Gödel: $\mathbf{ZFC} \nvdash \neg\mathbf{CH}$

(Gödel's "Silver Blaze" Theorem)

Part I: ZFC Review/Context

Selmer Bringsjord

IFLAI2
Nov 22 2021
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Troy NY USA
version 1122210845NY

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Rensselaer AI and Reasoning Lab

Context …

Gödel's Great Theorems (OUP)

by Selmer Bringsjord

- Introduction ("The Wager")
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
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Hilbert's problems

From Wikipedia, the free encyclopedia

Hilbert's problems are twenty-three problems in mathematics published by German mathematician David Hilbert in 1900. The problems were all unsolved at the time, and several of them were very influential for 20th-century mathematics. Hilbert presented ten of the problems (1, 2, 6, 7, 8, 13, 16, 19, 21, and 22) at the Paris conference of the International Congress of Mathematicians, speaking on August 8 in the Sorbonne. The complete list of 23 problems was published later, most notably in English translation in 1902 by Mary Frances Winston Newson in the *Bulletin of the American Mathematical Society*.[1]

David Hilbert

Contents [hide]

Gödel as logician/mathematican; Gödel as prophet.

Table of problems [edit]

Hilbert's twenty-three problems are (for details on the solutions and references, see the detailed articles that are linked to in the first column):

Problem	Brief explanation	Status	Year Solved
1st	The continuum hypothesis (that is, there is no set whose cardinality is strictly between that of the integers and that of the real numbers)	Proven to be impossible to prove or disprove within Zermelo–Fraenkel set theory with or without the Axiom of Choice (provided Zermelo–Fraenkel set theory is consistent, i.e., it does not contain a contradiction). There is no consensus on whether this is a solution to the problem.	1940, 1963
2nd	Prove that the axioms of arithmetic are consistent.	There is no consensus on whether results of Gödel and Gentzen give a solution to the problem as stated by Hilbert. Gödel's second incompleteness theorem, proved in 1931, shows that no proof of its consistency can be carried out within arithmetic itself. Gentzen proved in 1936 that the consistency of arithmetic follows from the well-foundedness of the ordinal ε_0.	1931, 1936
3rd	Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces that can be reassembled to yield the second?	Resolved. Result: No, proved using Dehn invariants.	1900
4th	Construct all metrics where lines are geodesics.	Too vague to be stated resolved or not.[h]	—
5th	Are continuous groups automatically differential groups?	Resolved by Andrew Gleason, assuming one interpretation of the original statement. If, however, it is understood as an equivalent of the Hilbert–Smith conjecture, it is still unsolved.	1953?
6th	Mathematical treatment of the axioms of physics (a) axiomatic treatment of probability with limit theorems for foundation of statistical physics (b) the rigorous theory of limiting processes "which lead from the atomistic view to the laws of motion of continua"	Partially resolved depending on how the original statement is interpreted.[9] Items (a) and (b) were two specific problems given by Hilbert in a later explanation.[1] Kolmogorov's axiomatics (1933) is now accepted as standard. There is some success on the way from the "atomistic view to the laws of motion of continua."[10]	1933–2002?
7th	Is a^b transcendental, for algebraic $a \neq 0,1$ and irrational algebraic b ?	Resolved. Result: Yes, illustrated by Gelfond's theorem or the Gelfond–Schneider theorem.	1934

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Hilbert's First Problem

Let *A* be an infinite set of real numbers. Then we need to prove that *A* is in one-to-one correspondence either with the set of natural numbers, or with the set of *all* real numbers (i.e. with *the continuum*).

Alternatively (with transfinite numbers allowed):

$$\aleph_1 = \mathscr{P}(\mathbb{N}) = 2^{\aleph_0}:$$

Admission: We "AI-got" (ie obtained automated proofs of) G1, G-RT, G2, GST, but getting this quite another story!

“Shorthand” History, & The Admission Again

Hilbert’s #1 (1900): “very plausible theorem”: **CH**

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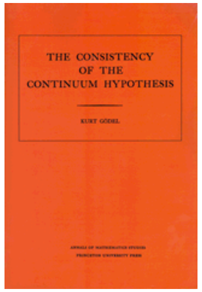

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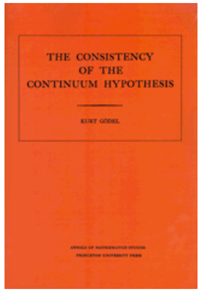

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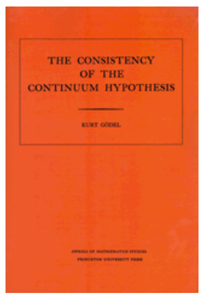

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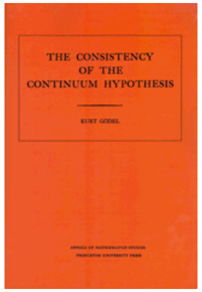

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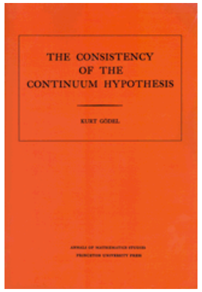

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GCI-based approach to enabling an AI to discover and prove Gödel’s First Incompleteness Theorem:

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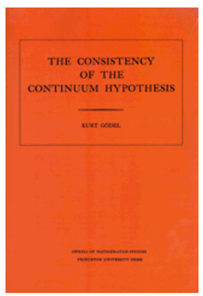

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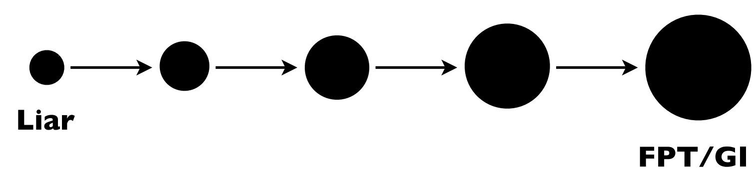

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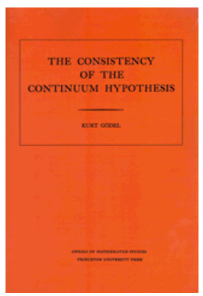

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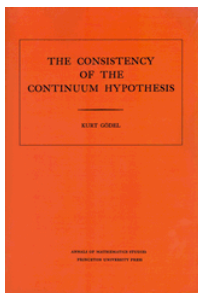

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Won't work on this theorem of Gödel's!

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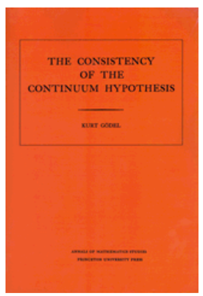

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Note: Gödel (1940): Not like G2!

Cohen (1963):
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GCI-based approach to enabling an AI to discover and prove Gödel’s First Incompleteness Theorem:

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ZFC to the Rescue …

Friday's Hill, Haslemere, 16 June 1902

Dear colleague,

For a year and a half I have been acquainted with your *Grundgesetze der Arithmetik*, but it is only now that I have been able to find the time for the thorough study I intended to make of your work. I find myself in complete agreement with you in all essentials, particularly when you reject any psychological element [[Moment]] in logic and when you place a high value upon an ideography [[Begriffsschrift]] for the foundations of mathematics and of formal logic, which, incidentally, can hardly be distinguished. With regard to many particular questions, I find in your work discussions, distinctions, and definitions that one seeks in vain in the works of other logicians. Especially so far as function is concerned (§ 9 of your *Begriffsschrift*), I have been led on my own to views that are the same even in the details. There is just one point where I have encountered a difficulty. You state (p. 17 [[p. 23 above]]) that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer its opposite follows. Therefore we must conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [[Menge]] does not form a totality.

I am on the point of finishing a book on the principles of mathematics and in it I should like to discuss your work very thoroughly.[1] I already have your books or shall buy them soon, but I would be very grateful to you if you could send me reprints of your articles in various periodicals. In case this should be impossible, however, I will obtain them from a library.

The exact treatment of logic in fundamental questions, where symbols fail, has remained very much behind; in your works I find the best I know of our time, and therefore I have permitted myself to express my deep respect to you. It is very regrettable that you have not come to publish the second volume of your *Grundgesetze*; I hope that this will still be done.

Very respectfully yours,

Bertrand Russell

The above contradiction, when expressed in Peano's ideography, reads as follows:

$$w = \mathrm{cls} \cap x \ni (x \sim\varepsilon\, x) .\supset\colon w\, \varepsilon\, w \,.=.\, w \sim\varepsilon\, w.$$

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☑ RussellsLetter2Frege

The challenge here is to prove that from Russell's instantiation of Frege's doomed Axiom V a contradiction can be promptly derived. The letter has of course been examined in some detail by S Bringsjord (in the Mar 16 2020 lecture in the 2020 lecture lineup); it, along with an astoundingly soft-spoken reply from Frege, can be found here. Put meta-logically, your task in the present problem is to build a proof that confirms this:

$$\{\exists x \forall y((y \in x) \rightarrow (y \notin y))\} \vdash \zeta \wedge \neg\zeta.$$

Make sure you understand that the given here is an instantiation of Frege's Axiom V; i.e. it's an instantiation of

$$\exists x \forall y((y \in x) \rightarrow \phi(y)).$$

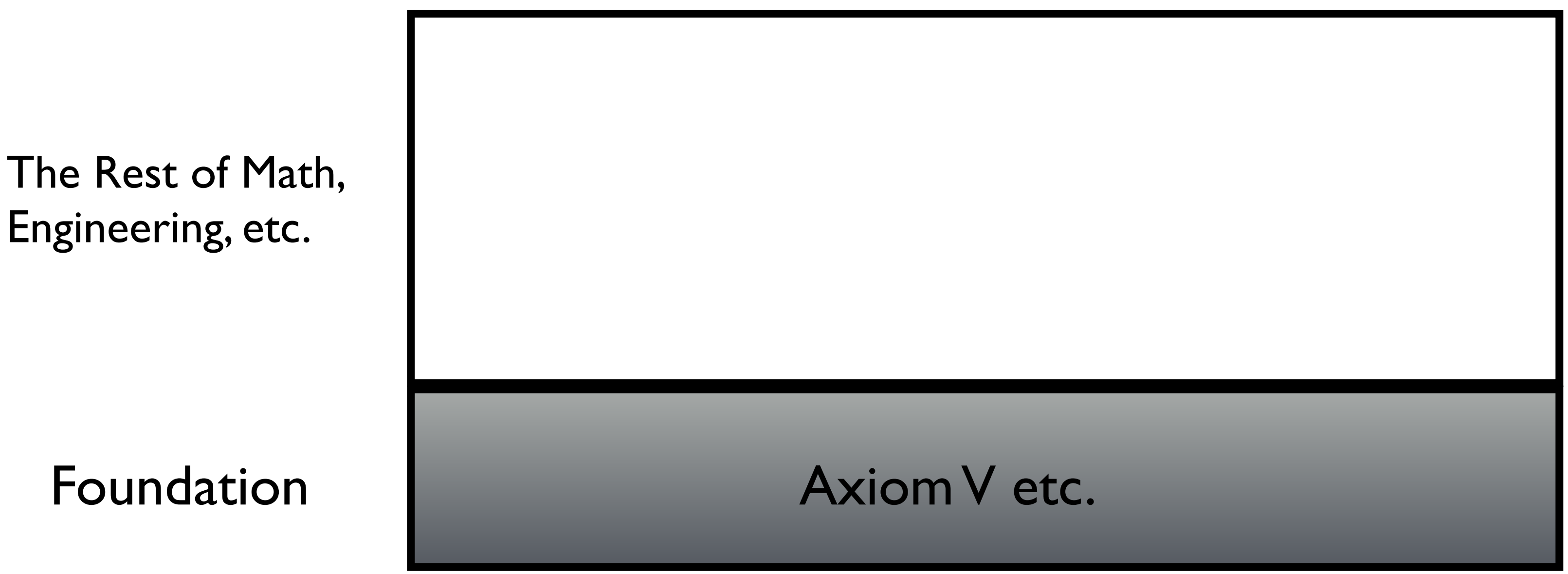
(The notation $\phi(y)$, recall, is the standard way in mathematical logic to say that y is free in ϕ.) **Note**: Your finished proof is allowed to make use the PC-provability oracle (but *only* that oracle).

(Now a brief remark on matters covered by in class by Bringsjord when second-order logic = $\mathscr{L}_2$ arrives on the scene: Longer term, and certainly constituting evidence of Frege's capacity for ingenius, intricate deduction, it has recently been realized that while Frege himself relied on Axiom V to obtain what is known as **Hume's Principle** (= HP), this reliance is avoidable. That from just HP we can deduce all of Peano Arithmetic (**PA**) (!) is a result Frege can be credited with showing; the result is known today as Frege's Theorem (= FT). Following the link just given will reward the reader with an understanding of HP, and how how to obtain **PA** from it.)

Deadline 22 Apr 2020 23:59:00 EST

Solve

The Foundation Crumbles



The Foundation Crumbles

The Rest of Math,
Engineering, etc.

Foundation

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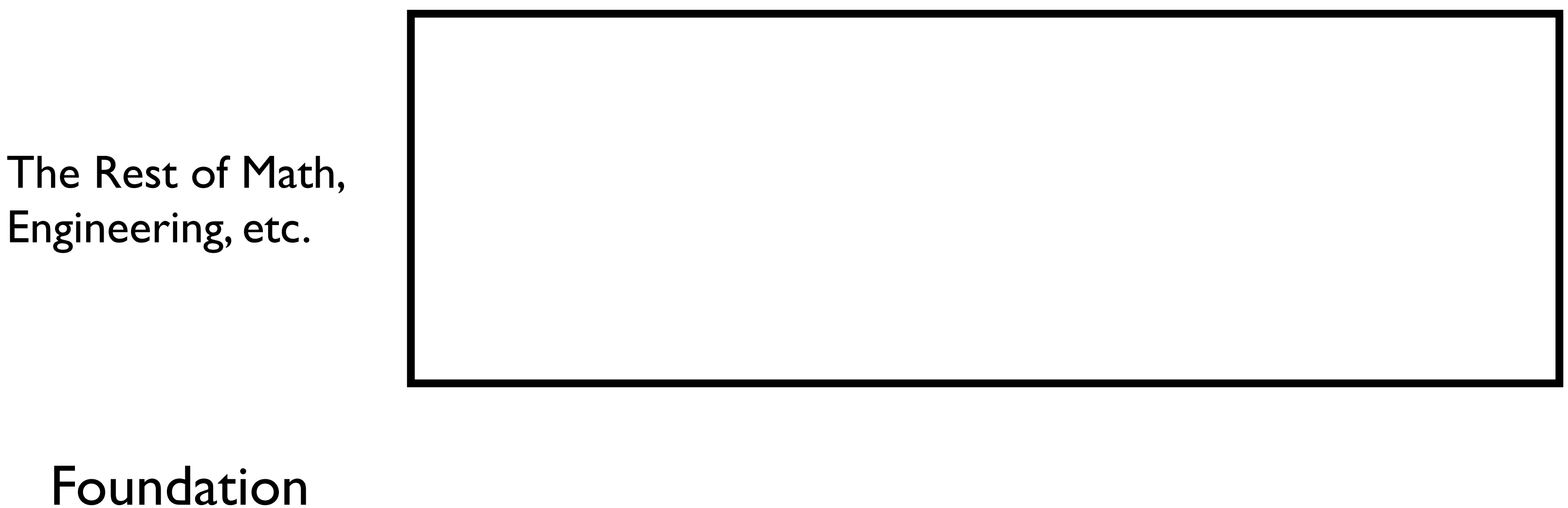

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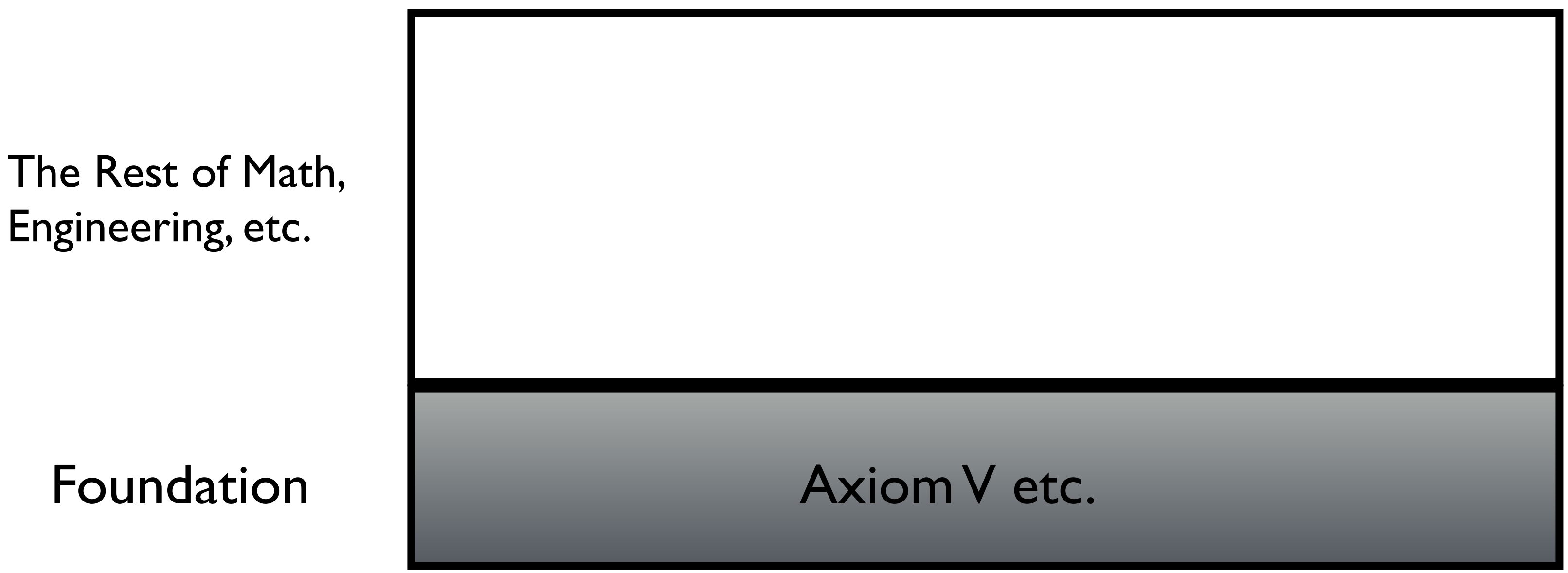

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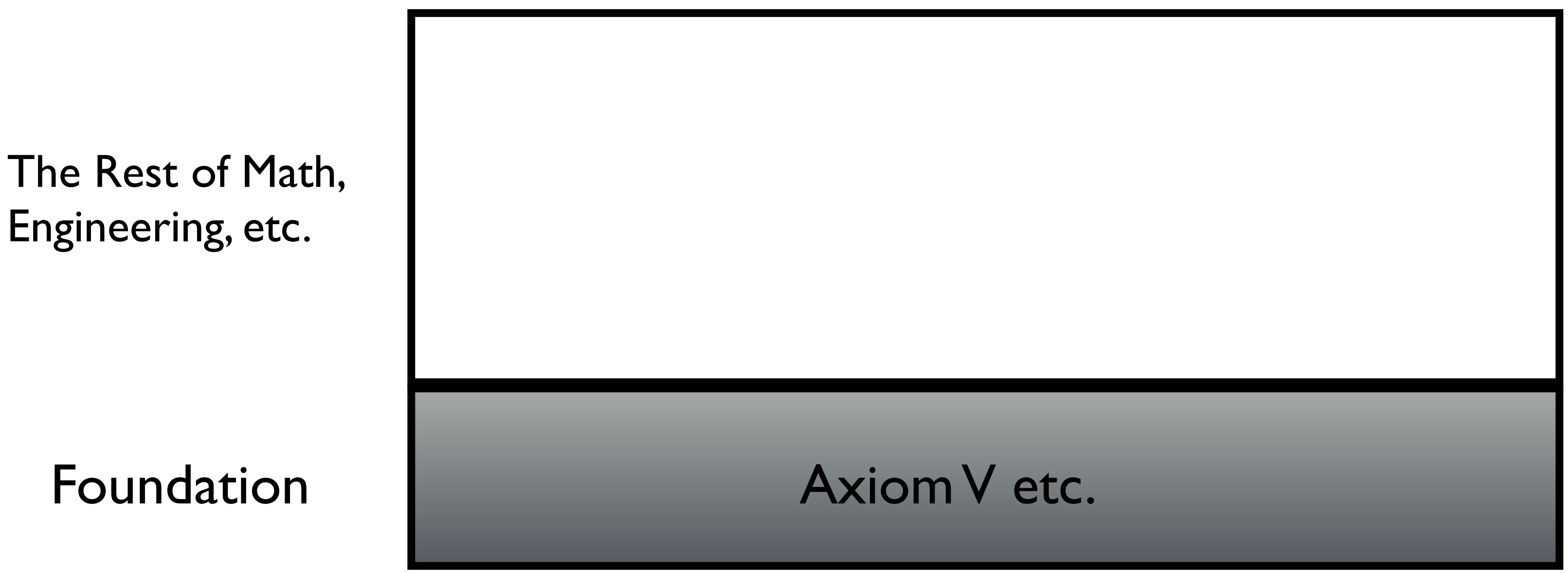

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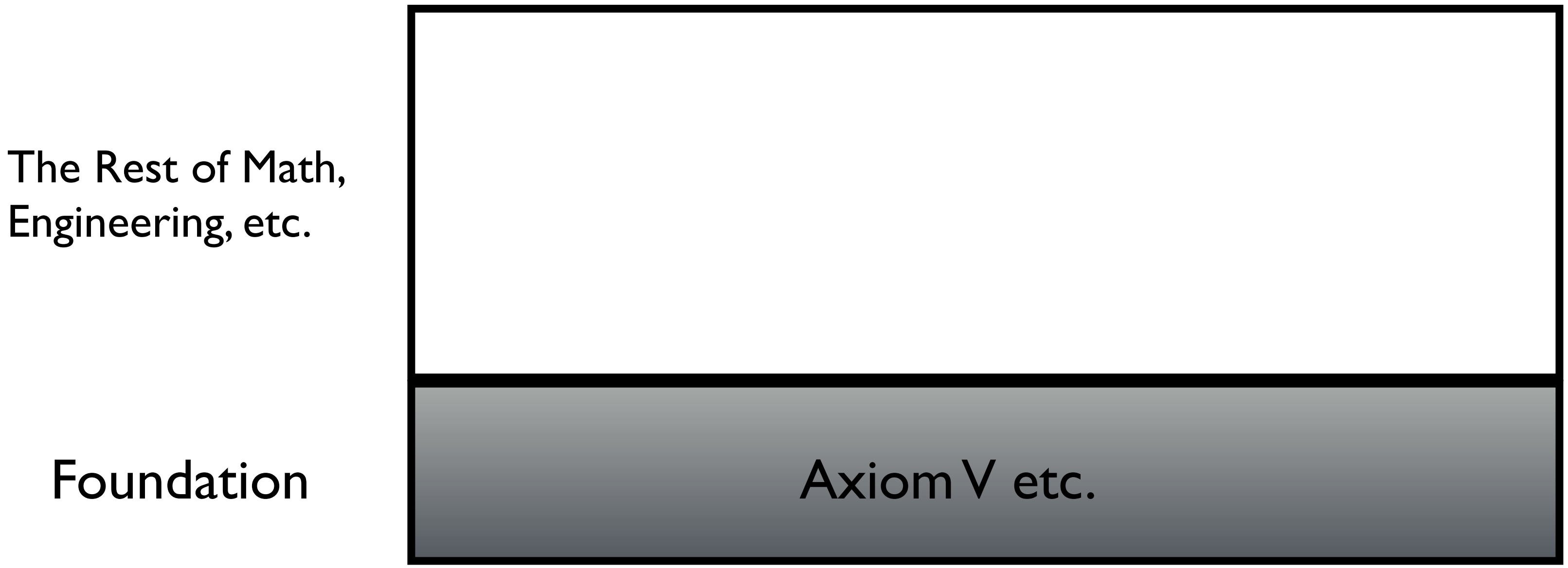

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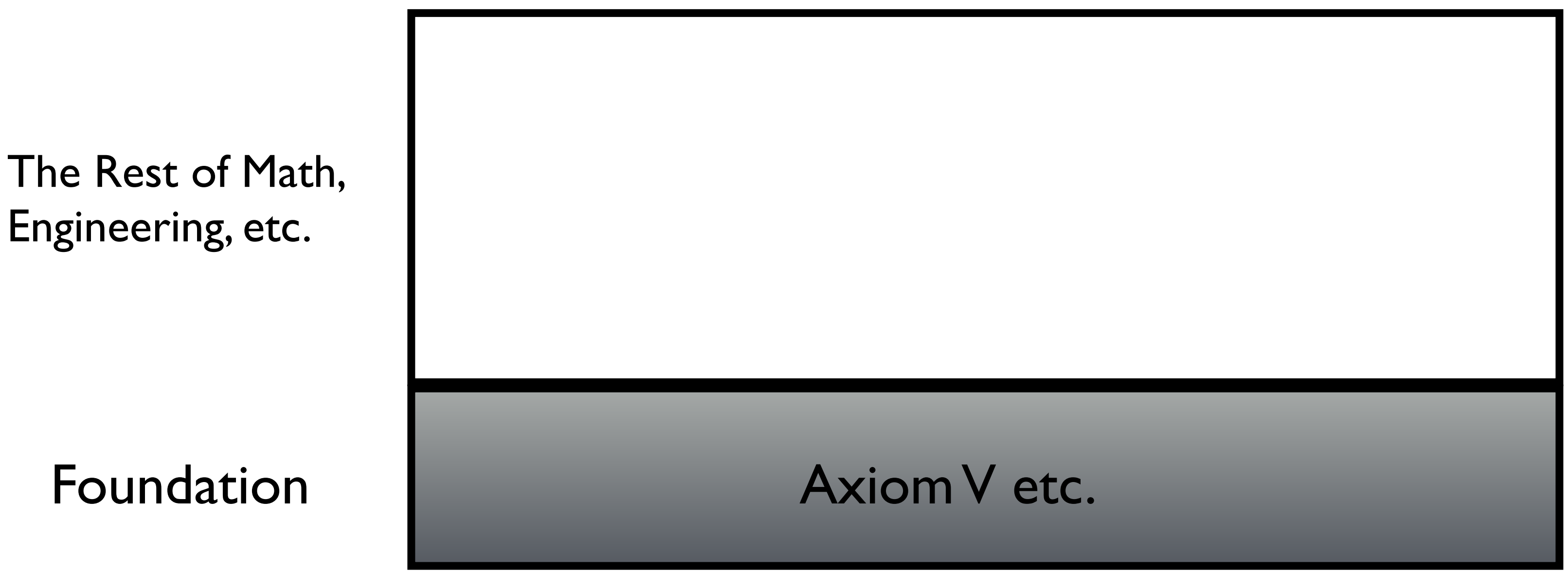
a formula of arbitrary size in which the variable *y* is free; this formula ascribes a property to *y*

The Foundation Crumbles



$$\text{Axiom V} \quad \exists x \forall y [y \in x \leftrightarrow \phi(y)]$$

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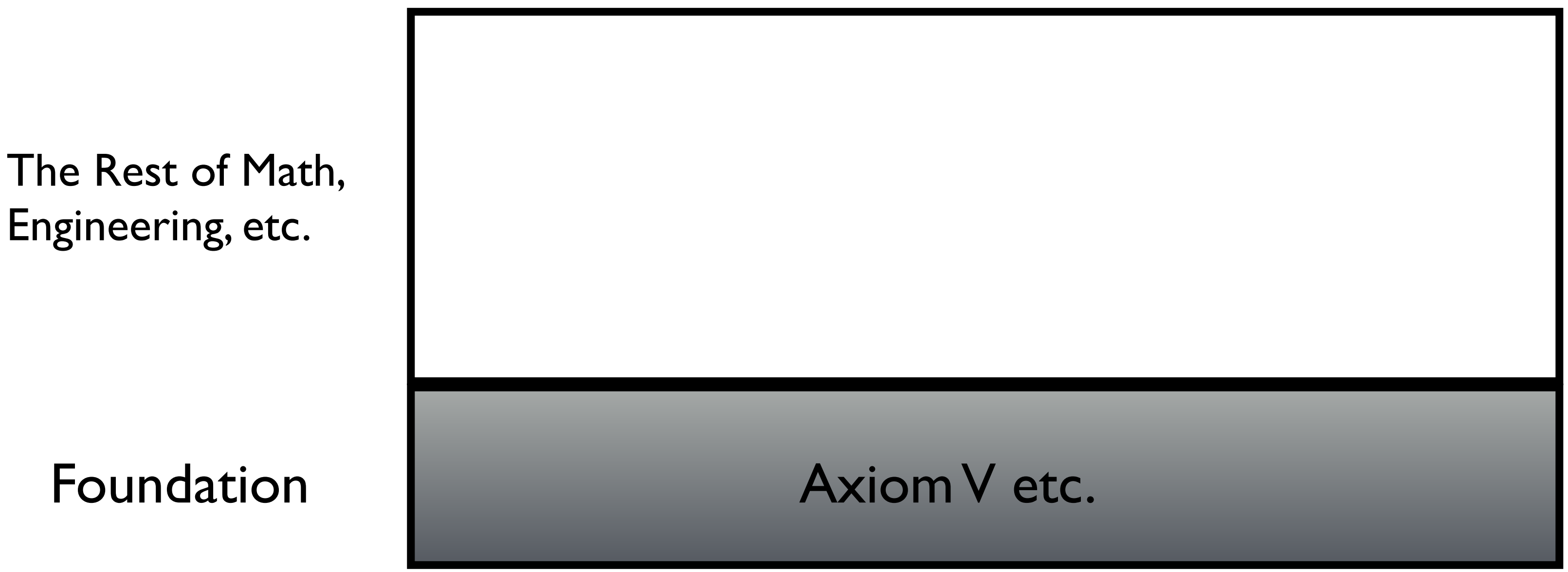
The Foundation Crumbles



The Foundation Crumbles

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The Rest of Math,
Engineering, etc.

Foundation

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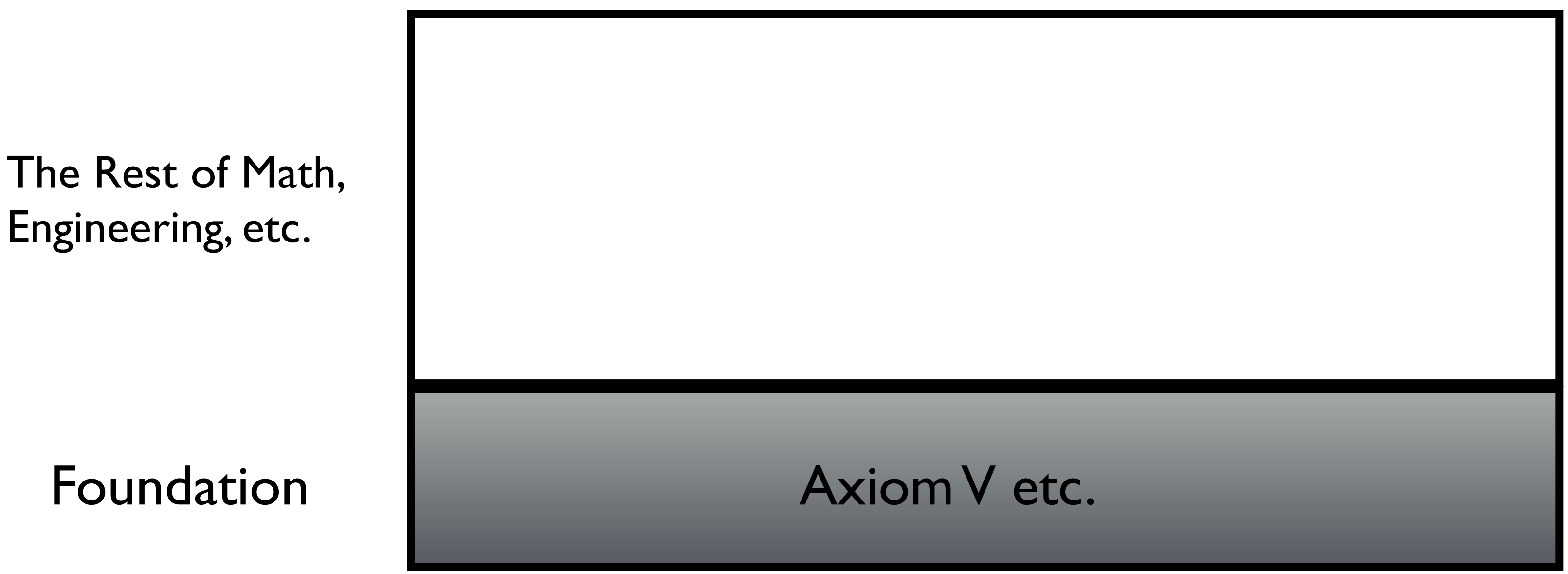

The Foundation Crumbles

The Foundation Rebuilt

The Rest of Math,
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New Foundation

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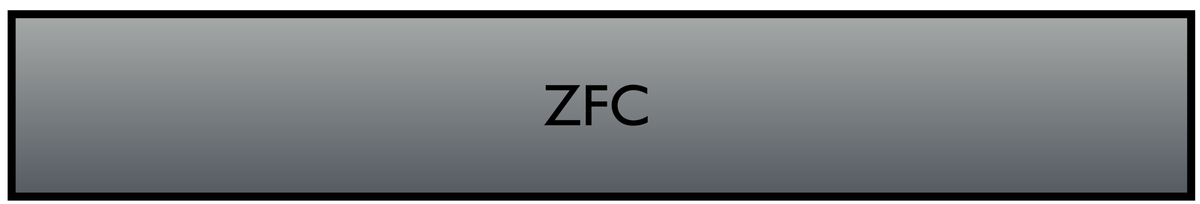

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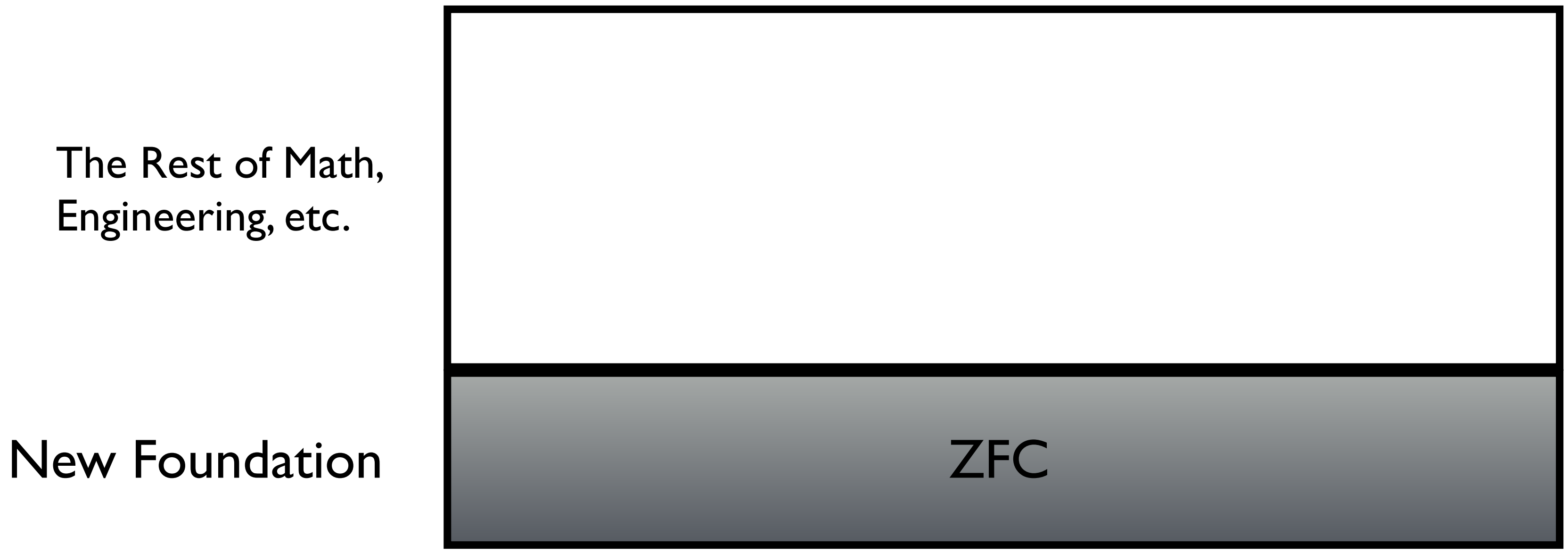

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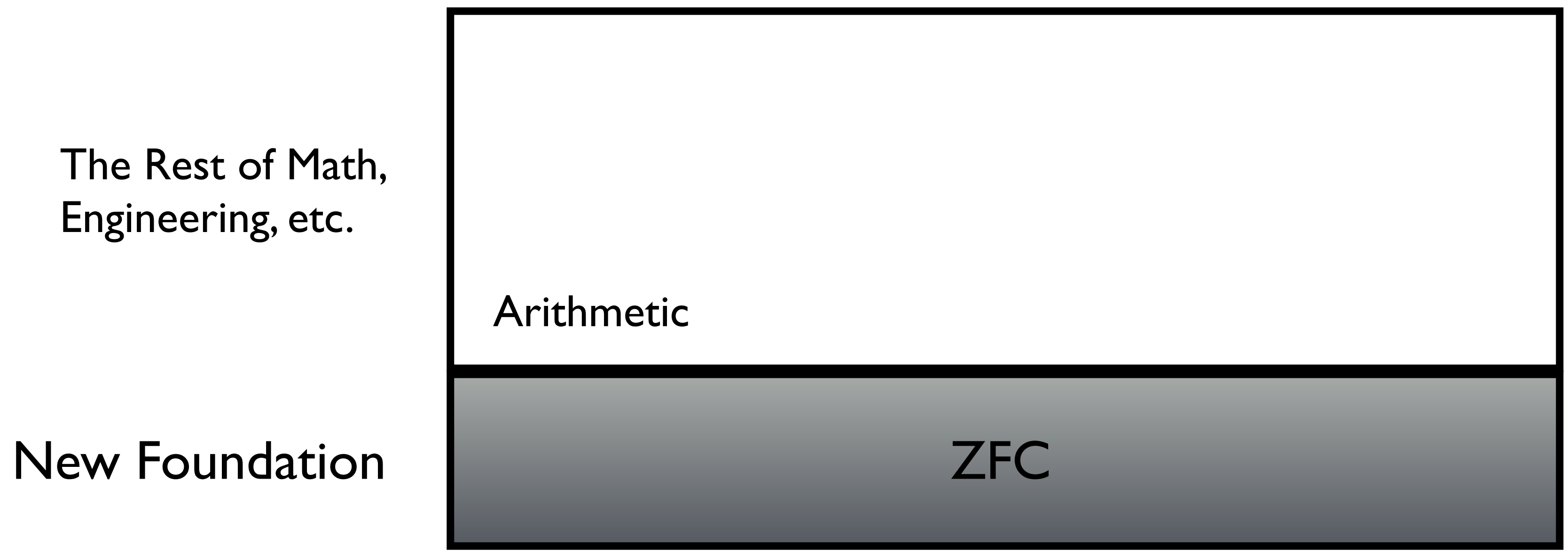

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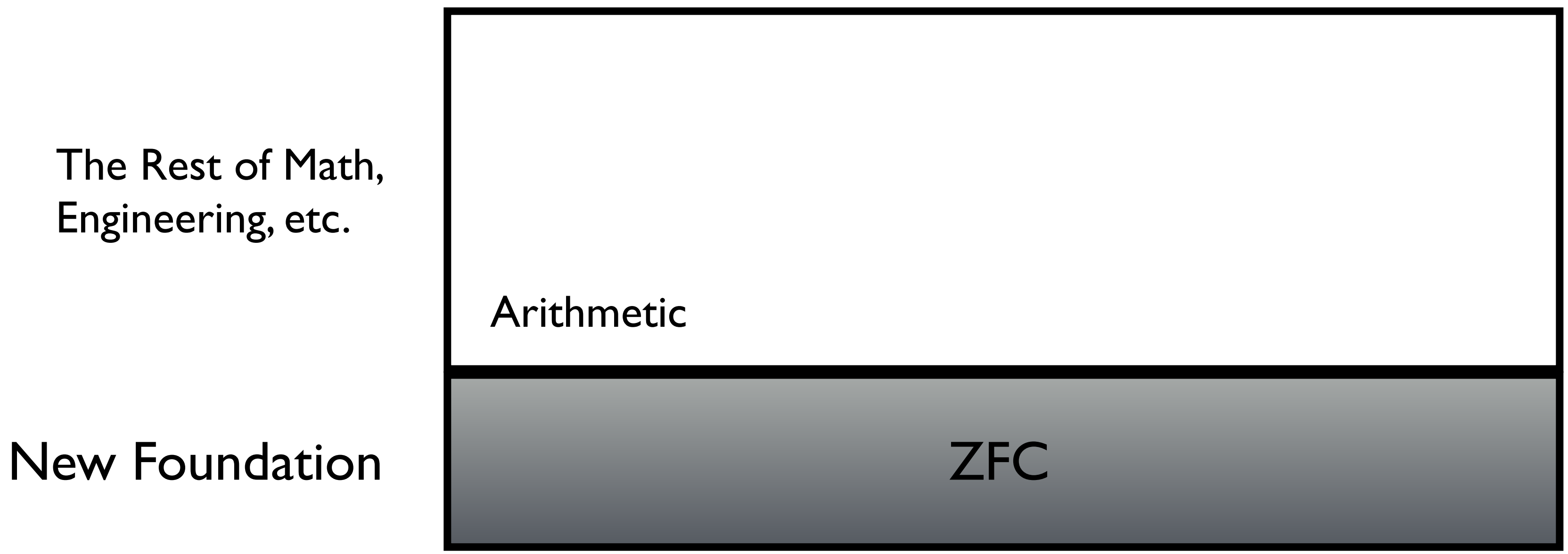

So what are the axioms in ZFC?

The Foundation Rebuilt

The Rest of Math,
Engineering, etc.

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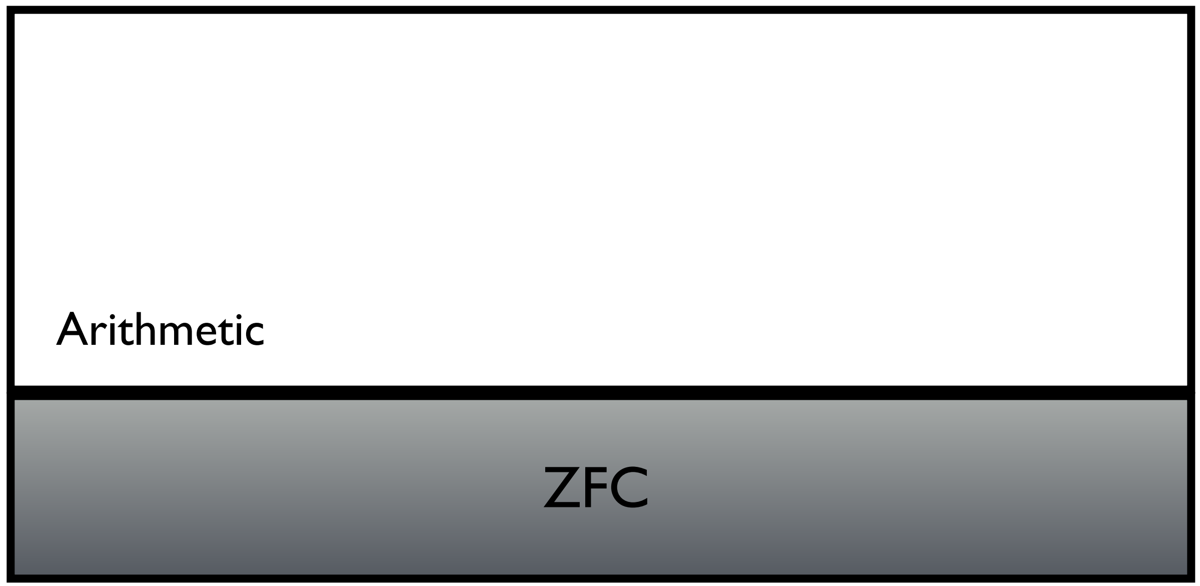

So what are the axioms in ZFC?

ZFC

6.4.1 ZFC

The Zermelo-Fraenkel Axioms for Set Theory, or just ‘ZFC’ for short, include the following nine axioms.[34]

Axiom of Extensionality

$$\forall x \forall y (\forall z (z \in x \longleftrightarrow z \in y) \to x = y)$$

Axiom Schema of Separation

$$\forall x_0 \ldots \forall x_{n-1} \forall x \exists y \forall z (z \in y \longleftrightarrow (z \in x \wedge \phi(z, x_0, \ldots, x_{n-1})))$$

Pair Set Axiom

$$\forall x \forall y \exists z \forall w (w \in z \longleftrightarrow (w = x \vee w = y))$$

Sum Set Axiom

$$\forall x \exists y \forall z (z \in y \longleftrightarrow \exists w (w \in x \wedge z \in w))$$

Power Set Axiom

$$\forall x \exists y \forall z (z \in y \longleftrightarrow \forall w (w \in z \to w \in x))$$

Axiom of Infinity

$$\exists x (\emptyset \in x \wedge \forall y (y \in x \to y \cup \{y\} \in x))$$

Axiom Schema of Replacement

$$\forall x_0 \ldots \forall x_{n-1} (\forall x \exists^{=1} y \, \phi(x, y, x_0, \ldots, x_{n-1}) \to \forall u \exists v \forall y (y \in v \longleftrightarrow \exists x (x \in u \wedge \phi(x, y, x_0, \ldots, x_{n-1}))))$$

Axiom of Choice

$$\forall x ((\emptyset \notin x \wedge \forall u \forall v ((u \in x \wedge v \in x \wedge u \neq v) \to u \cap v = \emptyset)) \to \exists y \forall w (w \in x \to \exists^{=1} z \, z \in w \cap y))$$

Axiom *Schema* of Separation (SEP)

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—

SEP

$$\forall x_1 \ldots \forall x_k \forall x \exists y \forall z [z \in y \leftrightarrow (z \in x \wedge \phi(z, x_1, \ldots, x_k))]$$

where x and y are distinct, and are both distinct from z and the x_i; and, as usual for us now, ϕ expresses a property using $\in$.

—

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"Given *beforehand* some set *x* and property $\mathscr{P}$ expressed by a formula ϕ that uses $\in$ for its relation, the set *y* composed of $\{z \in x : \mathscr{P}(z)\}$ exists."

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"Given *beforehand* some set *x* and property $\mathscr{P}$ expressed by a formula ϕ that uses $\in$ for its relation, the set *y* composed of $\{z \in x : \mathscr{P}(z)\}$ exists."

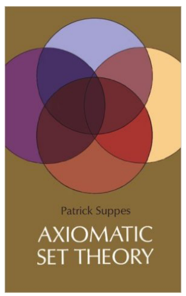
(How does this neutralize Russell's letter to Frege?)

Re. ZFC and HS®

$$\vdash \neg\exists x \forall y (y \in x \leftrightarrow y \notin y)$$

(Russell's Theorem; poor Frege!)

http://plato.stanford.edu/entries/russell-paradox/#HOTP



Supplant Cantor's/Frege's Axiom V with the Axiom Schema of Separation (& put on our thinking caps …) and you try to show Theorem 1 from Suppes:

$$\vdash \forall x (x \notin \emptyset)$$

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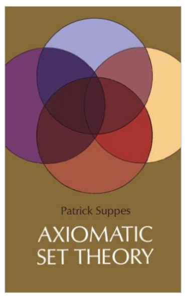

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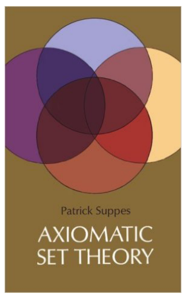

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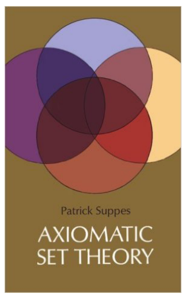

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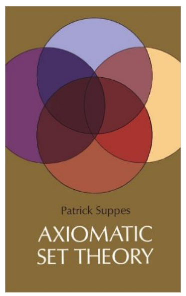

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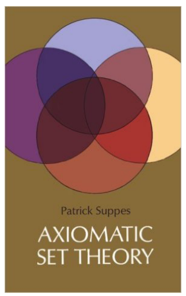

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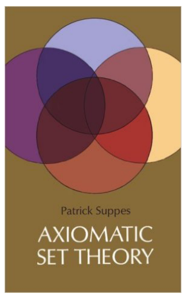

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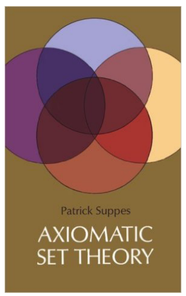

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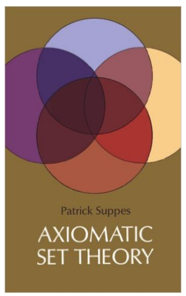

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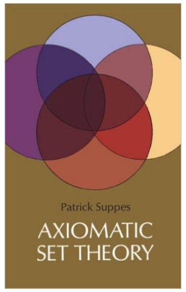
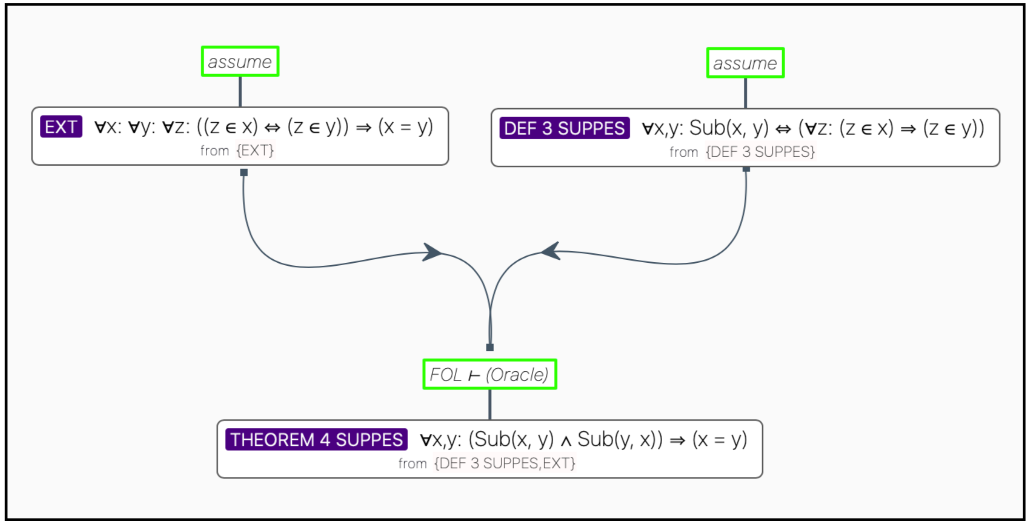
Re. Theorem 4 in HS®





ZFC (in some English)

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$\mathbb{N}$

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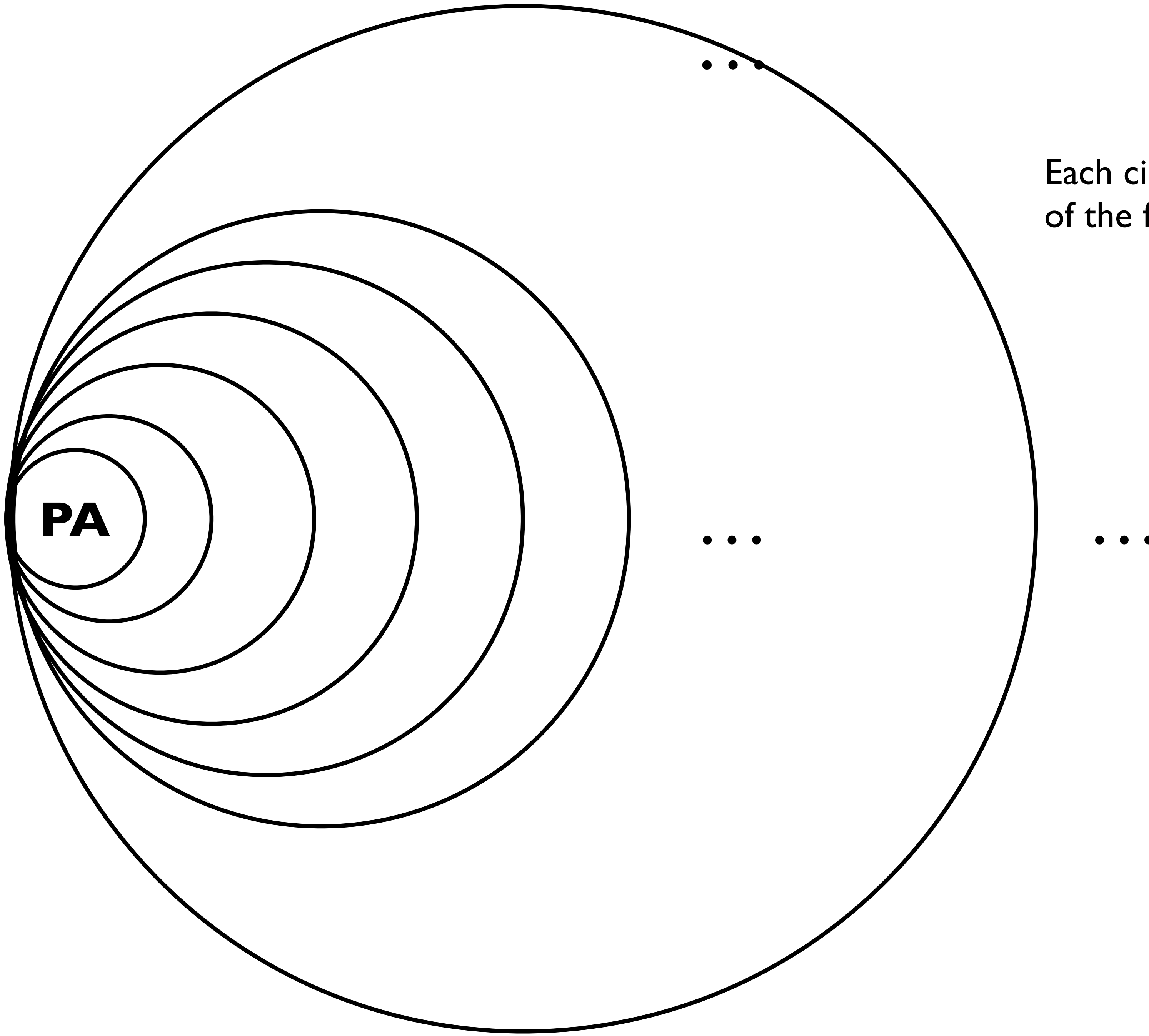
Better Overarching Pictorial Conception of ZF/ZFC …

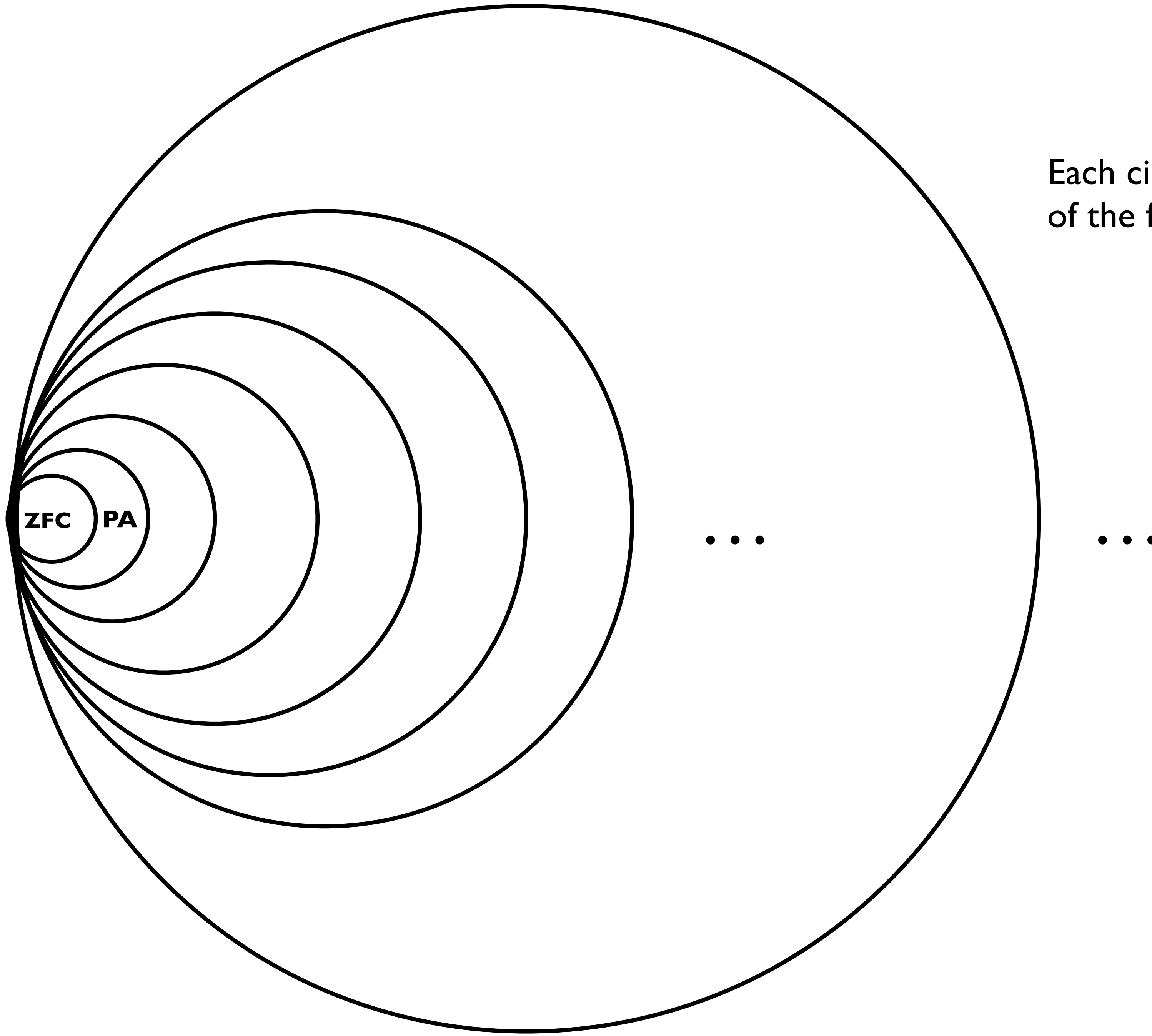
Arithmetic is Part of All Things Sci/Eng/Tech!

and courtesy of Gödel: We can't even prove all truths of arithmetic!



Each circle is a larger part of the formal sciences.

Actually, the true kernel is set theory!



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But these are easily misinterpreted pictures; so …

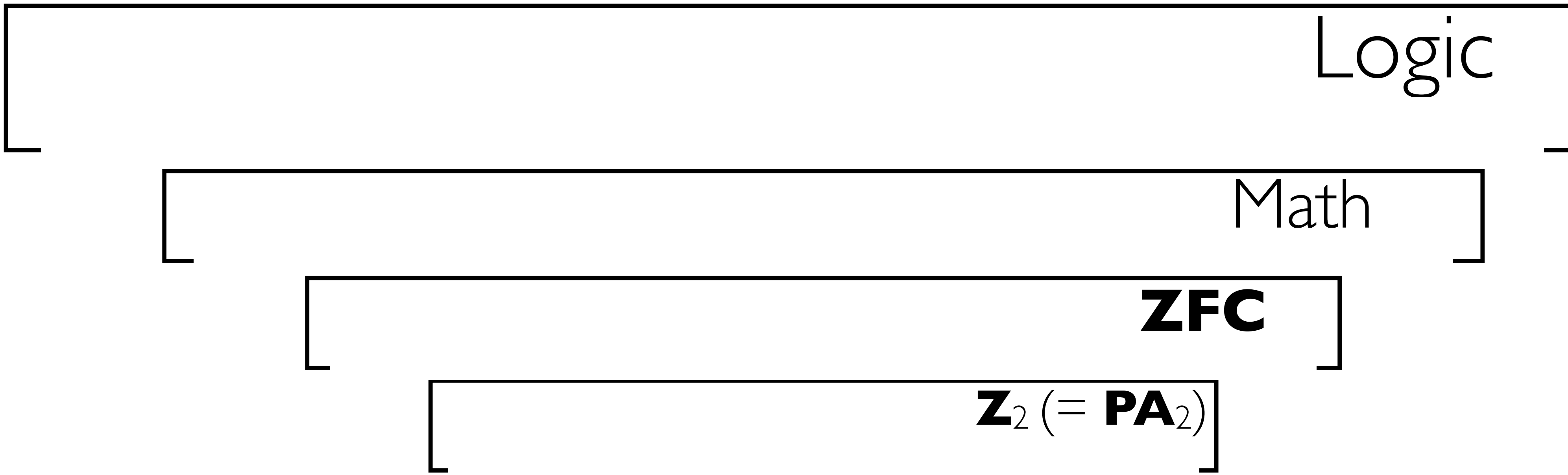
$PA_2 = Z_2$

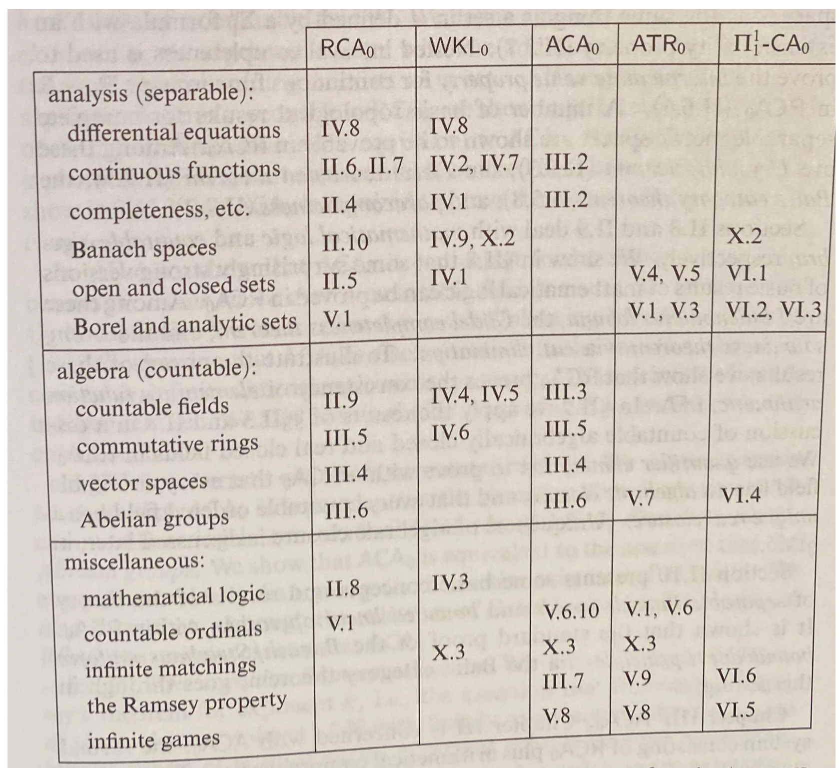
$$\begin{array}{ll} \text{A1} & \forall x(0 \neq s(x)) \\ \text{A2} & \forall x \forall y(s(x) = s(y) \rightarrow x = y) \\ \text{A3} & \forall x(x \neq 0 \rightarrow \exists y(x = s(y)) \\ \text{A4} & \forall x(x + 0 = x) \\ \text{A5} & \forall x \forall y(x + s(y) = s(x + y)) \\ \text{A6} & \forall x(x \times 0 = 0) \\ \text{A7} & \forall x \forall y(x \times s(y) = (x \times y) + x) \end{array}$$

Induction Axiom $\forall X[(X(0) \wedge \forall x(X(x) \rightarrow X(s(x)) \rightarrow \forall x X(x)]$

Comprehension Axioms $\exists X \forall x(x \in X \leftrightarrow \phi(x))$

“Reality”

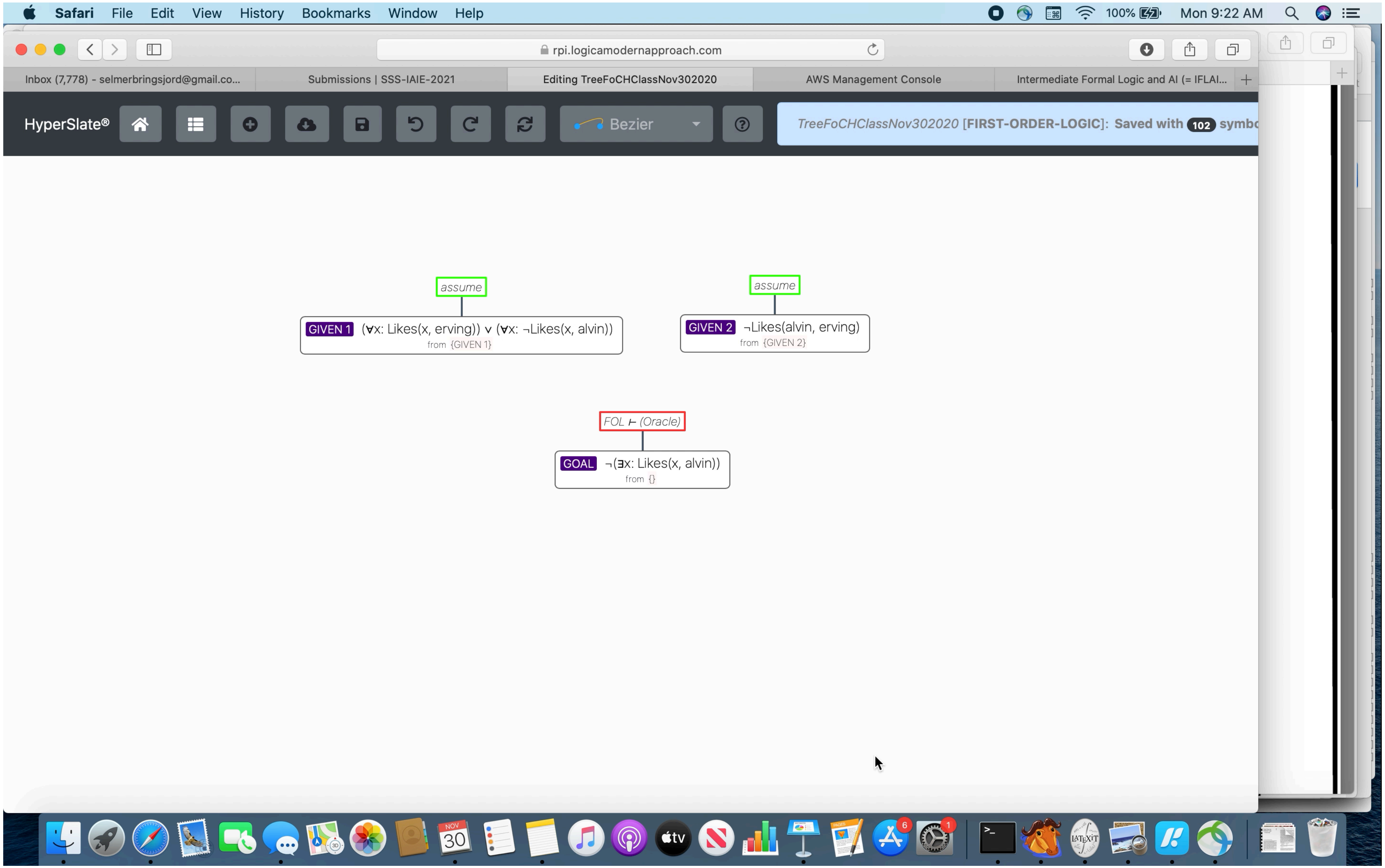




	RCA_0	WKL_0	ACA_0	ATR_0	Π^1_1-CA_0
analysis (separable):					
differential equations	IV.8	IV.8			
continuous functions	II.6, II.7	IV.2, IV.7	III.2		
completeness, etc.	II.4	IV.1	III.2		
Banach spaces	II.10	IV.9, X.2			X.2
open and closed sets	II.5	IV.1		V.4, V.5	VI.1
Borel and analytic sets	V.1			V.1, V.3	VI.2, VI.3
algebra (countable):					
countable fields	II.9	IV.4, IV.5	III.3		
commutative rings	III.5	IV.6	III.5		
vector spaces	III.4		III.4		
Abelian groups	III.6		III.6	V.7	VI.4
miscellaneous:					
mathematical logic	II.8	IV.3			
countable ordinals	V.1		V.6.10	V.1, V.6	
infinite matchings		X.3	X.3	X.3	
the Ramsey property			III.7	V.9	VI.6
infinite games			V.8	V.8	VI.5

Proofs of Non-Entailment in HS® …

We Have Entailment



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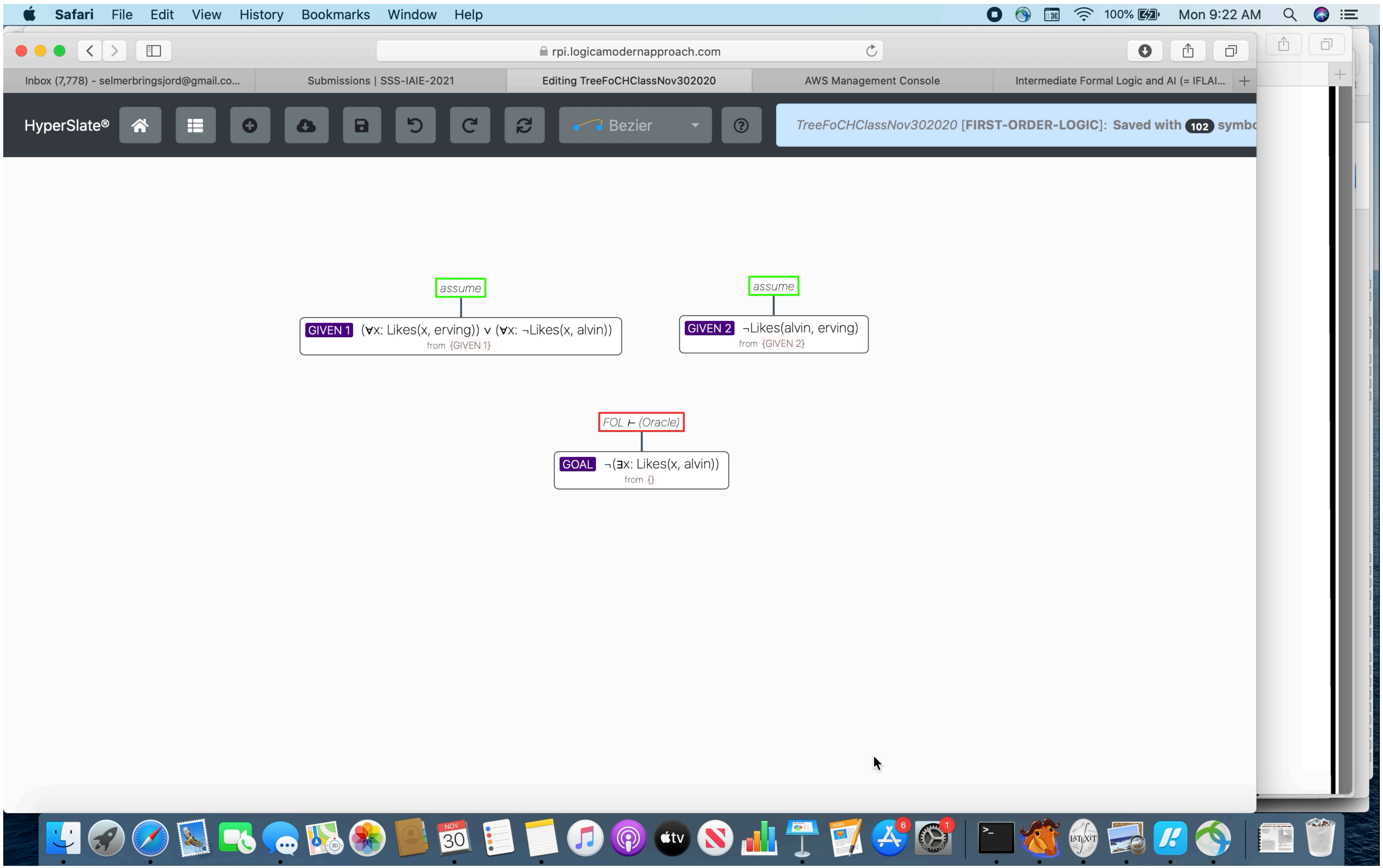
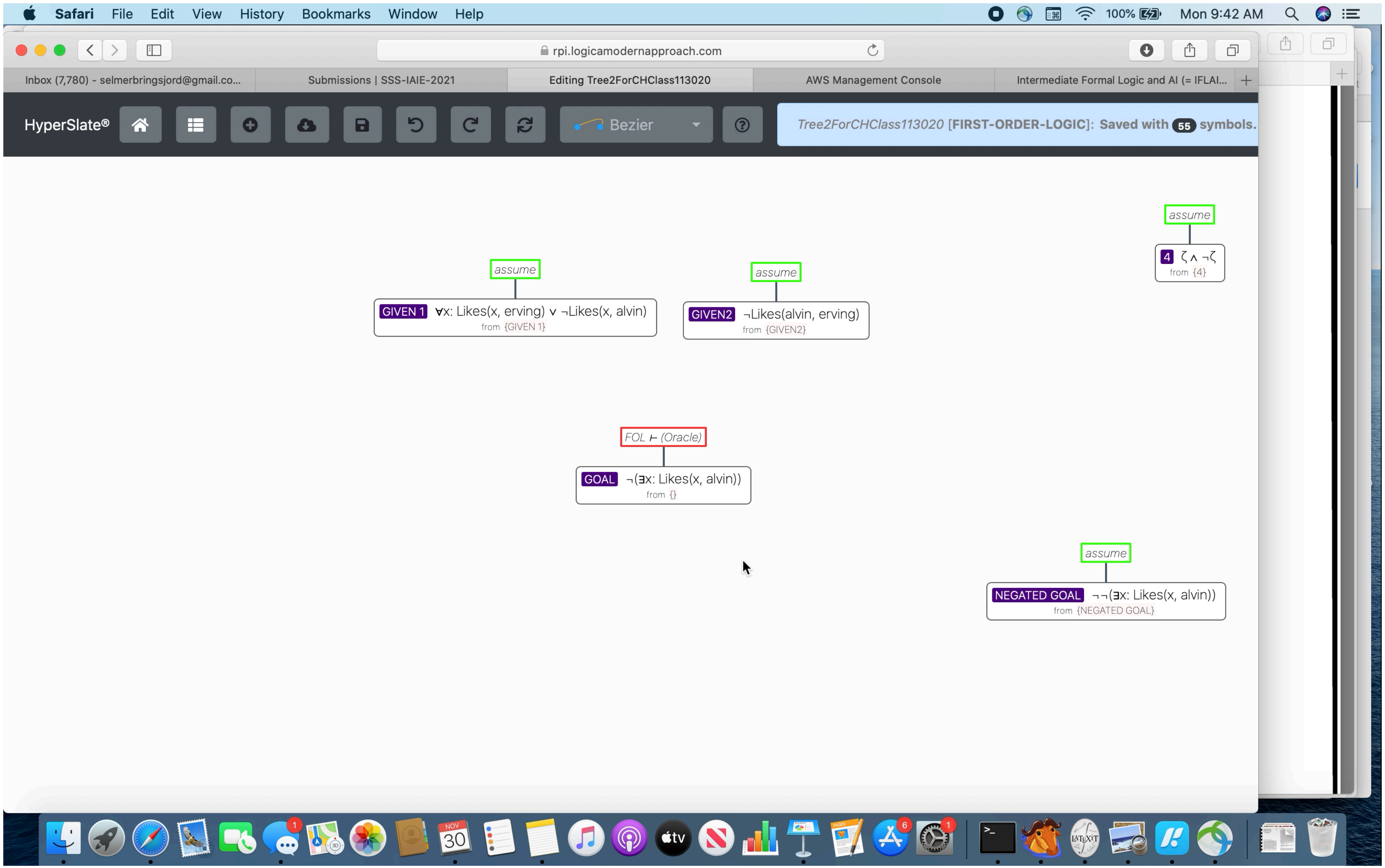


We *Don't* Have Entailment



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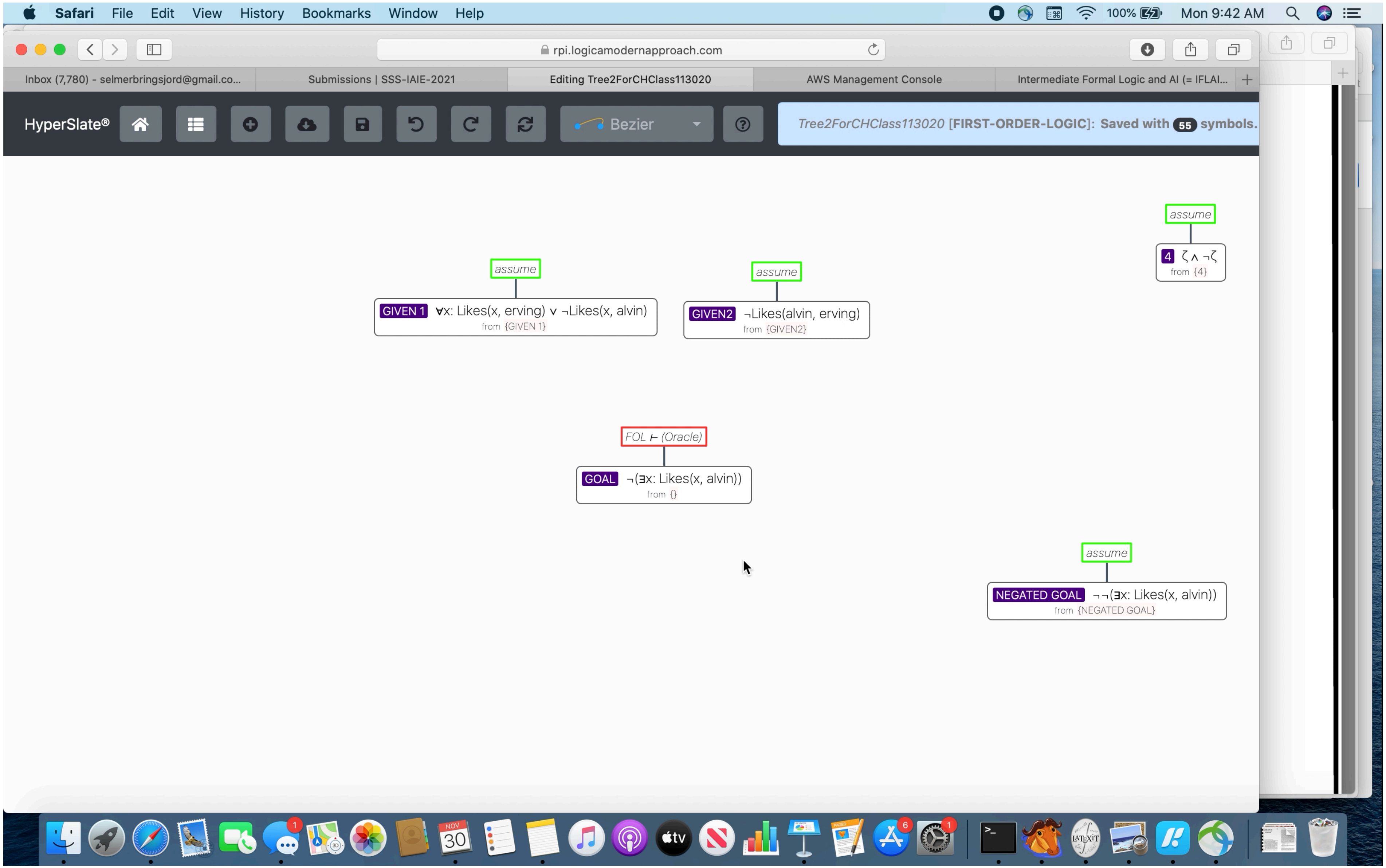

Cantorian Context …

Cantor’s Theorem

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Cantor (1878):

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$$\mathcal{P}(\mathbb{N}) > \mathbb{N}$$

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The power set of the natural numbers is larger than than the natural numbers!

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Cantor (1878):

$$\mathcal{P}(\mathbb{N}) > \mathbb{N}$$

The power set of the natural numbers is larger than than the natural numbers!

How do we know this???!???

Continuum Hypothesis

(*sans* use of ordinal or cardinal numbers)

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$$\mathrm{CH} : \forall S[(S \subset \mathbb{R} \wedge \neg \mathbf{Fin}(S)) \rightarrow (S \sim \mathbb{N} \vee S \sim \mathbb{R})]$$

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Every infinite subset of the reals is either the same size as the natural numbers or the same size as the reals.

Generalized Continuum Hypothesis

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For every infinite set $S, \mathcal{P}(S) > S$.

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For every infinite set $S, \mathcal{P}(S) > S$.

Generalized Continuum Hypothesis (GCH):

There's no set (size-wise) between *S* and *P*(*S*).

Single slide summary

Cantor (1878):

$$\mathcal{P}(\mathbb{N}) > \mathbb{N}$$

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