Propositional Logic & Hyperslate

James T. Oswald

Schedule For Today

- Introduction to Formal Propositional Logic (The Logic PC)
- Review of P0 Inference Rules in Hyperslate & Exercises
- Exercises with common theorems & Can LLMs solve them?

To follow along, this slide deck is available at: http://bit.ly/3MKU0cr

What is Propositional Logic? First What is a Logic?

Definition: A logic L is *formal system* that defines a notion of truth

Definition: A *formal system* is a mathematical object that contains:

- A *formal language,* a set of strings or rules for generating a set of strings that are considered to be "Well formed" in the case of logic, these are "Well formed formulas"
- A *formal semantics,* a set of rules assigning a *meaning* to well formed statements. In logic, the "meaning" of a formula is closely tied to its truth value.

Formal Language of Propositional Logic

This is *formal language* is defined via a *formal grammar* a set of rules that define the infinite set of well formed propositional formulae.

$$
\begin{aligned}
atom &::= < \text{Any Letter or Word} > \\
formula &::= atom \\
&\quad \mid \quad (formula) \\
&\quad \mid \quad \neg formula \\
&\quad \mid \quad formula \land formula \\
&\quad \mid \quad formula \lor formula \\
&\quad \mid \quad formula \rightarrow formula \\
&\quad \mid \quad formula \leftrightarrow formula
\end{aligned}
$$

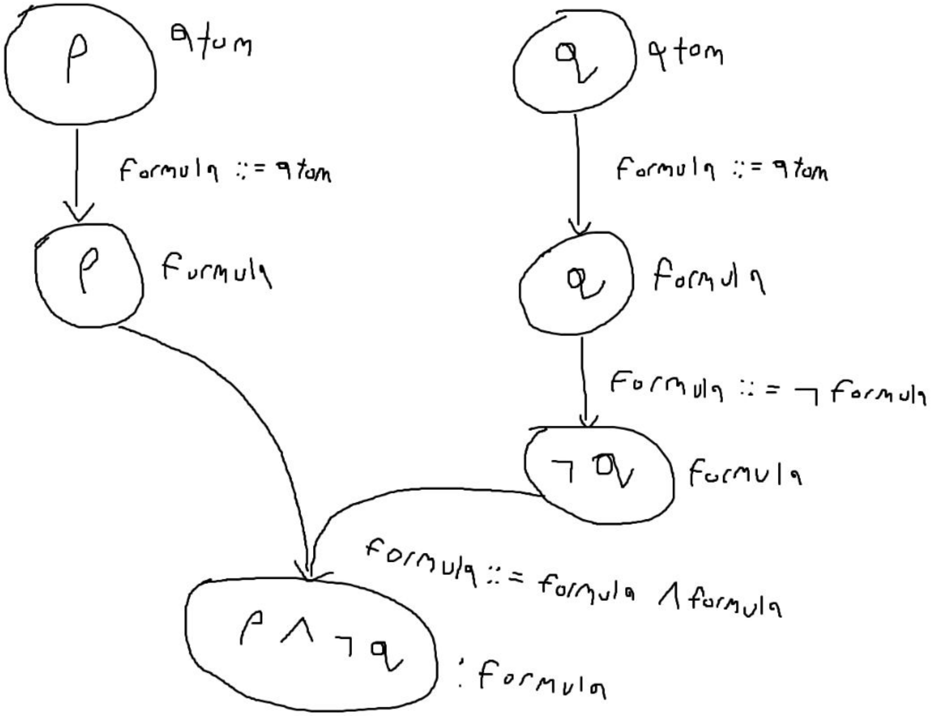
1. Is $p \land \neg q$ well formed?
2. Is $p \land \neg \rightarrow \neg\neg q$ well formed?

Syntax Tree

Well-formed statements have syntax trees!

A syntax tree builds a formula by applying the rules from the formal grammar.

Definition (Well-formed): A formula is *well formed* if and only if you can build a syntax tree for it.



A syntax tree for $p \wedge \neg q$

Exercises

Are the following well formed? If so draw a syntax tree else explain where they fail.

1. $(P \wedge Q) \rightarrow R$
2. $(\text{Cold} \vee (P \rightarrow Q \wedge r)$
3. $\neg(\text{Hot} \rightarrow (p \vee q))$
4. $\text{Rain}\neg \rightarrow (\text{Cloudy} \vee p)$
5. $(\text{Wind} \wedge (Q \vee r \rightarrow \neg))$
6. $(S \vee \neg P) \rightarrow Q$
7. $\neg\neg(p \wedge Q) \vee r$

Reminder….

$$
\begin{aligned}
atom &::= < \text{Any Letter or Word} > \\
formula &::= atom \\
&\quad |\quad (formula) \\
&\quad |\quad \neg formula \\
&\quad |\quad formula \wedge formula \\
&\quad |\quad formula \vee formula \\
&\quad |\quad formula \rightarrow formula \\
&\quad |\quad formula \leftrightarrow formula
\end{aligned}
$$

Formal Semantics of Propositional Logic

Well-formed formulae are just collections of symbols, but we care about their meanings, particularly the notion of *truth.* When is a well formed formula true?

In this class the a notion of truth of a formula is defined syntactically via a system called *natural-deduction proof-theoretic semantics (*NDPTS*).*

Definition (Truth in NDPTS): Under NDPTS, a well-formed formulae φ said to be true iff there exists a natural deduction proof of φ.

Notation (⊦ φ): We use the notation "⊦ φ" for any formula φ to mean "φ is provable" or equivalently "there exists a natural deduction proof of φ". Under NDPTS "⊦ φ" can also be read "φ is a theorem".

More on Semantics

In natural deduction proofs we make assumptions and discharge them to prove statements. It is useful to have a notion of

Notation ($\Gamma \vdash \varphi$) :We use the notation "$\Gamma \vdash \varphi$" for any formula φ and set of formulae Γ to mean "φ is provable under the assumption Γ".

Example: $\{a, b\} \vdash a \wedge b$ is read:

"assuming a is true and assuming b is true it is provable that a and b".

Note that: "$\{\} \vdash \varphi$", read "assuming nothing φ is provable" is identical to "$\vdash \varphi$" and thus "$\{\} \vdash \varphi$" can be read "φ is a theorem" or any other reading of "$\vdash \varphi$".

Inference Rules

Definition (Inference Rules): Inference rules in natural deduction (ND) define what is provable (They literally define “⊦”). They operate purely syntactically. They take premices, analyze their syntax, and return a conclusion.

Inference rules in ND have two parts: *premises* and a *conclusion*

Examples:

$$\frac{\begin{array}{c}\text{Premise 1}\\ \text{Premise 2}\end{array}}{\text{Conclusion}}$$

$$\frac{\varphi \quad \psi}{(\varphi \wedge \psi)}$$

Reads “for any formula phi or psi if we have phi is provable and psi probable than phi and psi is provable” (See And Intro slide)

Explicit Provability & Assumptions

If you look up natural deduction inference rules online you may see examples like the last slide. Due to how we keep track of and discharge assumptions in natural deduction, it can be better for clarity to show explicitly how assumptions are passed from premicies to conclusions, or discarded.

$$\frac{\varphi \quad \psi}{(\varphi \land \psi)} \qquad \frac{\Gamma \vdash \psi \quad \Sigma \vdash \varphi}{\Gamma \cup \Sigma \vdash \varphi \land \psi} \land I$$

Both are the same but the second one makes it explicit that assumptions for proving phi and psi are all used as assumptions for proving the conjunction of phi and psi.

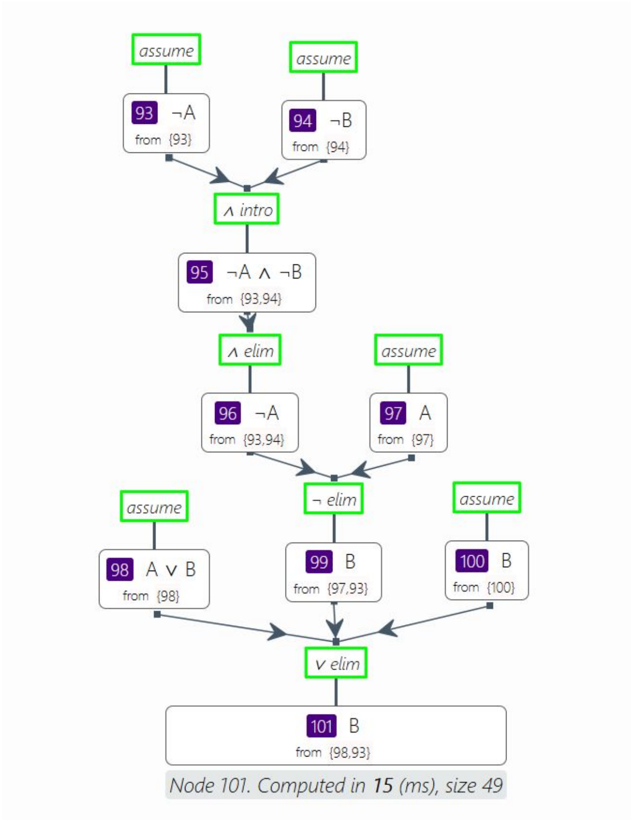
What is a Natural Deduction Proof?

A proof is a series of inference rule applications leading to some final conclusion

If the final conclusion is in the form “{} ⊦ φ” than we say φ is a theorem.

Hyperslate is an interactive node-based natural deduction
proof builder.

$$\dfrac{\dfrac{}{\{A\lor B\}\vdash A\lor B}\,\mathrm{A} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\{\neg A\}\vdash\neg A}\,\mathrm{A}\quad\dfrac{}{\{\neg B\}\vdash\neg B}\,\mathrm{A}}{\{\neg B,\neg A\}\vdash\neg A\land\neg B}\,\land I}{\{\neg B,\neg A\}\vdash\neg A}\,\land E_l \quad \dfrac{}{\{A\}\vdash A}\,\mathrm{A}}{\{\neg A, A\}\vdash B}\,\neg E \quad \dfrac{}{\{B\}\vdash B}\,\mathrm{A}}{\{A\lor B,\neg A\}\vdash B}\,\lor E$$



Inference Rule: Assumption

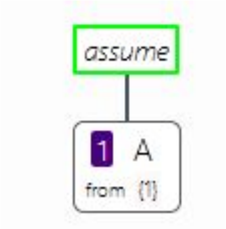
No premicies

Conclude that:

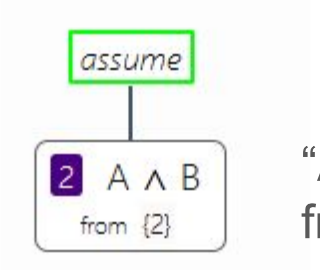
“assuming φ it is provable that φ”.

“φ is provable from φ”

$$\frac{}{\{\varphi\} \vdash \varphi}\ \mathrm{A}$$



“A is provable from A”



“A and B is provable from A and B”

In Hyperslate you can assume anything! But note the “from {1}” and “from {2}”

Assumption Exercises

In Hyperslate represent the following:

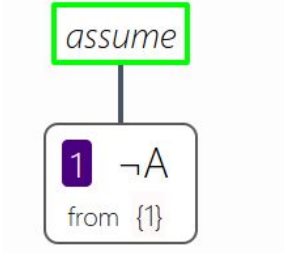
1. “Assuming not A It is provable that not A”
2. “A or B is provable from A or B”
3. “If A or B then not C It is provable from If A or B then not C”

Assumption Exercise Solutions

In Hyperslate represent the following:

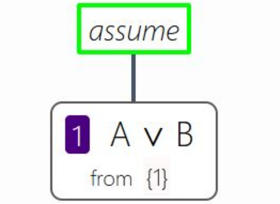
“Assuming not A It is provable that not A”

```
(not A)
```



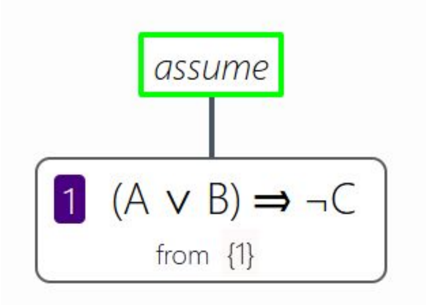
“A or B is provable from A or B”

```
(or A B)
```



“If A or B then not C It is provable from If A or B then not C”

```
(if (or A B) (not C))
```

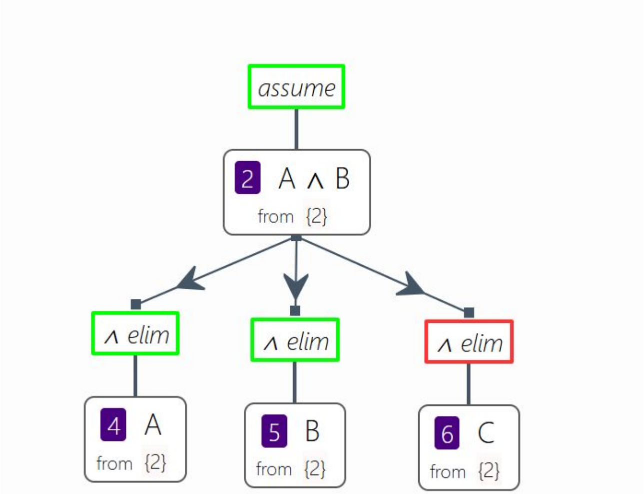


Inference Rule(s): Conjunction Elimination

Premicie: “φ and ψ is provable from Γ”

Conclusions: “φ is provable from Γ”
“ψ is provable from Γ”

Typically this is represented as two inference rules
But Hyperslate allows either conclusion.

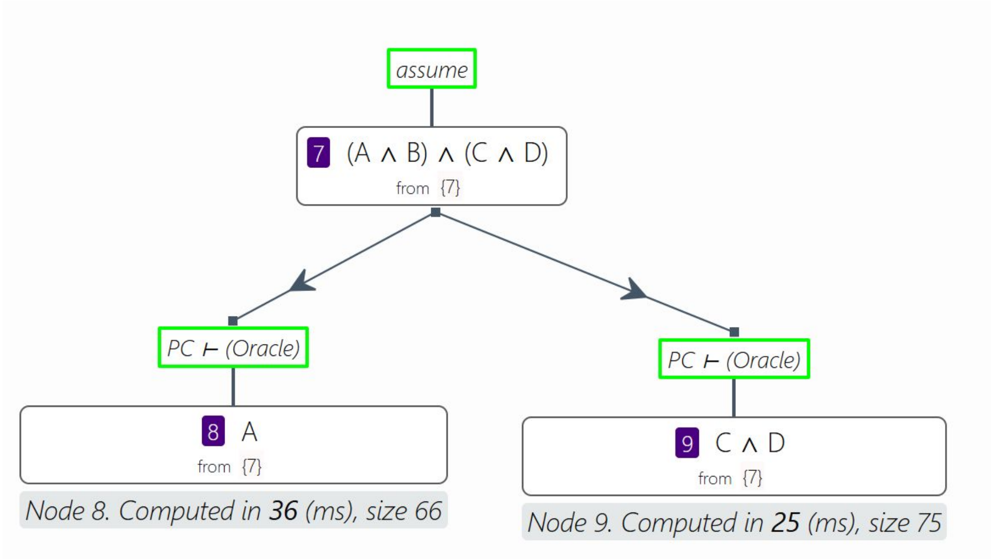


$$\frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \varphi} \land E_l \qquad \frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \psi} \land E_r$$

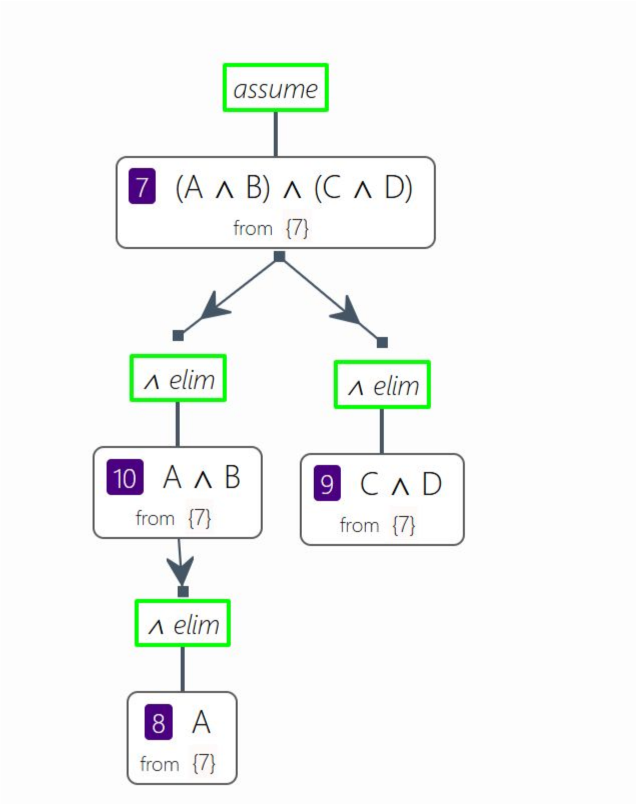
Conjunction Elimination Exercises

Assume (and (and A B) (and C D))

Use conjunction elimination to prove A from this and (and C D) from this.



Conjunction Elimination Exercise Solution



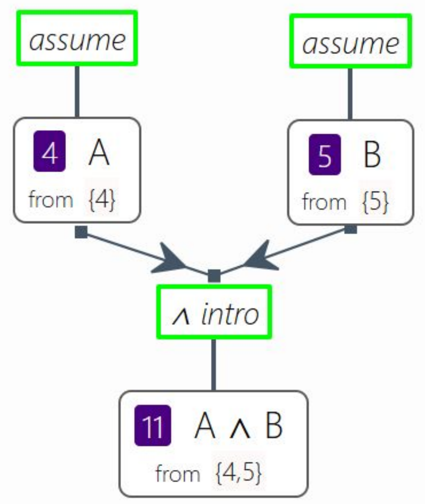
Conjunction Introduction

Premicies: "φ is provable from Γ"

"ψ is provable from Σ"

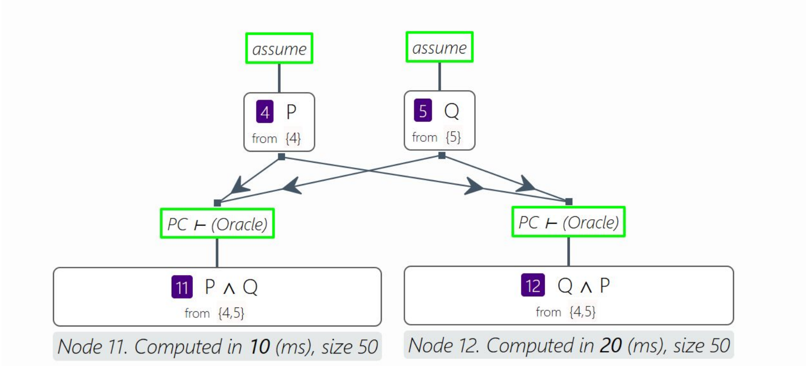
Conclusions: "φ and ψ is provable from Γ ∪ Σ"

$$\frac{\Gamma \vdash \psi \qquad \Sigma \vdash \varphi}{\Gamma \cup \Sigma \vdash \varphi \wedge \psi} \wedge I$$

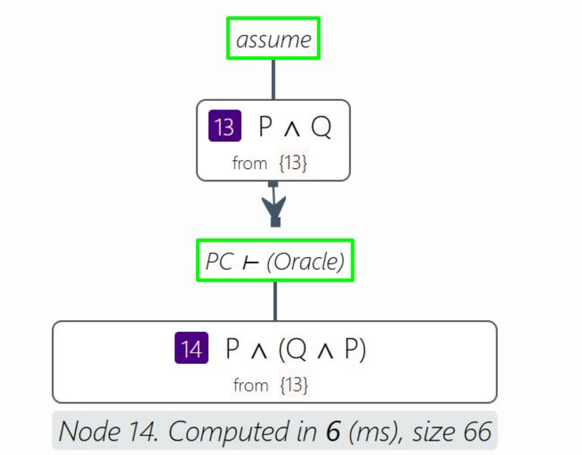


Conjunction Introduction Exercises

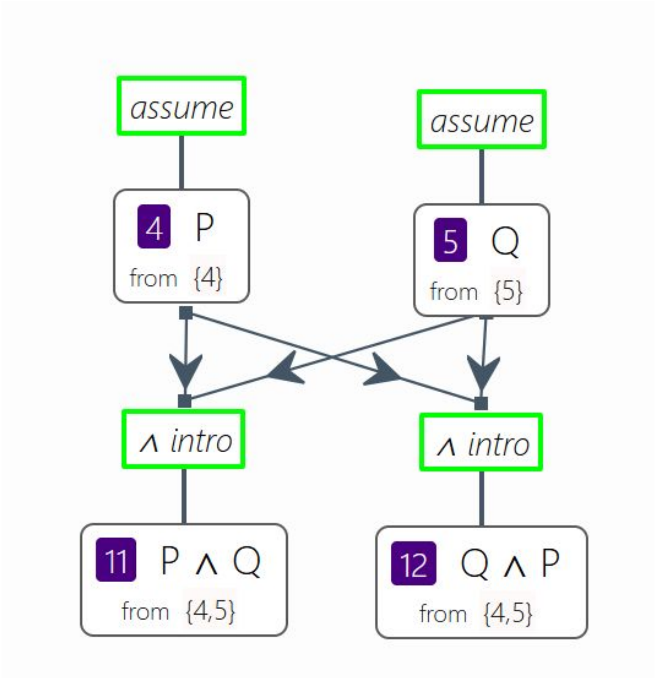
1) Assume P and Assume Q
   a) (and P Q)
   b) (and Q P)

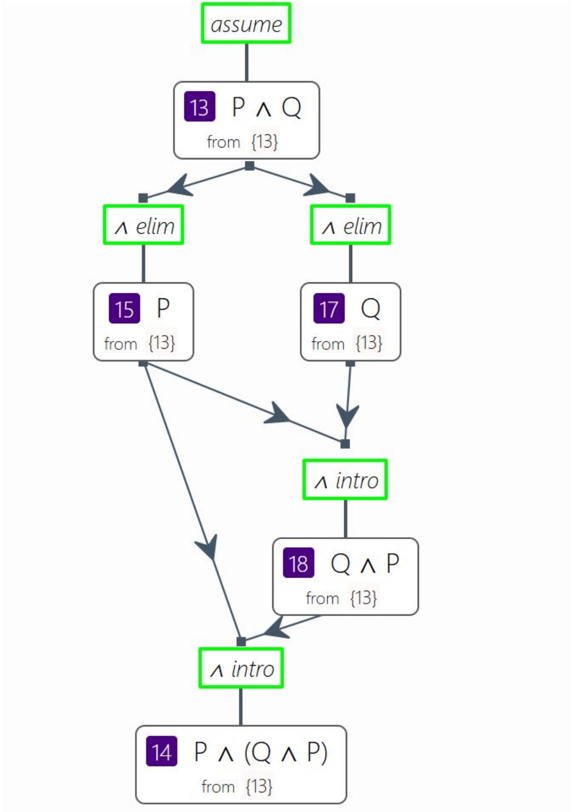


2) Assume (and P Q) prove (and P (and Q P))
   Hint: use a conjunction elimination as well



Conjunction Introduction Exercise Solutions



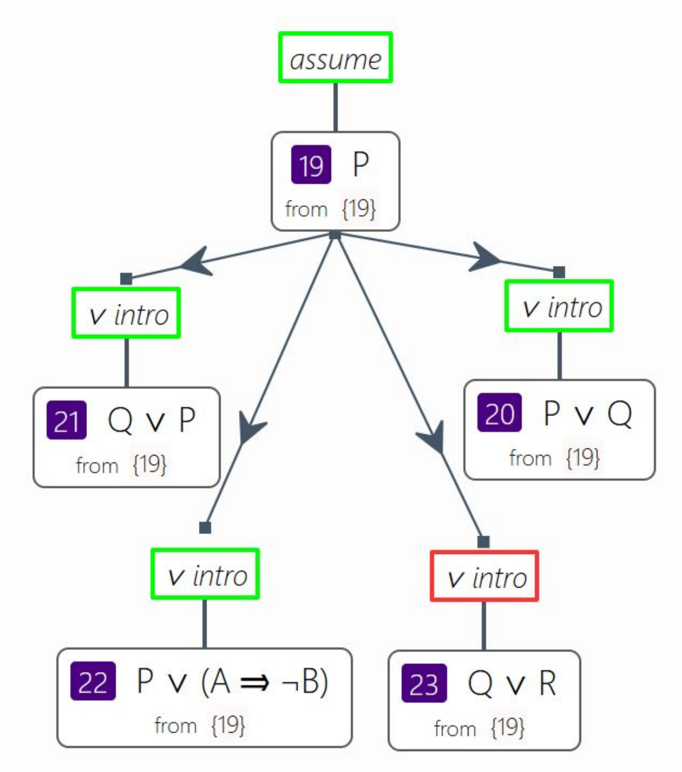


Inference Rule(s): Disjunction Introduction

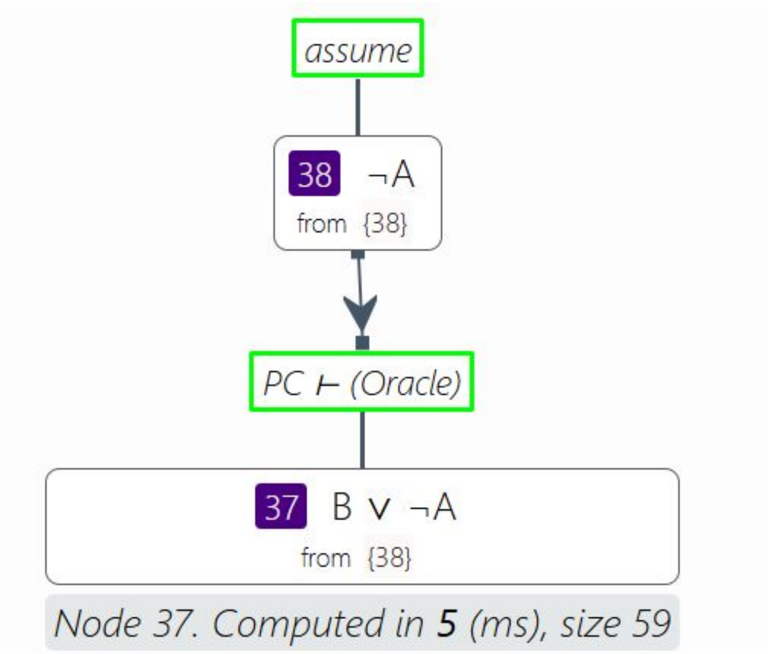
Premicie: “φ is provable from Γ”

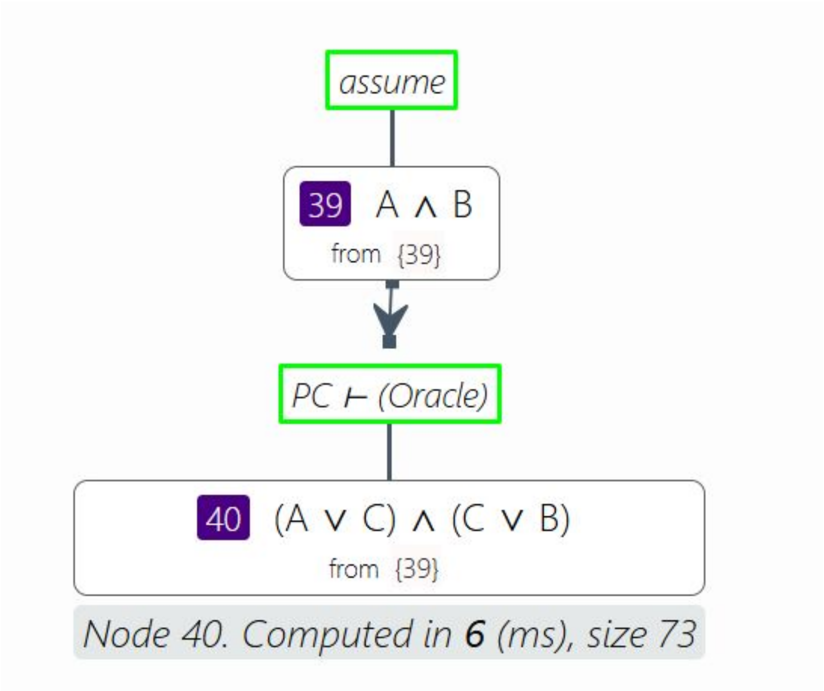
Conclusions: “φ or ψ is provable from Γ”
“ψ or φ is provable from Γ”

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \psi \vee \varphi}\ \vee I_l \qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ \vee I_r$$

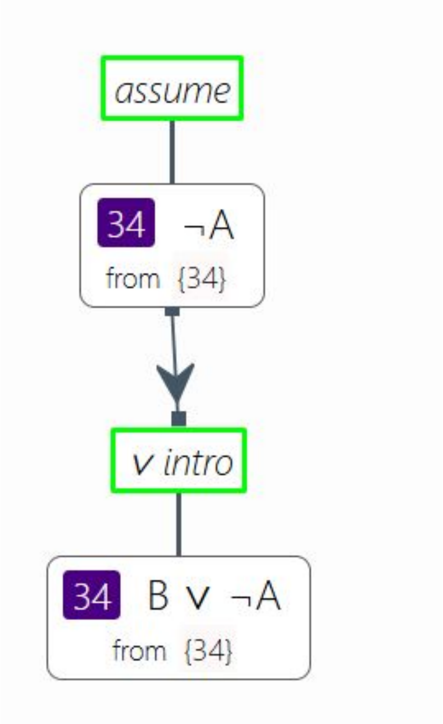


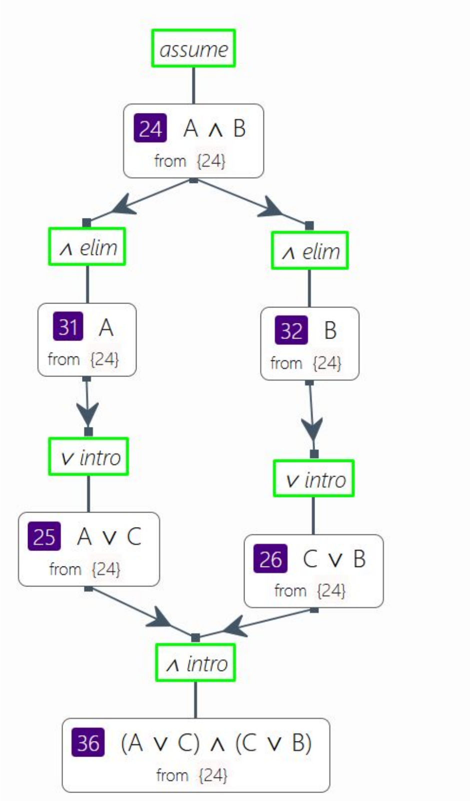
Disjunction Introduction Exercises





Disjunction Introduction Exercise Solutions





Inference Rule Disjunction Elimination

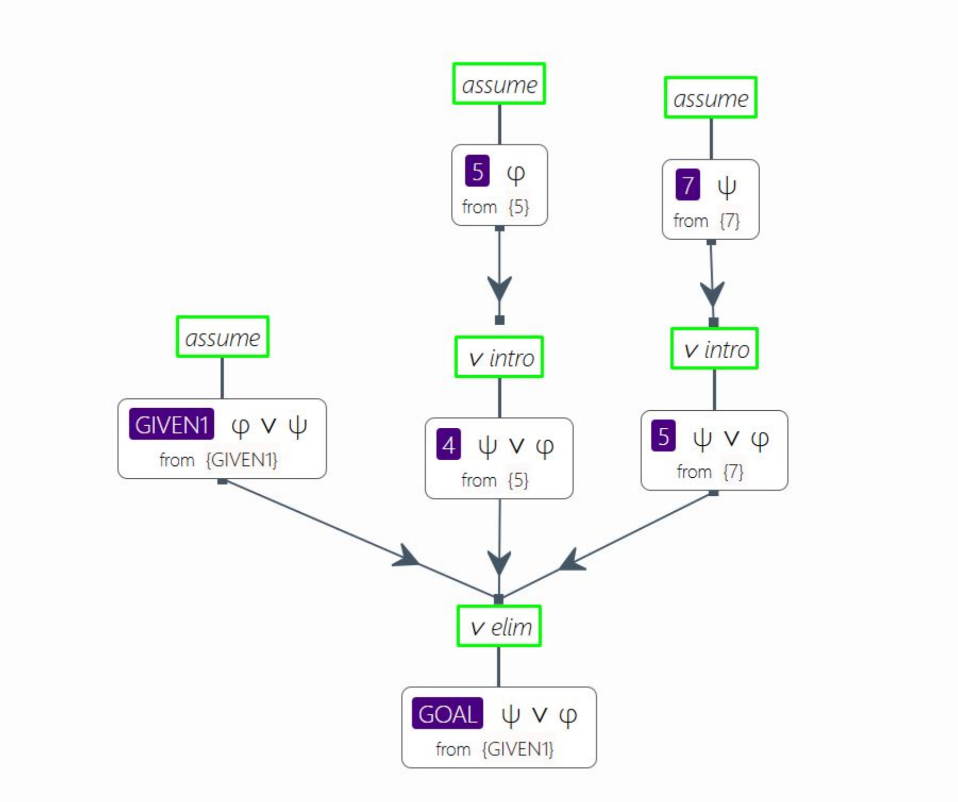
Premicies: “χ is provable from Γ and φ”
“χ is provable from Σ and ψ”
“φ or ψ is provable from Δ”

Conclusions: “χ is provable from Γ ∪ Σ ∪ Δ”

While this rule looks very scary it encodes the simple idea that if something follows from A and it follows from B separately, then it follows from A or B.

$$\frac{\Delta \vdash \psi \vee \varphi \quad \Gamma \cup \{\psi\} \vdash \chi \quad \Sigma \cup \{\varphi\} \vdash \chi}{\Delta \cup \Gamma \cup \Sigma \vdash \chi} \vee E$$

Disjunction Elimination Example: Switching Disjunct Order



More complex examples require other inference rules, we will return to use it more in later examples.

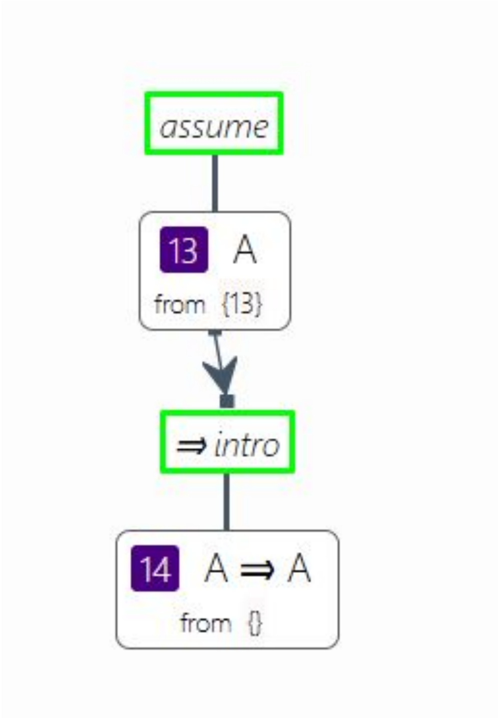
Implication Introduction

Premicies: “ψ is provable from Γ and φ”

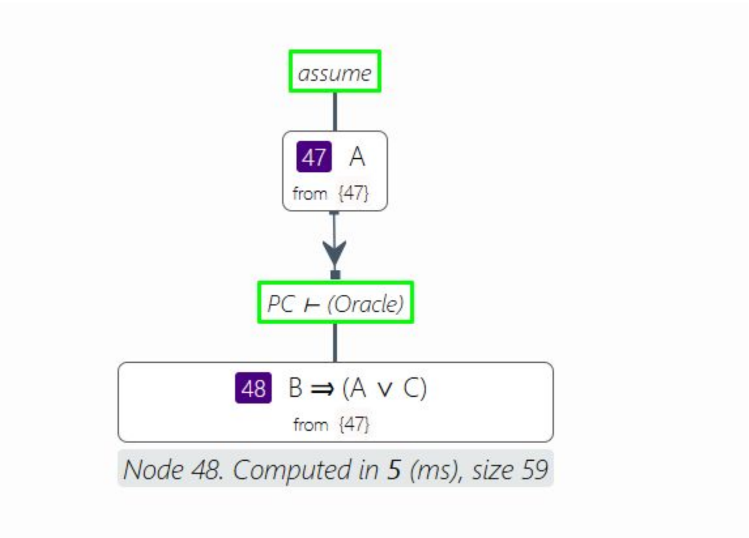
Conclusions: “φ implies ψ is provable from Γ”

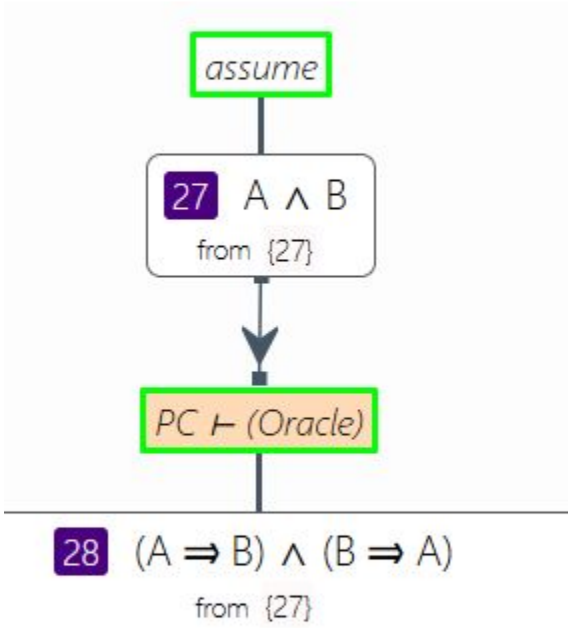
Classic Example of an assumption discharging rule: φ is taken out of the set of assumptions and moved into the formula.

$$\frac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} \rightarrow I$$

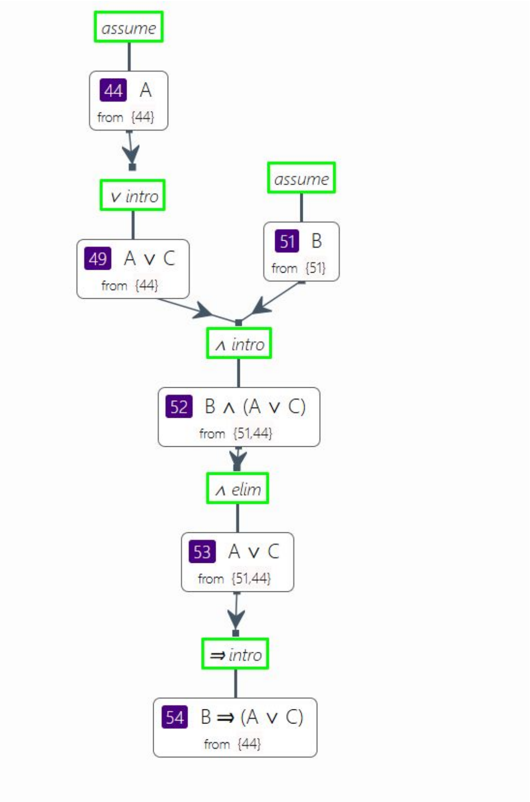


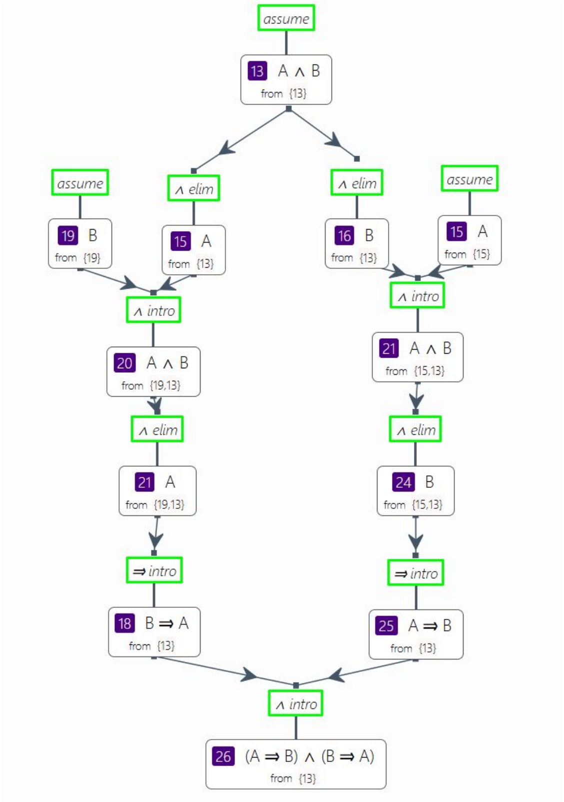
Implication Introduction Exercises





Implication Introduction Solutions





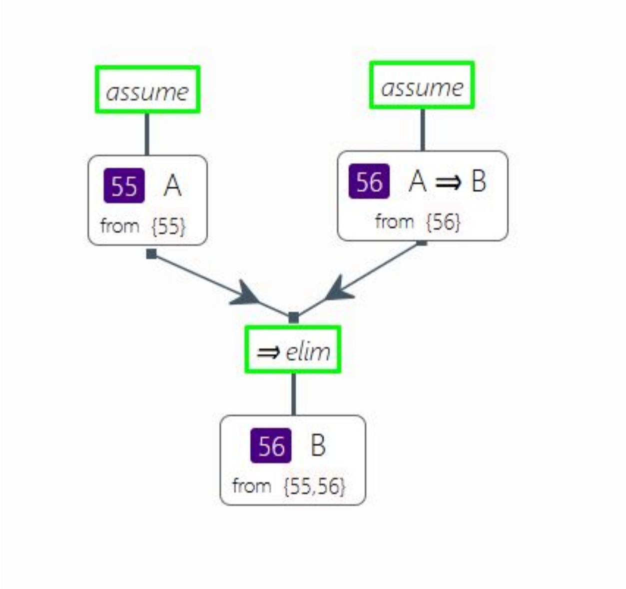
Implication Elimination (Modus Ponens)

Premicies: “φ is provable from Γ”
“φ implies ψ is provable from Σ”

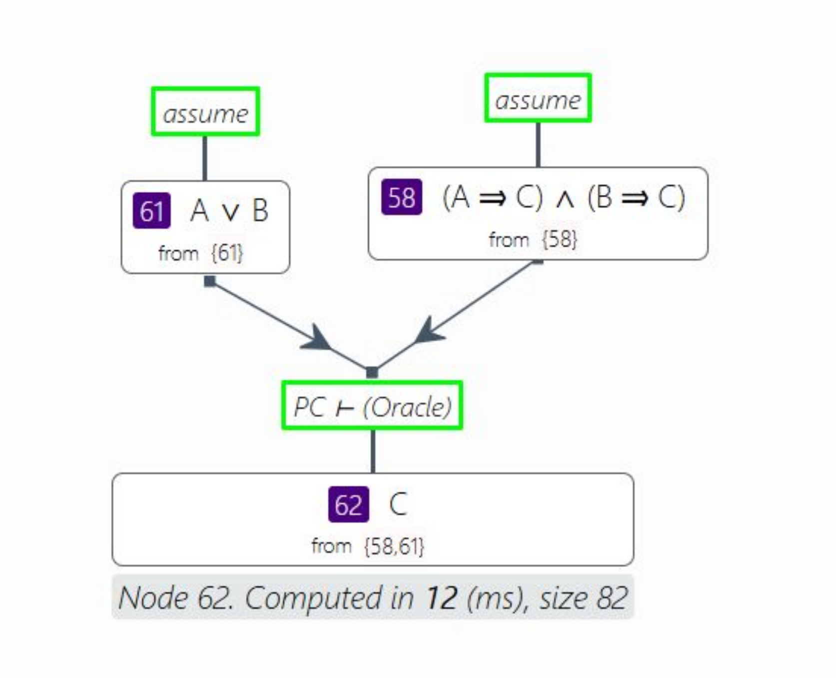
Conclusions: “ψ is provable from Γ and Σ”

Encodes the idea that:
“Given A and A implies B, conclude B.”

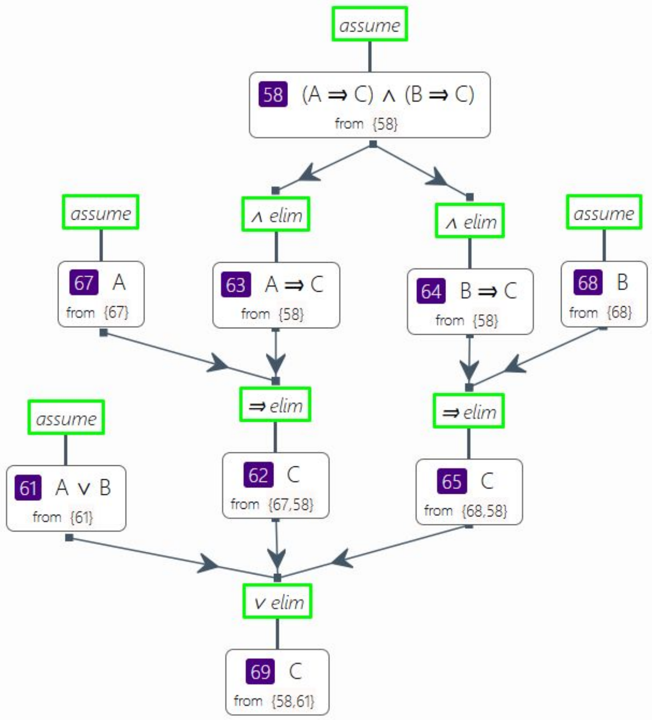
$$\frac{\Gamma \vdash \varphi \quad \Sigma \vdash \varphi \to \psi}{\Gamma \cup \Sigma \vdash \psi} \to E$$



Implication Elimination Exercise



Implication Elimination Exercise Solution



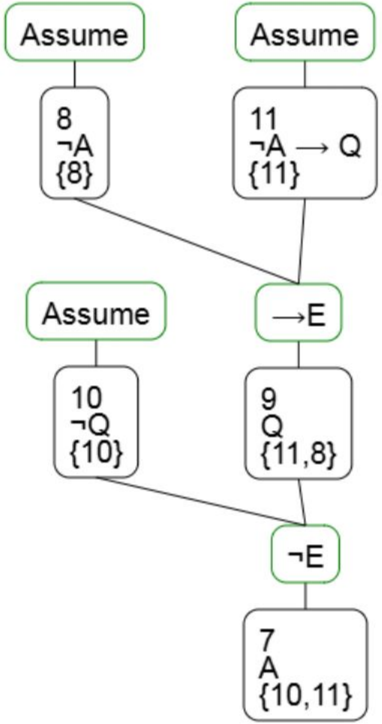
Negation Introduction

Premicies: “ψ is provable from Γ and φ”
“not ψ is provable from Σ”

Conclusions: “not φ is provable from Γ and Σ”

Encodes the idea that:
“If φ leads to a contradiction, φ must be false”

$$\frac{\Gamma \cup \{\varphi\} \vdash \psi \qquad \Sigma \vdash \neg\psi}{\Gamma \cup \Sigma \vdash \neg\varphi}\ \neg I$$



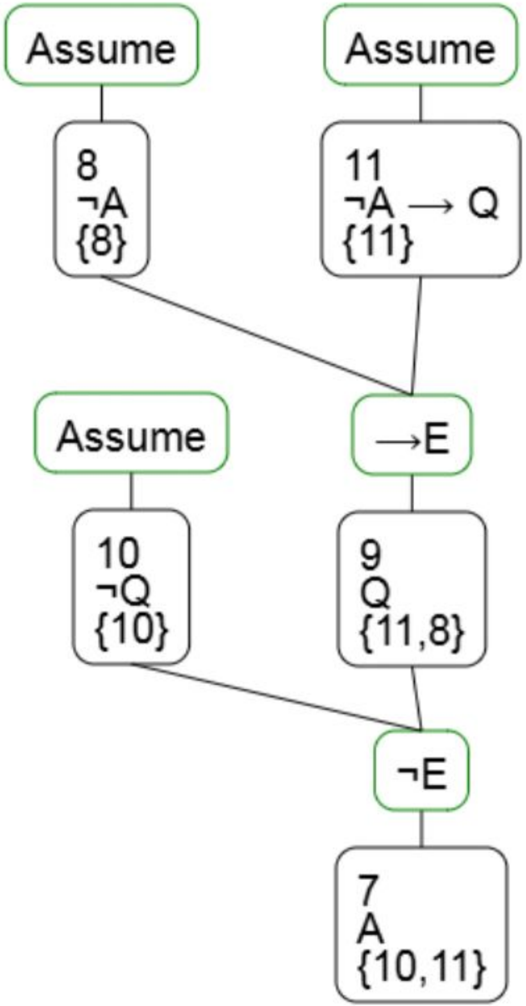
Negation Elimination

Premicies: “ψ is provable from Γ and not φ”
“not ψ is provable from Σ”

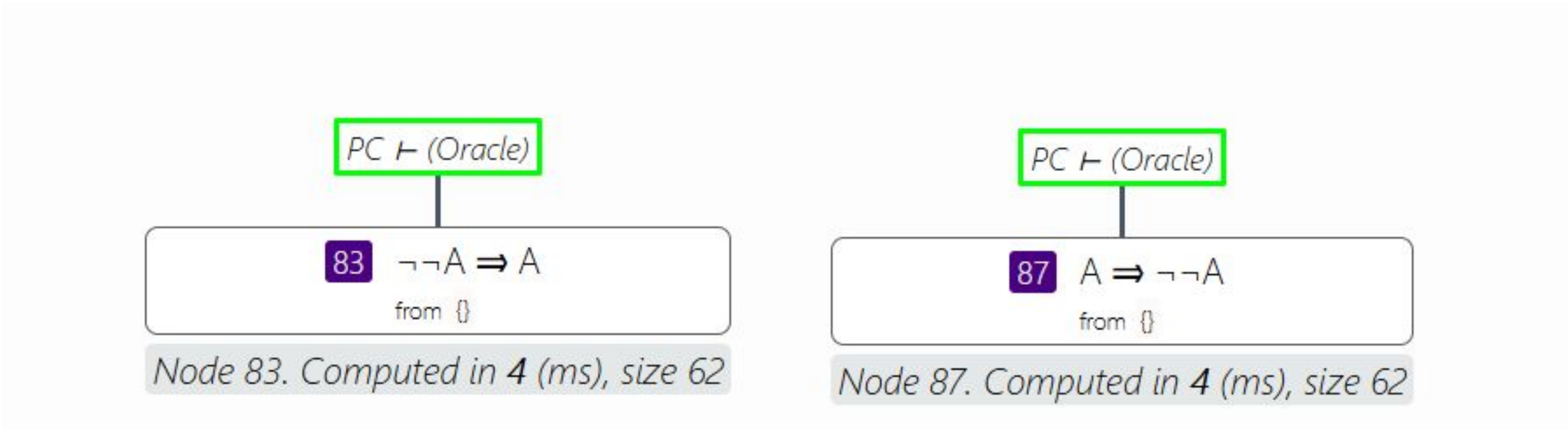
Conclusions: “φ is provable from Γ and Σ”

Encodes the idea that:
“If not φ leads to a contradiction, φ must be true”

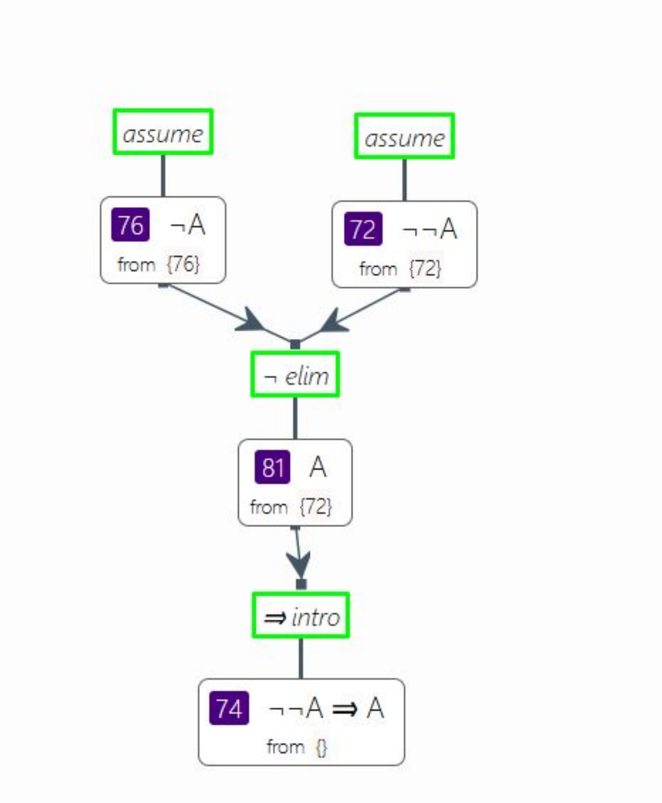
$$\frac{\Gamma \cup \{\neg\varphi\} \vdash \psi \quad \Sigma \vdash \neg\psi}{\Gamma \cup \Sigma \vdash \varphi}\ \neg E$$

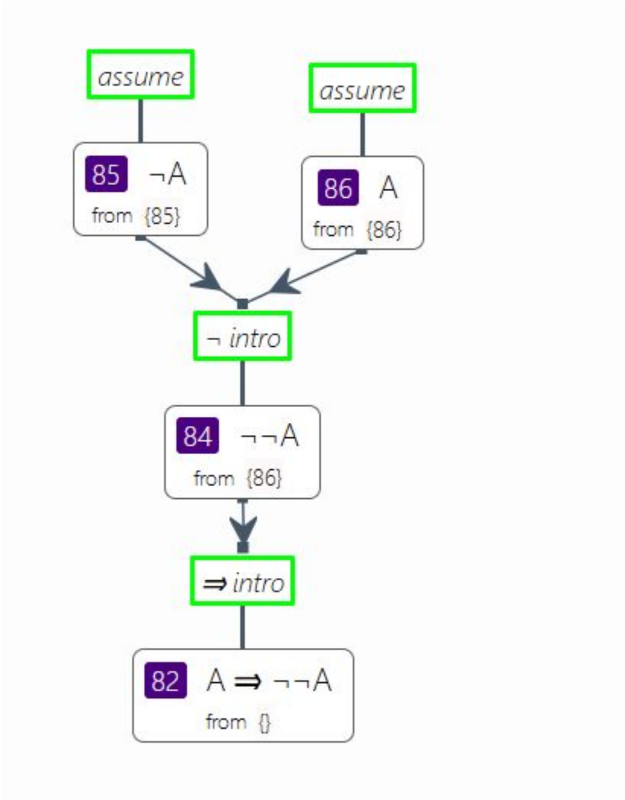


Negation Introduction and Elimination Exercises



Negation Introduction and Elimination Exercises

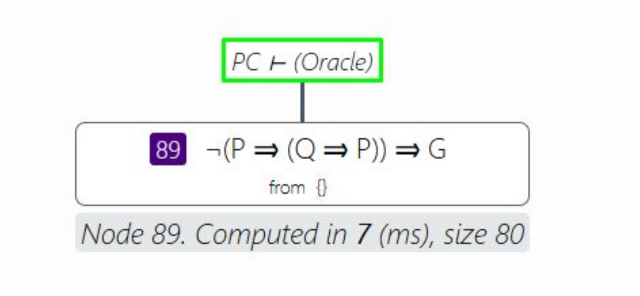




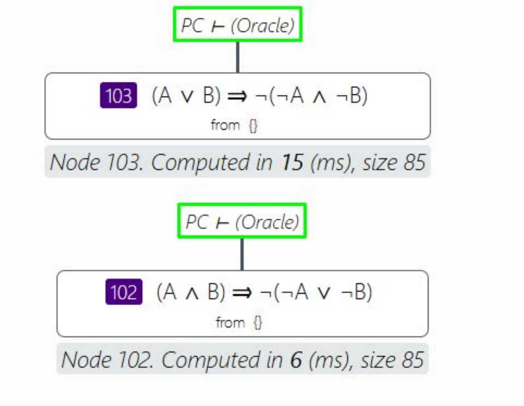
For the Rest of Class:

If you have never used Hyperslate, work on these problems:

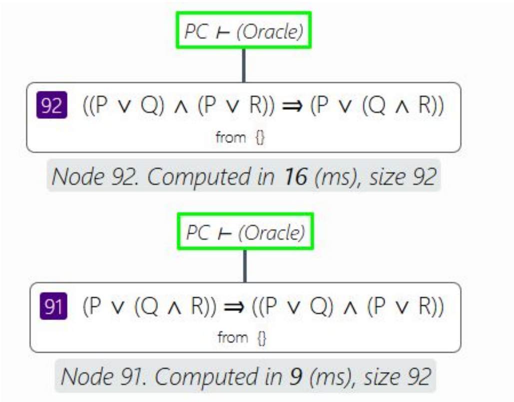
Green Cheese Moon



Subset of De Morgan's laws



Distribution of Or over And



If you are a Hyperslate master and the above look easy to you:

Go back to previous problems (http://bit.ly/3MKU0cr), can ChatGPT provide correct natural deduction proofs for them? Can Bard? In your prompt feel free to include the list of rules in our natural deduction system in the ReadMe of lazyslate (https://github.com/RAIRLab/lazyslate). Email me your results, oswalj@rpi.edu