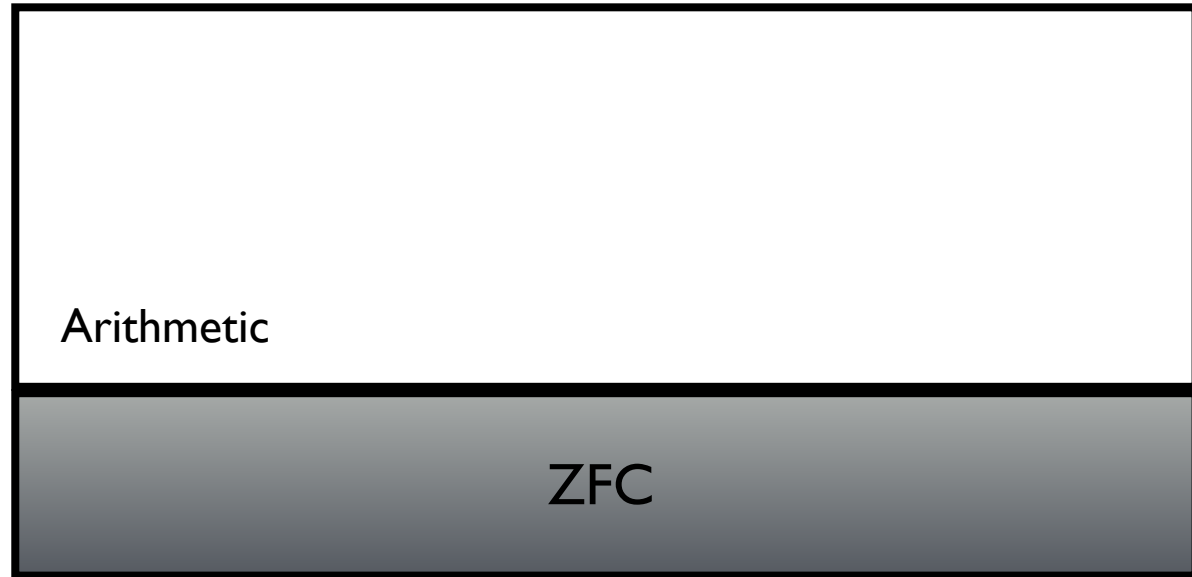
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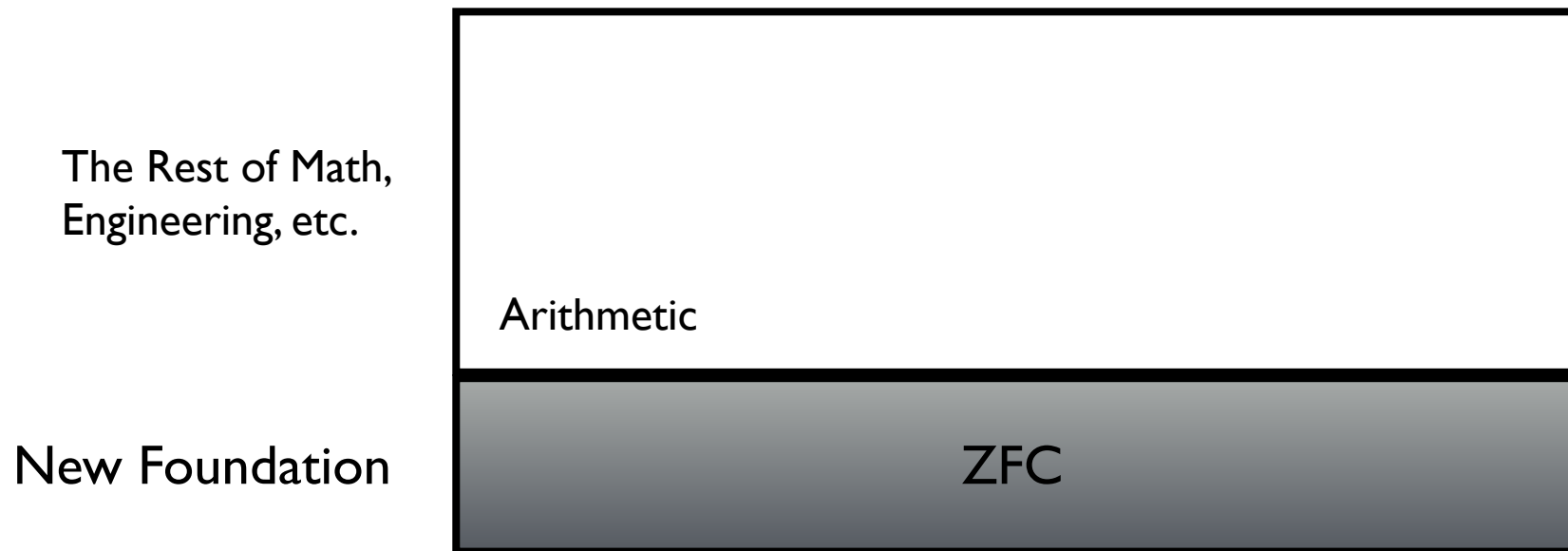
Arithmetic

New Foundation

ZFC

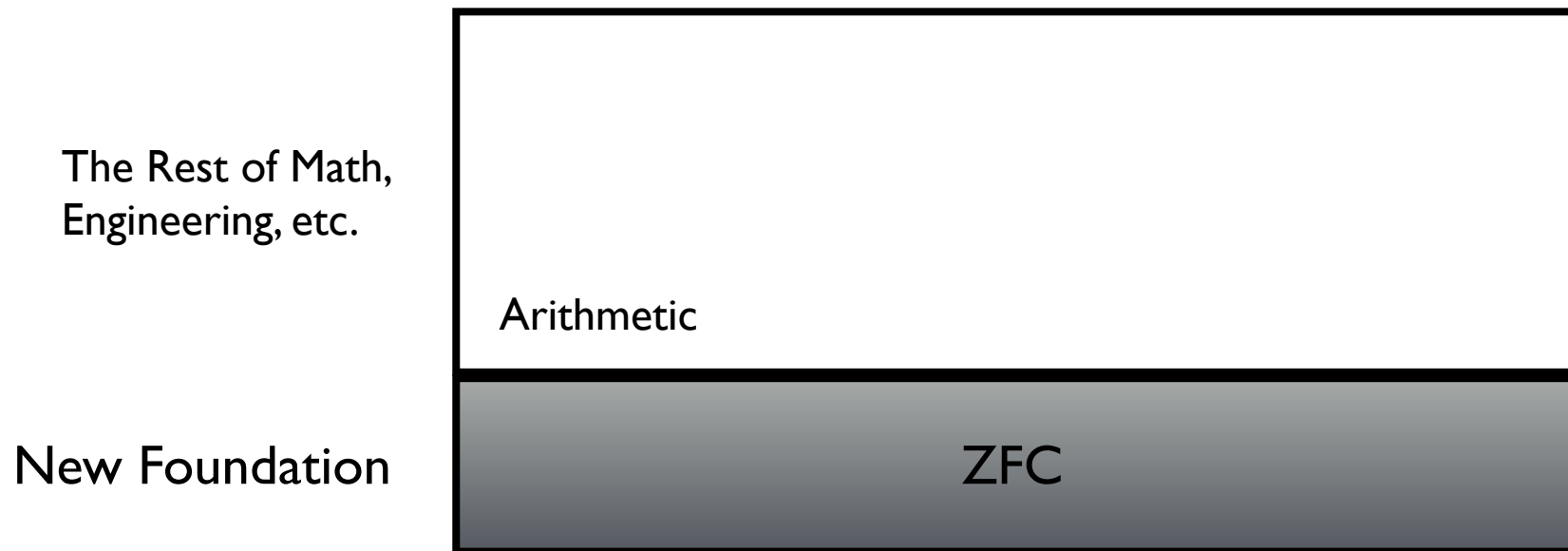


# The Foundation Rebuilt



So what are the axioms in ZFC?

# The Foundation Rebuilt



So what are the axioms in ZFC?

# ZFC

## 6.4.1 ZFC

The Zermelo-Fraenkel Axioms for Set Theory, or just 'ZFC' for short, include the following nine axioms.<sup>34</sup>

### Axiom of Extensionality

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

### Axiom Schema of Separation

$$\forall x_0 \dots \forall x_{n-1} \forall x \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge \phi(z, x_0, \dots, x_{n-1})))$$

### Pair Set Axiom

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y))$$

### Sum Set Axiom

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \wedge z \in w))$$

### Power Set Axiom

$$\forall x \exists y \forall z (z \in y \leftrightarrow \forall w (w \in z \rightarrow w \in x))$$

### Axiom of Infinity

$$\exists x (\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup \{y\} \in x))$$

### Axiom Schema of Replacement

$$\forall x_0 \dots \forall x_{n-1} (\forall x \exists^1 y \phi(x, y, x_0, \dots, x_{n-1}) \rightarrow \forall u \exists v \forall y (y \in v \leftrightarrow \exists x (x \in u \wedge \phi(x, y, x_0, \dots, x_{n-1}))))$$

### Axiom of Choice

$$\forall x ((\emptyset \notin x \wedge \forall u \forall v ((u \in x \wedge v \in x \wedge u \neq v) \rightarrow u \cap v = \emptyset)) \rightarrow \exists y \forall w (w \in x \rightarrow \exists^1 z z \in w \cap y))$$

# Axiom *Schema* of Separation (SEP)

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**SEP**

$$\forall x_1 \dots \forall x_k \forall x \exists y \forall z [z \in y \leftrightarrow (z \in x \wedge \phi(z, x_1, \dots, x_k))]$$

where  $x$  and  $y$  are distinct, and are both distinct from  $z$  and the  $x_i$ ;  
and, as usual for us now,  $\phi$  expresses a property using  $\in$ .

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“Given *beforehand* some set  $x$  and property  $\mathcal{P}$   
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(How does this neutralize  
Russell’s letter to Frege?)



# Re. ZFC and HS<sup>®</sup>

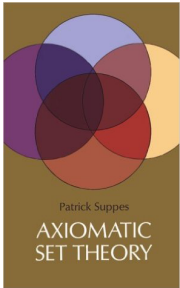
$$\vdash \neg \exists x \forall y (y \in x \leftrightarrow y \notin y)$$

(Russell's Theorem; poor Frege!)

<http://plato.stanford.edu/entries/russell-paradox/#HOTP>

Supplant Cantor's/Frege's Axiom V with the Axiom Schema of Separation (& put on our thinking caps ...) and you try to show Theorem I from Suppes:

$$\vdash \forall x (x \notin \emptyset)$$



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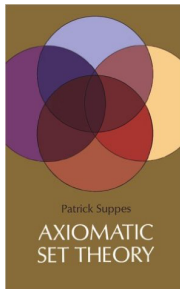
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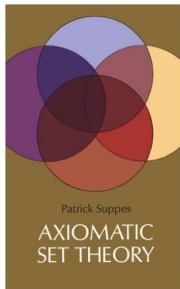
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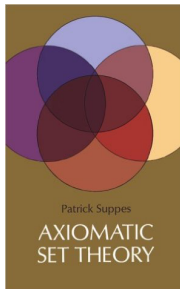


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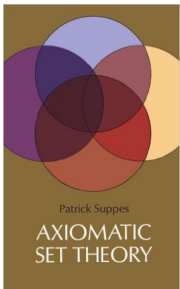
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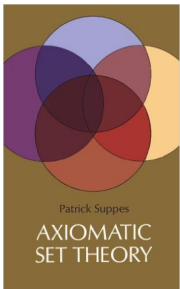
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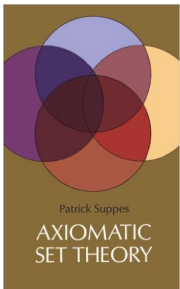
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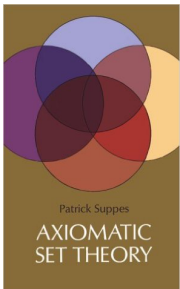
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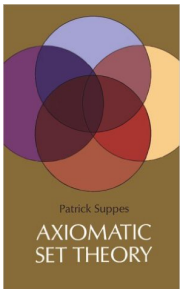
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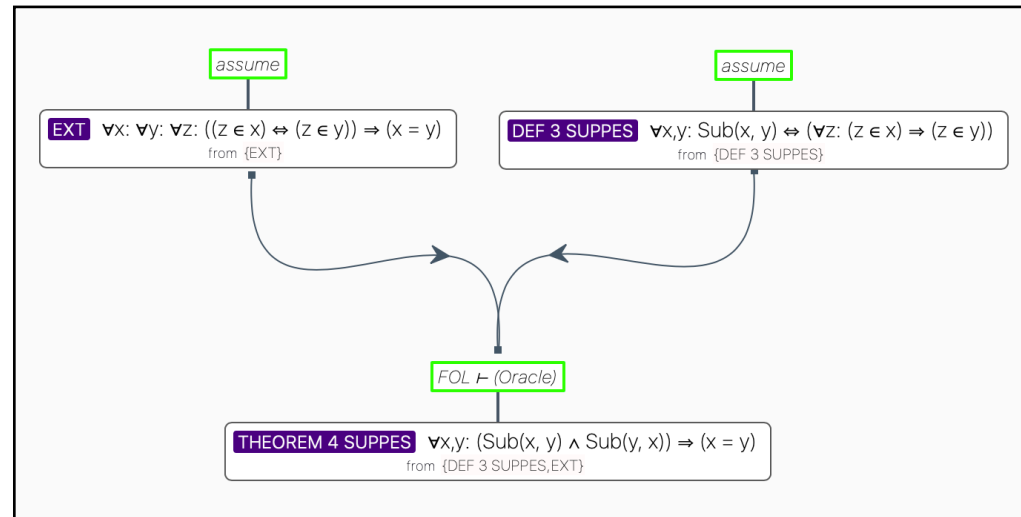
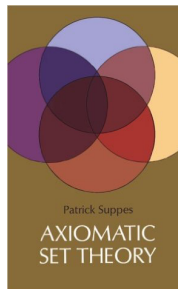
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# Re.Theorem 4 in HS<sup>®</sup>



**ZFC** (in some English)

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$\aleph$

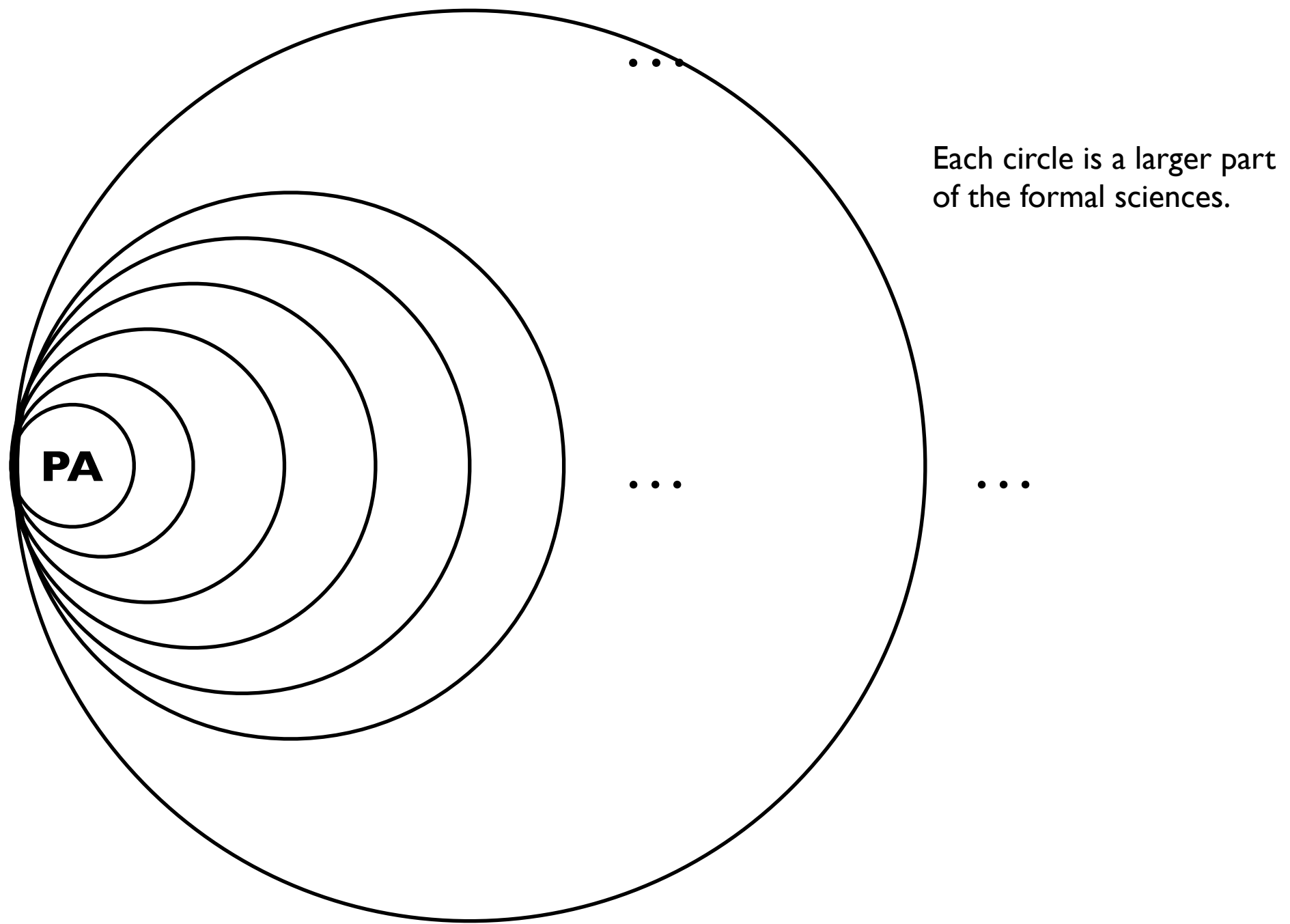
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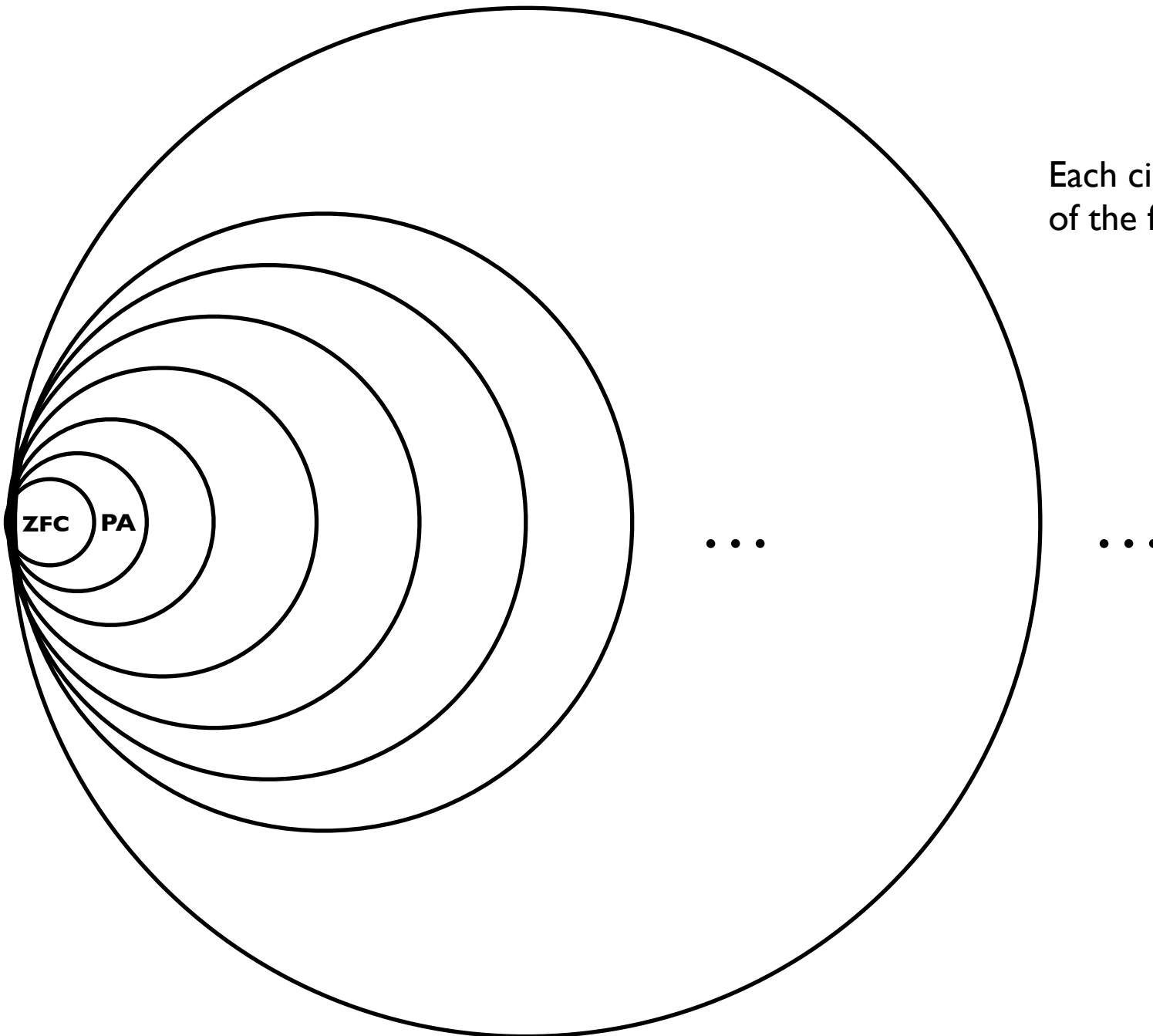
Better Overarching  
Pictorial Conception of  
ZF/ZFC ...

# Arithmetic is Part of All Things Sci/Eng/Tech!

and courtesy of Gödel: We can't even prove all truths of arithmetic!



# Actually, the true kernel is set theory!



Each circle is a larger part of the formal sciences.

But these are easily  
misinterpreted pictures; so ...

$$\mathbf{PA}_2 = \mathbf{Z}_2$$

$$\mathbf{A1} \quad \forall x(0 \neq s(x))$$

$$\mathbf{A2} \quad \forall x \forall y (s(x) = s(y) \rightarrow x = y)$$

$$\mathbf{A3} \quad \forall x (x \neq 0 \rightarrow \exists y (x = s(y)))$$

$$\mathbf{A4} \quad \forall x (x + 0 = x)$$

$$\mathbf{A5} \quad \forall x \forall y (x + s(y) = s(x + y))$$

$$\mathbf{A6} \quad \forall x (x \times 0 = 0)$$

$$\mathbf{A7} \quad \forall x \forall y (x \times s(y) = (x \times y) + x)$$

$$\mathbf{Induction Axiom} \quad \forall X [(X(0) \wedge \forall x (X(x) \rightarrow X(s(x))) \rightarrow \forall x X(x)]$$

$$\mathbf{Comprehension Axioms} \quad \exists X \forall x (x \in X \leftrightarrow \phi(x))$$

# Re question from Can Mekik. Hmm. Not sure what what to say, yet ...

## **Inconsistency Robustness in Foundations: Mathematics self proves its own Consistency and Other Matters**

Carl Hewitt

*This article is dedicated to Alonzo Church, Stanislaw Jaśkowski,  
Ludwig Wittgenstein, and Ernst Zermelo.*

### **Abstract**

Inconsistency Robustness is performance of information systems with pervasively inconsistent information. Inconsistency Robustness of the community of professional mathematicians is their performance repeatedly repairing contradictions over the centuries. In the Inconsistency Robustness paradigm, deriving contradictions have been a progressive development and not “game stoppers.” Contradictions can be helpful instead of being something to be “swept under the rug” by denying their existence, which has been repeatedly attempted by Establishment Philosophers (beginning with some Pythagoreans). Such denial has delayed mathematical development. This article reports how considerations of Inconsistency Robustness have recently influenced the foundations of mathematics for Computer Science continuing a tradition developing the sociological basis for foundations.<sup>1</sup>

Classical Direct Logic is a foundation of mathematics for Computer Science, which has a foundational theory (for convenience called “Mathematics”) that can be used in any other theory. A bare turnstile is used for Mathematics so that  $\vdash\Psi$  means that  $\Psi$  is a mathematical proposition that is a theorem of Mathematics and  $\Phi\vdash\Psi$  means that  $\Psi$  can be inferred from  $\Phi$  in Mathematics.

The current common understanding is that Gödel proved “Mathematics cannot prove its own consistency, if it is consistent.” However, the consistency of mathematics can be proved by a simple argument using standard rules of Mathematics including the following:

- rule of Proof by Contradiction, *i.e.*,  $(\neg\Phi\Rightarrow(\Theta\wedge\neg\Theta))\vdash\Phi$
- and the rule of Soundness, *i.e.*,  $(\vdash\Phi)\Rightarrow\Phi$

*Formal Proof.* By definition,

$\text{Consistent}\Leftrightarrow\neg\exists[\Psi:\text{Proposition}]\rightarrow\vdash(\Psi\wedge\neg\Psi)$ . By Existential Elimination, there is some proposition  $\Psi_0$  such that  $\neg\text{Consistent}\Rightarrow\vdash(\Psi_0\wedge\neg\Psi_0)$  which by Soundness and transitivity of implication means  $\neg\text{Consistent}\Rightarrow(\Psi_0\wedge\neg\Psi_0)$ . Substituting for  $\Phi$  and  $\Theta$ , in the rule for Proof by Contradiction, it follows that  $(\neg\text{Consistent}\Rightarrow(\Psi_0\wedge\neg\Psi_0))\vdash\text{Consistent}$ . Thus,  $\vdash\text{Consistent}$ .

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# “Reality”

Logic

Math

**ZFC**

**$Z_2 (= PA_2)$**

	$RCA_0$	$WKL_0$	$ACA_0$	$ATR_0$	$\Pi_1^1-CA_0$
analysis (separable):					
differential equations	IV.8	IV.8			
continuous functions	II.6, II.7	IV.2, IV.7	III.2		
completeness, etc.	II.4	IV.1	III.2		
Banach spaces	II.10	IV.9, X.2			X.2
open and closed sets	II.5	IV.1		V.4, V.5	VI.1
Borel and analytic sets	V.1			V.1, V.3	VI.2, VI.3
algebra (countable):					
countable fields	II.9	IV.4, IV.5	III.3		
commutative rings	III.5	IV.6	III.5		
vector spaces	III.4		III.4		
Abelian groups	III.6		III.6	V.7	VI.4
miscellaneous:					
mathematical logic	II.8	IV.3			
countable ordinals	V.1		V.6.10	V.1, V.6	
infinite matchings		X.3	X.3	X.3	
the Ramsey property			III.7	V.9	VI.6
infinite games			V.8	V.8	VI.5

# Proofs of Non-Entailment in HS<sup>®</sup> ...

# We Have Entailment

The screenshot shows a web browser window with the URL `rpi.logicmodernapproach.com`. The browser's address bar and tabs are visible. The main content area displays a logic proof in HyperSlate. The proof consists of three nodes:

- GIVEN 1** (purple box):  $(\forall x: \text{Likes}(x, \text{erving})) \vee (\forall x: \neg \text{Likes}(x, \text{alvin}))$   
from {GIVEN 1}
- GIVEN 2** (purple box):  $\neg \text{Likes}(\text{alvin}, \text{erving})$   
from {GIVEN 2}
- GOAL** (purple box):  $\neg(\exists x: \text{Likes}(x, \text{alvin}))$   
from {}

Each given node is connected to an **assume** label (green box) above it. The goal node is connected to an **FOL  $\vdash$  (Oracle)** label (red box) above it. The browser's dock at the bottom shows various application icons, including Safari, Mail, Calendar, and others.

# We Have Entailment

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- GIVEN 1:** A box containing the formula  $(\forall x: \text{Likes}(x, \text{erving})) \vee (\forall x: \neg \text{Likes}(x, \text{alvin}))$  with the text "from {GIVEN 1}" below it. A green box labeled "assume" is connected to this box by a vertical line.
- GIVEN 2:** A box containing the formula  $\neg \text{Likes}(\text{alvin}, \text{erving})$  with the text "from {GIVEN 2}" below it. A green box labeled "assume" is connected to this box by a vertical line.
- GOAL:** A box containing the formula  $\neg(\exists x: \text{Likes}(x, \text{alvin}))$  with the text "from {}" below it. A red box labeled "FOL  $\vdash$  (Oracle)" is connected to this box by a vertical line.

The browser's dock at the bottom shows various application icons, including Safari, Mail, Calendar, and others. The system status bar at the top right indicates the time as "Mon 9:22 AM" and the battery level as "100%".

# We Don't Have Entailment

The screenshot shows a web browser window with the address bar at `rpi.logicmodernapproach.com`. The browser tabs include "Inbox (7,780) - selmerbringsjord@gmail.co...", "Submissions | SSS-IAIE-2021", "Editing Tree2ForCHClass113020", "AWS Management Console", and "Intermediate Formal Logic and AI (= IFLAI...". The interface features a toolbar with icons for home, list, add, save, undo, redo, and a Bezier tool. A status bar at the top right indicates "Tree2ForCHClass113020 [FIRST-ORDER-LOGIC]: Saved with 55 symbols."

The main workspace displays a logic proof tree diagram with the following components:

- GIVEN 1**:  $\forall x: \text{Likes}(x, \text{erving}) \vee \neg \text{Likes}(x, \text{alvin})$  from {GIVEN 1}. An "assume" label is above it.
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- Top Node**:  $\zeta \wedge \neg \zeta$  from {4}. An "assume" label is above it.

The diagram illustrates a logical derivation process where a goal is derived from a set of assumptions, and a negated goal is also derived from a set of assumptions, leading to a contradiction.

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The diagram illustrates a logical derivation process where assumptions and goals are used to reach a contradiction, demonstrating that the system does not have entailment.

**Cantorian Context ...**

# Cantor's Theorem



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(*sans* use of ordinal or cardinal numbers)

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For every infinite set  $S$ ,  $\mathcal{P}(S) > S$ .

*Generalized* Continuum Hypothesis (GCH):

There's no set (size-wise) between  $S$  and  $P(S)$ .

Cantor (1878):

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The power set of the natural numbers is larger than than the natural numbers!

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Re “The Adventure of  
Silver Blaze” ...

# The Adventure of Silver Blaze

From Wikipedia, the free encyclopedia

*For the 1937 film, see [Silver Blaze \(1937 film\)](#). For the 1977 film, see [Silver Blaze \(1977 film\)](#).*

"**The Adventure of Silver Blaze**", one of the 56 [Sherlock Holmes](#) short stories written by Sir [Arthur Conan Doyle](#), is one of 12 in the cycle collected as *The Memoirs of Sherlock Holmes*. It was first published in *The Strand Magazine* in December 1892.<sup>[1]</sup>

Doyle ranked "Silver Blaze" 13th in a list of his 19 favourite Sherlock Holmes stories.<sup>[2]</sup> One of the most popular Sherlock Holmes short stories, "Silver Blaze" focuses on the disappearance of the eponymous [race horse](#) (a famous winner, owned by a Colonel Ross) on the eve of an important race and on the apparent murder of its trainer. The tale is distinguished by its atmospheric [Dartmoor](#) setting and late-Victorian sporting milieu. It also features some of Conan Doyle's most effective plotting, hingeing on the "curious incident of the dog in the night-time":

Gregory ([Scotland Yard](#) detective): Is there any other point to which you would wish to draw my attention?

Holmes: To the curious incident of the dog in the night-time.

Gregory: The dog did nothing in the night-time.

Holmes: That was the curious incident.

## Contents [hide]

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- 2 [Publication history](#)
- 3 [Adaptations](#)
  - 3.1 [Film and television](#)
  - 3.2 [Radio](#)
- 4 [In popular culture](#)
- 5 [References](#)
- 6 [External links](#)

## "The Adventure of Silver Blaze"



1892 illustration by [Sidney Paget](#) in *The Strand Magazine*

<b>Author</b>	Arthur Conan Doyle
<b>Country</b>	Great Britain
<b>Language</b>	English
<b>Series</b>	<i>The Memoirs of Sherlock Holmes</i>
<b>Genre(s)</b>	Detective story
<b>Published in</b>	December 1892
<b>Preceded by</b>	" <a href="#">The Adventure of the Copper Beeches</a> "
<b>Followed by</b>	" <a href="#">The Adventure of the Cardboard Box</a> "

With Sherlock Holmes in the picture ...

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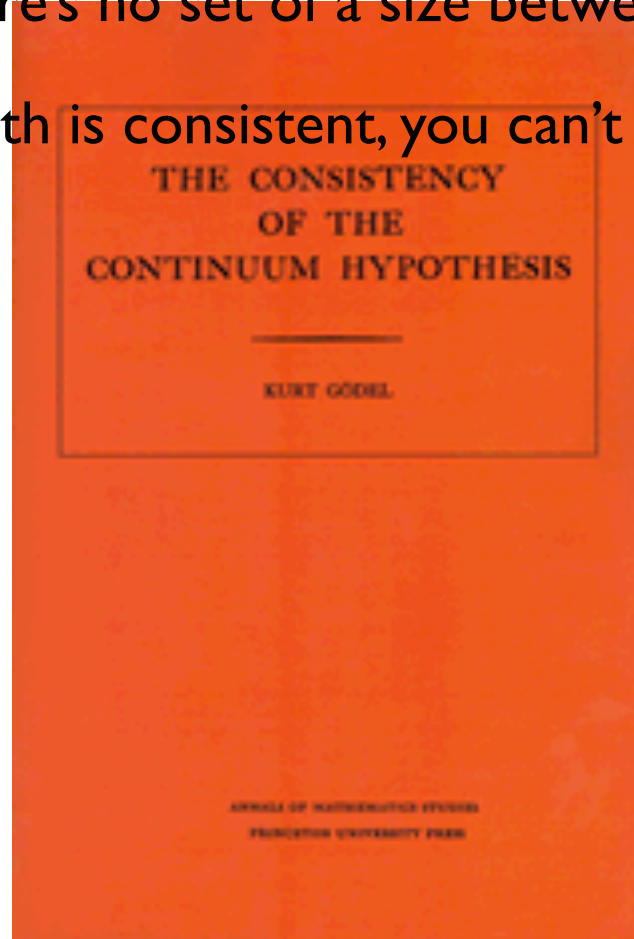
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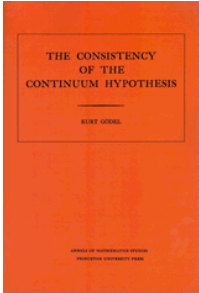
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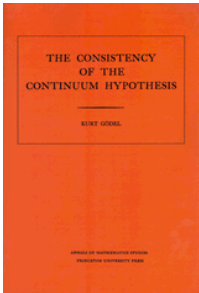
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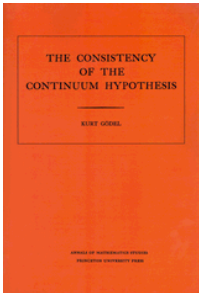
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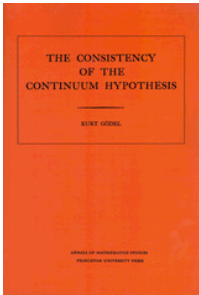
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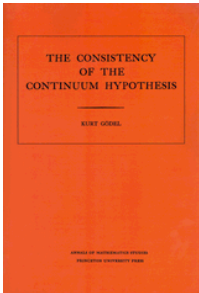
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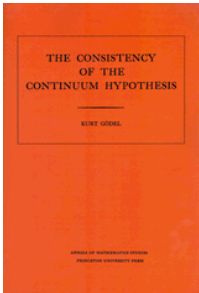
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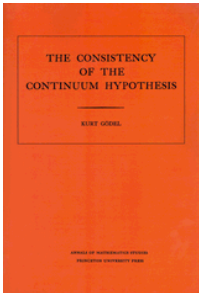
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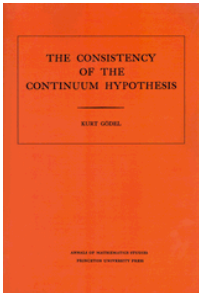
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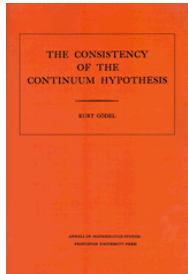
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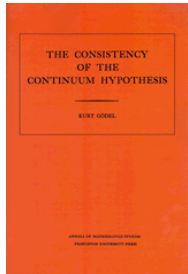
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Facts F ...

# Scenario G

## (in honor of Inspector Gregory)

"And yet," said I, "even now I fail to understand what the theory of the police can be."

"I am afraid that whatever theory we state has very grave objections to it," returned my companion. "The police imagine, I take it, that this Fitzroy Simpson, having drugged the lad, and having in some way obtained a duplicate key, opened the stable door and took out the horse, with the intention, apparently, of kidnapping him altogether. His bridle is missing, so that Simpson must have put this on. Then, having left the door open behind him, he was leading the horse away over the moor, when he was either met or overtaken by the trainer. A row naturally ensued. Simpson beat out the trainer's brains with his heavy stick without receiving any injury from the small knife which Straker used in self-defense, and then the thief either led the horse on to some secret hiding-place, or else it may have bolted during the struggle, and be now wandering out on the moors. That is the case as it appears to the police, and improbable as it is, all other explanations are more improbable still. However, I shall very quickly test the matter when I am once upon the spot, and until then I cannot really see how we can get much further than our present position."



# Sherlock as Logician

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F = facts of the case

G = Inspector Gregory's scenario

H = Holmes's scenario

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Holmes: “Gregory's claim is G (Simpson is guilty, & other details re. what he did). We have F, the facts of the case, disputed by no one (and obviously consistent). The question is:  $F \vdash G$ ? The answer is clearly No, for my scenario H, which entails  $\neg G$ , is consistent with the facts [Con ( $F \cup H$ )], and H entails  $\neg G$ . Here's the proof: Suppose for *reductio* that  $F \vdash G$ . Then  $F \cup H$  are inconsistent — contradiction! Hence  $F \not\vdash G$ .”

**Proof sketch ...**

# Questions:

How many  $\mathcal{L}_1$  formulae are there?

And is each formula finite in length?



**Proof-sketch:** The complete set-theoretic universe, known as  $V$ , stratified and built up step by step by iterated use of the power-set operator  $\mathcal{P}$ , is the complete collection of all sets sanctioned by **ZFC**. (Details require transfinite numbers.)

Let's construct, said Gödel in Holmes-in-“Silver Blaze” fashion, our *own* scenario: viz. a hierarchy  $L$  of sets (**the inner model of constructible sets**), likewise building up gradually level by level, starting with  $\emptyset$ , but only taking one step upwards to the set  $Def_{\mathcal{L}_1}(A)$  of all subsets of  $A$  definable in  $\mathcal{L}_1 =$  first-order logic using only the two-place relation  $\in$ . We can show that  $L$  gives us a model that renders **ZFC** true; and all that is true on this model is consistent with **ZFC**. In particular,  $V = L$  is consistent, assuming that **ZFC** is (and we made this assumption to get started in the first place).

But note along with the assumption that  $V = L$ , there are only a countable number of defining  $\mathcal{L}_1$  formulae. Since by hypothesis every real number occurs on some level of the  $L$ -hierarchy, and each jump up the hierarchy requires a first-order formula, there are only as many real numbers as the first *uncountable* “size” we have ( $= \omega_1 = \mathcal{P}(\mathbb{N})$ ). Hence, CH holds. **QED**

**Proof sketch, if we had some  
command of cardinal and  
ordinal numbers ...**



# Gödel's “Holmesian” Proof

ZFC  $\not\equiv$  GCH

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**Proof-Sketch:** GCH is equivalent to the statement that there are at most  $c^*$  subsets of  $c$ , where ' $c$ ' is any cardinal number, and  $c^*$  is the cardinal after  $c$  (i.e.,  $|P(c)| \leq c^*$ ). Gödel showed by employing transfinite recursion (!) that there are at most  $c^*$  constructible subsets of  $c$ ; hence if all sets are constructible, GCH follows. Gödel created a scenario in which all sets *are* constructible (viz.,  $C$ ), and showed that this scenario is consistent with ZFC (assuming, as we are, that ZFC is consistent). Therefore, by parallel to Holmes's reasoning, GCH can never be disproved from ZFC! Here's the reasoning: Suppose for reductio that  $ZFC \vdash \neg GCH$ . We know that  $C \vdash GCH$ . Therefore  $C$  and ZFC are inconsistent; but this contradicts the consistency that Gödel proved. **QED**

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(Of course, in "Silver Blaze," Sherlock did proceed to show that his scenario was true. The analogue to that doesn't hold in the Gödel story.)



*slutten*