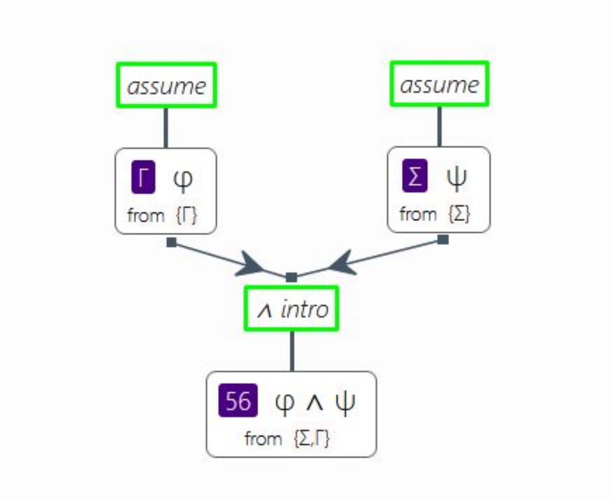
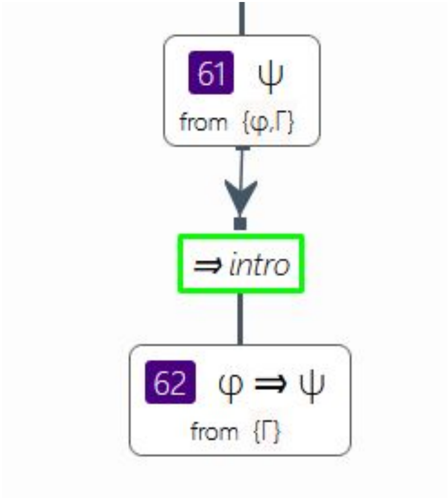
Hyperslate: Modal Logic

James Oswald

A Quick Reminder on ND Sequent Notation



$$\frac{\Gamma \vdash \psi \quad \Sigma \vdash \phi}{\Gamma \cup \Sigma \vdash \phi \wedge \psi} \wedge I$$



$$\frac{\Gamma \cup \{\phi\} \vdash \psi}{\Gamma \vdash \phi \to \psi} \to I$$

All Propositional Hyperslate ND Sequents

$$\frac{}{\{\phi\} \vdash \phi}\ \mathrm{A} \qquad \frac{\Gamma \vdash \psi \quad \Sigma \vdash \phi}{\Gamma \cup \Sigma \vdash \phi \wedge \psi}\ \wedge I \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}\ \wedge E_l \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}\ \wedge E_r$$

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \psi \vee \phi}\ \vee I_l \qquad \frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi}\ \vee I_r \qquad \frac{\Delta \vdash \psi \vee \phi \quad \Gamma \cup \{\psi\} \vdash \chi \quad \Sigma \cup \{\phi\} \vdash \chi}{\Delta \cup \Gamma \cup \Sigma \vdash \chi}\ \vee E$$

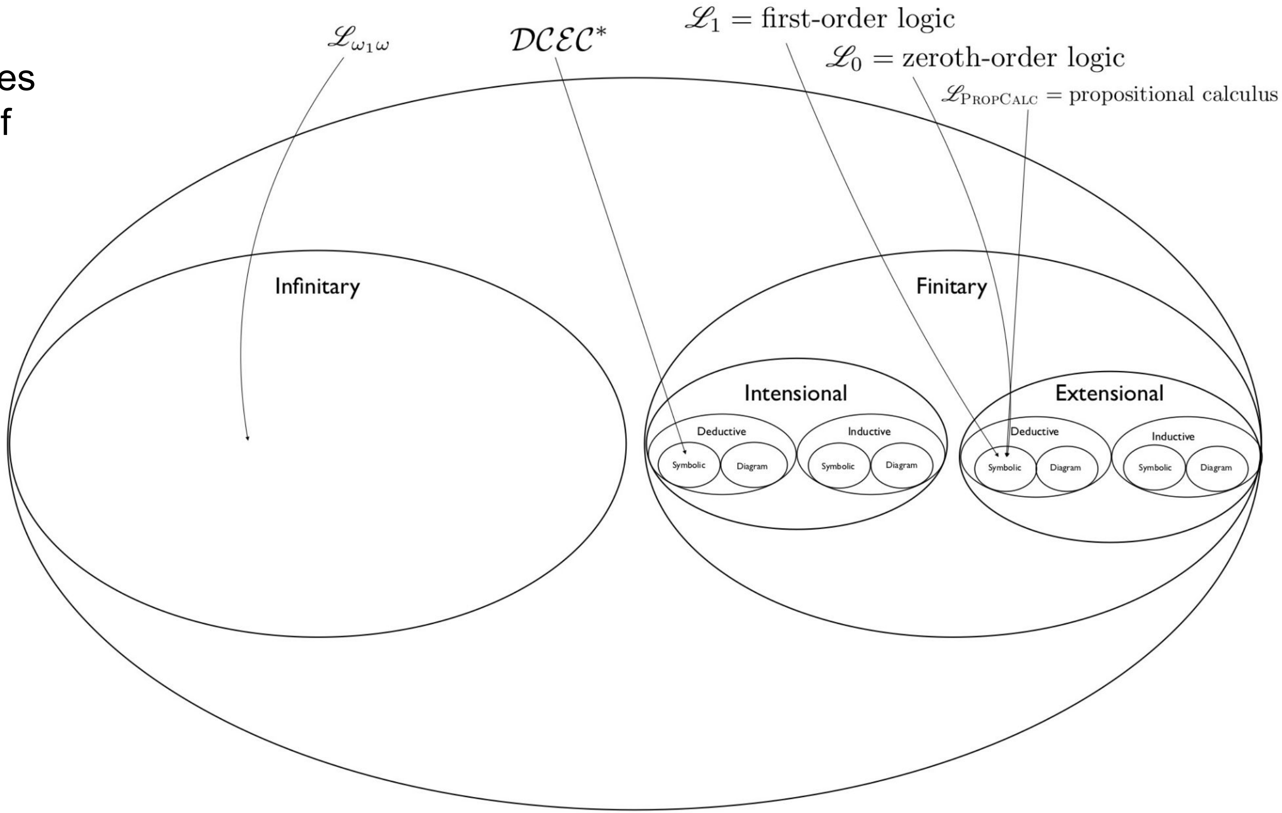
$$\frac{\Gamma \cup \{\phi\} \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \rightarrow I \qquad \frac{\Gamma \vdash \phi \quad \Sigma \vdash \phi \rightarrow \psi}{\Gamma \cup \Sigma \vdash \psi} \rightarrow E$$

$$\frac{\Gamma \cup \{\phi\} \vdash \psi \quad \Sigma \vdash \neg\psi}{\Gamma \cup \Sigma \vdash \neg\phi}\ \neg I \qquad \frac{\Gamma \cup \{\neg\phi\} \vdash \psi \quad \Sigma \vdash \neg\psi}{\Gamma \cup \Sigma \vdash \phi}\ \neg E$$

$$\frac{\Gamma \cup \{\phi\} \vdash \psi \quad \Sigma \cup \{\psi\} \vdash \phi}{\Gamma \cup \Sigma \vdash \phi \leftrightarrow \psi} \leftrightarrow I \qquad \frac{\Gamma \vdash \phi \quad \Sigma \vdash \phi \leftrightarrow \psi}{\Gamma \cup \Sigma \vdash \psi} \leftrightarrow E_l \qquad \frac{\Gamma \vdash \psi \quad \Sigma \vdash \phi \leftrightarrow \psi}{\Gamma \cup \Sigma \vdash \phi} \leftrightarrow E_r$$

The Universe of Logics

Where are we today? Where does modal logic fall in the universe of logics?



A Linguistic Perspective: Modals

Modal logic provides a family of logics for reasoning about the truth of propositions under *linguistic modals.*

Linguistic modals are words that modify the truth value of propositional sentences in a way that does not fully truth-functionally depend on the truth of the underlying sentence.

- "Is is raining"
- "It is possible that it is raining" or "It is possibly raining"
- "John believes it is raining" or
- "It is always raining"
- "It should be raining"

Representing Modal Formulae

To represent modals we will use a unary box □ and diamond ◊

For now we will read:

- □ as the modal "necessarily"
- ◊ as the modal "possibly"

HYPERSLATE TIP:
(nec F) for box □F
(pos F) for diamond ◊F

"$\Box\varphi$" reads "It is necessary that φ"

"$\Box(\varphi \wedge \psi)$" reads "It is necessary that φ and ψ"

"$\Diamond \varphi \Rightarrow \neg\Box\psi$" reads
"If φ is possible then ψ is not necessary"

"$\Diamond\Box\neg\varphi$" reads "It is possibly necessary that not φ"

"$\neg\Diamond\neg\varphi$" reads "It is not possible that not φ"

"$\neg\Box\neg\varphi$" reads "It is not necessary that not φ"

Modal Duals

Many modals have natural **<u>modal duals</u>** that correspond linguistically to the negation of the modal of the negated sentence. Some examples:

It is not possible that it is not raining = It is necessary that it is raining

It is not necessarily not raining = It possibly raining

It will not eventually not be raining = It will always be raining

It will not always not be raining = It will eventually be raining

Modal Duals Symbolically

Modal Duals give us our first fundamental way box and diamond are related.

Any box can be rewritten as not diamond not

Any diamond can be rewritten as not box not

$$\Diamond\psi = \neg\Box\neg\psi$$

$$\Box\psi = \neg\Diamond\neg\psi$$

Our first modal corollary

If we have these replacement rules
and we have double negation elimination…

$$\Diamond\psi = \neg\Box\neg\psi$$
$$\Box\psi = \neg\Diamond\neg\psi$$

We can derive the following simple rewrites:

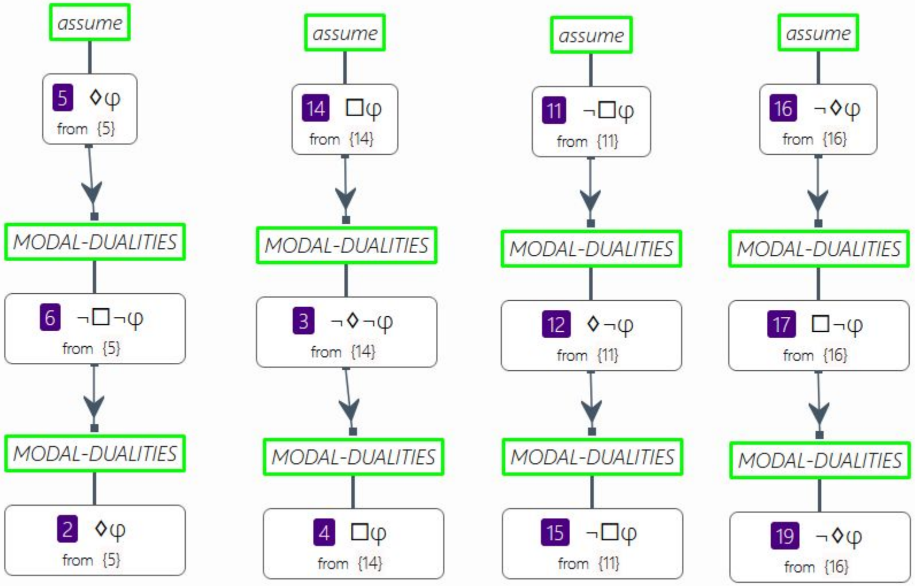
$\neg\Diamond\varphi = \neg\neg\Box\neg\varphi = \Box\neg\varphi$ i.e “It is not possible that φ” = “it is necessary that not φ”

$\neg\Box\varphi = \neg\neg\Diamond\neg\varphi = \Diamond\neg\varphi$ i.e “It is not necessary that φ” = “it is possible that not φ”

$$\neg\Diamond\psi = \Box\neg\psi$$
$$\neg\Box\psi = \Diamond\neg\psi$$

Your First Modal Hyperslate Rule

The MODAL-DUALITIES rule allows you to rewrite a top level modal formula or negated modal formula in terms of its dual.



$$\Diamond\psi = \neg\Box\neg\psi$$
$$\Box\psi = \neg\Diamond\neg\psi$$
$$\neg\Diamond\psi = \Box\neg\psi$$
$$\neg\Box\psi = \Diamond\neg\psi$$

Exercise

Create a new hyperslate file with the logic “K”, name it “1023lastnamefirstname1”

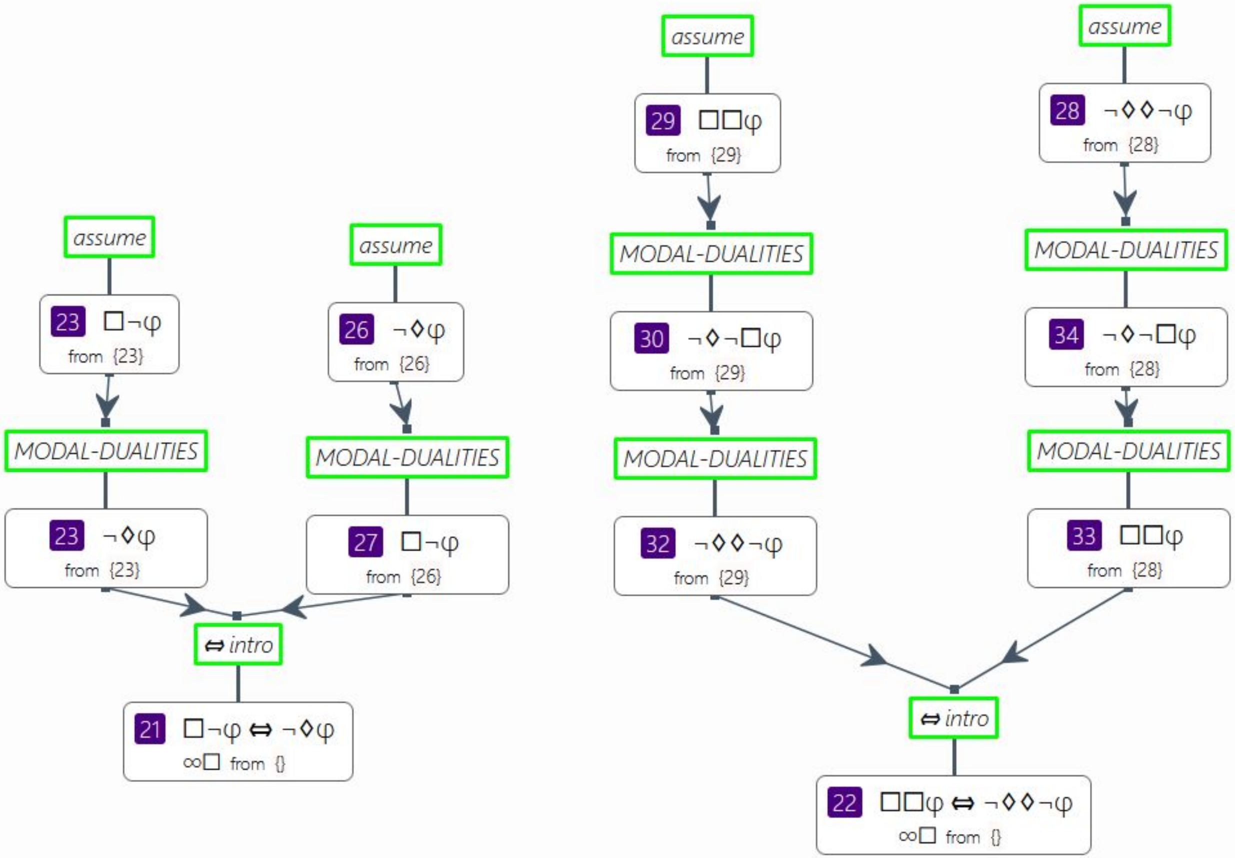
Using ONLY MODAL-DUALITIES and propositional logic rules, prove the following without the oracle:

$$\Diamond\psi = \neg\Box\neg\psi$$

$$\Box\psi = \neg\Diamond\neg\psi$$

$$\neg\Diamond\psi = \Box\neg\psi$$

$$\neg\Box\psi = \Diamond\neg\psi$$



$$\Diamond\psi = \neg\Box\neg\psi$$
$$\Box\psi = \neg\Diamond\neg\psi$$
$$\neg\Diamond\psi = \Box\neg\psi$$
$$\neg\Box\psi = \Diamond\neg\psi$$

Semantics of Necessity

Modal logic was originally designed to deal with reasoning about necessary and contingent truths. To understand more rules about □, we need to understand what it means for something to necessarily true.

Necessary vs. Contingent Truths

- **Necessary truths** are true in all possible worlds (e.g., "2 + 2 = 4")

  "Things could not have been any other way" (logical truths, mathematical truths)

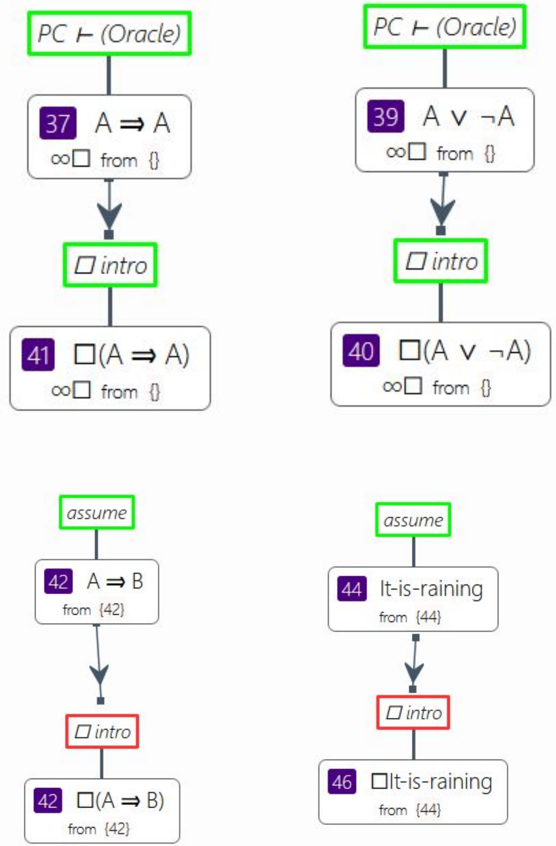
- **Contingent truths** are true in the actual world but could have been false (e.g., "It is raining today").

  "Things are this way, but could have not been this way"

Box Introduction, Necessitation

If something is a theorem of propositional logic (a tautology, from {}) then it is necessary.

$$\frac{\vdash \phi}{\vdash \Box\phi}\ \Box I_{theorem}$$



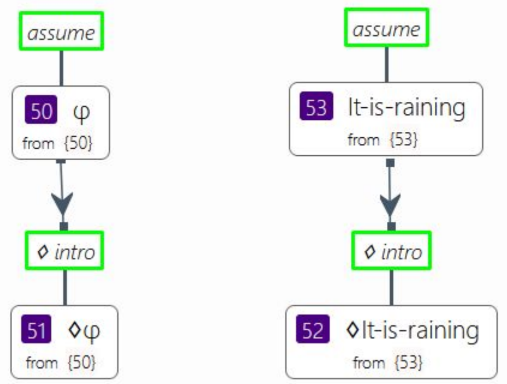
The $\infty\Box$ symbol denotes that you can introduce an infinite number of boxes in front of the formula on this node with box intro.

Possibility (Diamond) Introduction

If something is true under some assumptions (Gamma), it is possible under those assumptions.

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \Diamond\phi}\ \Diamond I$$

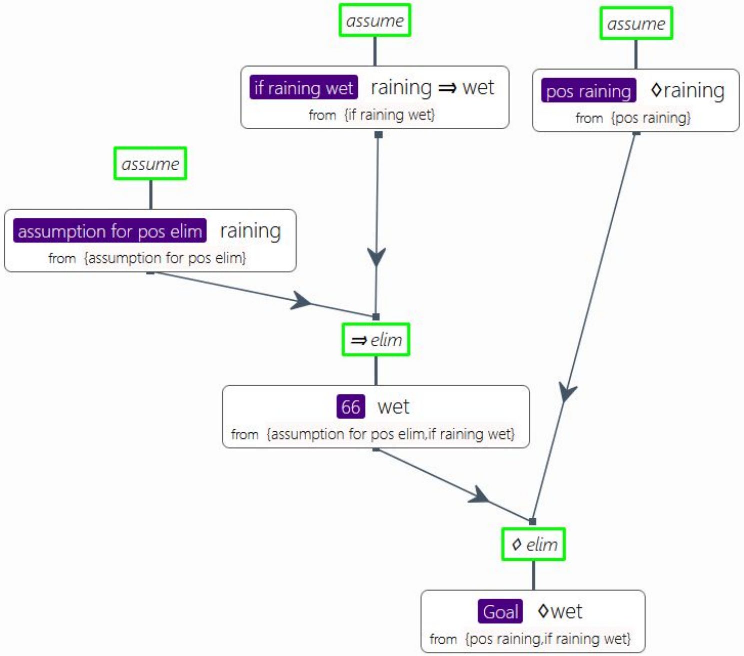
This is the simplest rule, it just says you can put a diamond in front of anything.



Diamond Elimination

$$\frac{\Gamma \cup \{\phi\} \vdash \psi \quad \Delta \vdash \Diamond\phi}{\Gamma \cup \Delta \vdash \Diamond\psi} \Diamond E$$

Given
1) A proof of psi from phi (and assumptions Gamma)
2) The possibility of psi (from assumptions Delta),
We can derive that psi is possible (Assuming Gamma and Delta).



Contrary to the name, this does not allow you to eliminate diamonds, it allows you to reason about formulae inside diamonds, but you need to put the diamond back when you finish.

Systems of Modal Logic

So far we have looked at properties of box and diamond that are considered to be universal. However, we may want finer grained control over what box really means, for example, are the following statements true or false?

If it is necessarily raining, it is raining. $\Box A \to A$

If it is necessarily raining, it is possibly raining. $\Box A \to \Diamond A$

If it is necessarily raining, it is necessary that it is necessarily raining. $\Box A \to \Box\Box A$

Answer: It Depends!

So far none of the rules we have shown allow us to prove

$$\Box A \to \Diamond A \qquad \Box A \to A \qquad \Box A \to \Box\Box A$$

Investigating questions like this has lead to the creation of the standard family of modal logics, in which we take as axiom that various statements like these are true. In Hyperslate, we add new rules of inference such that these can be proven.

The Standard Subsumption Cube

K: $\Box(A \to B) \to (\Box A \to \Box B)$

T: $\Box A \to A$

D: $\Box A \to \Diamond A$

5: $\Diamond A \to \Box\Diamond A;$

4: $\Box A \to \Box\Box A$

B: $A \to \Box\Diamond A.$

5 is equivalent to having both 4 and B.

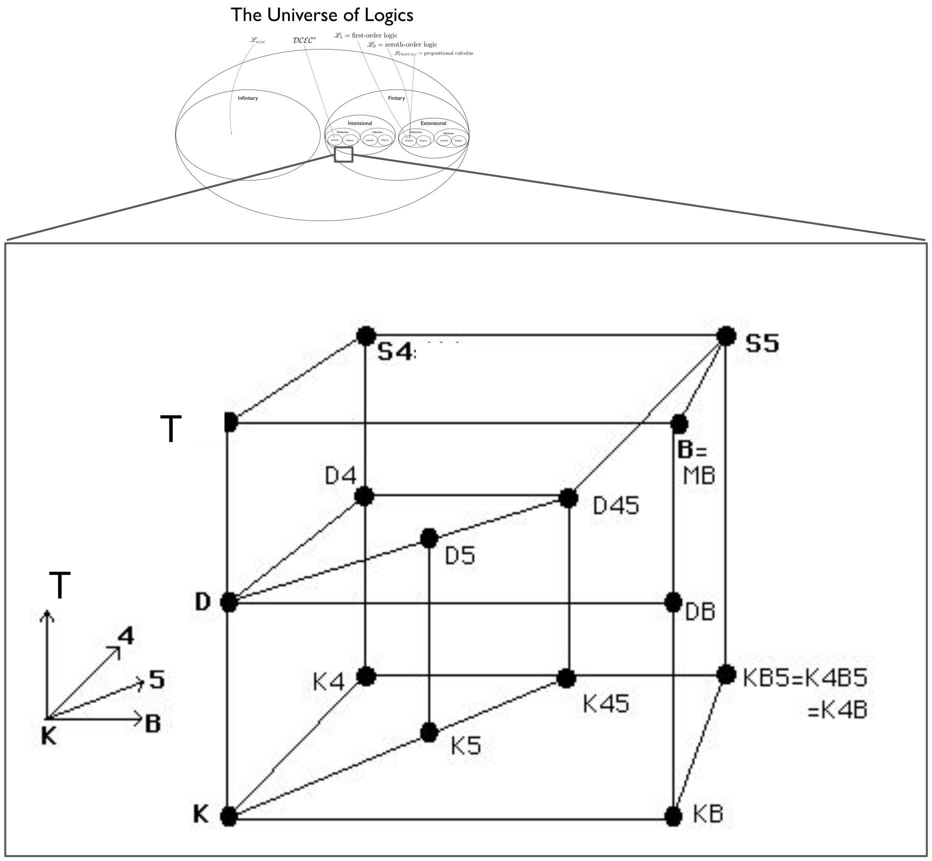
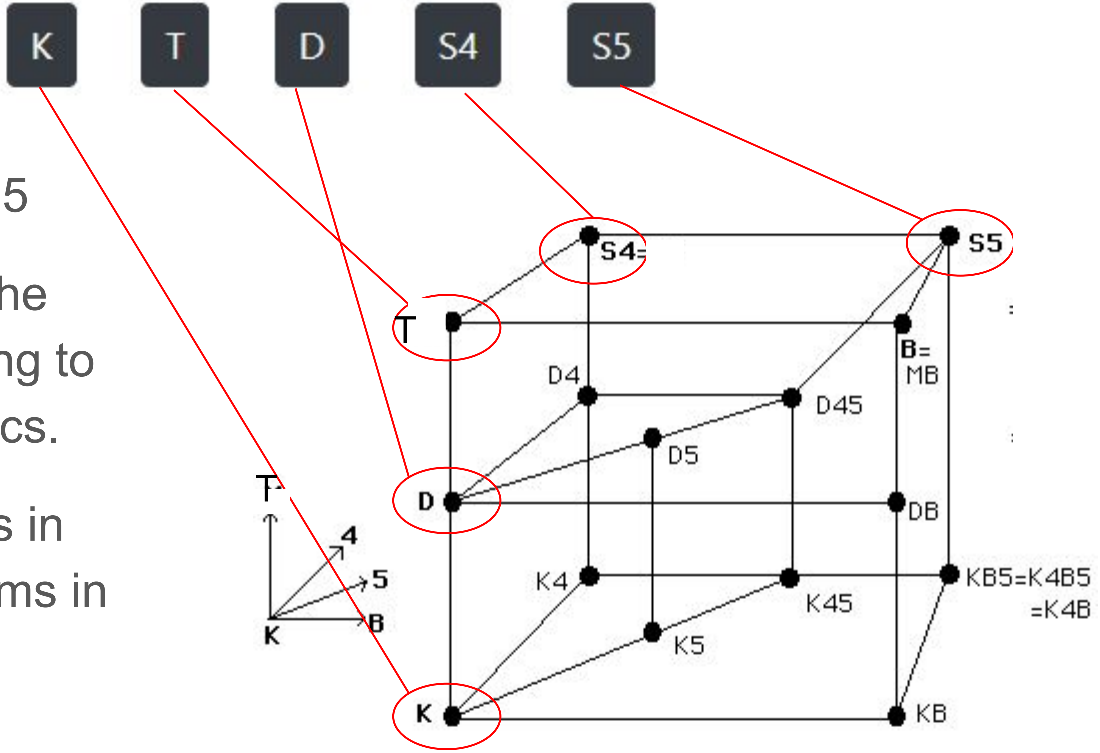


Figure From [Modal Logic (Stanford Encyclopedia of Philosophy)]

What do we have in Hyperslate?



Hyperslate offers K, D, T, S4, S5

Trace a path around edges of the modal subsumption cube leading to increasingly more powerful logics.

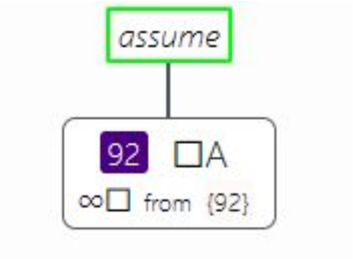
All Theorems in K are theorems in D, all theorems in D are theorems in T, and so on.

Extra Box Introduction Rules

Added In S4

$$\frac{\Gamma \vdash \Box\phi}{\Gamma \vdash \Box\Box\phi}\ \Box I_4$$

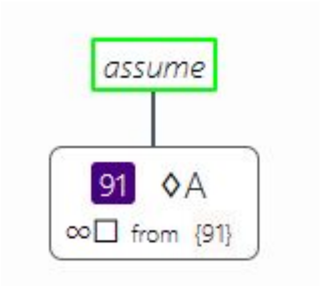
If something is necessary under some assumptions (Gamma), then it is necessarily necessary under those assumptions.



Added In S5

$$\frac{\Gamma \vdash \Diamond\phi}{\Gamma \vdash \Box\Diamond\phi}\ \Box I_5$$

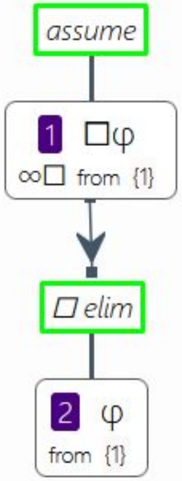
If something is possible under some assumptions (Gamma), then it is necessarily possible under those assumptions.



Box Elimination (In T, S4, and S5)

If something is necessary under assumptions (Gamma), it is true under those assumptions.

$$\frac{\Gamma \vdash \Box\phi}{\Gamma \vdash \phi}\ \Box E$$



For brevity we won't discuss box elimination in K or D, which exists, but is more complicated due to needing to numbers of sanctioned boxes. See 5.3 in the textbook for more.

Exercise: Prove the Axioms of S5

Create a new hyperslate file with the logic “S5”, name it “1023lastnamefirstname2”

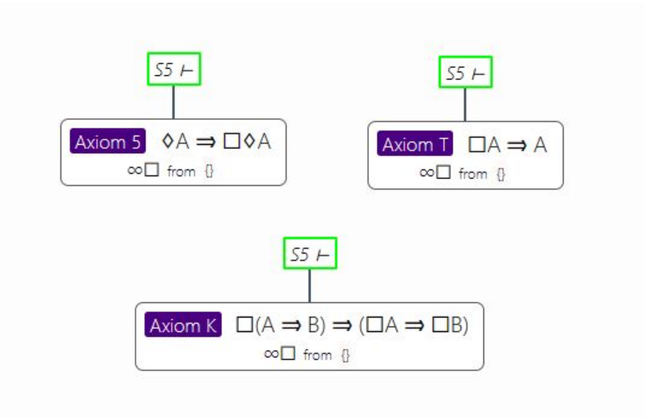
Using all inference rules, prove the following axioms hold:

K: $\Box(A \to B) \to (\Box A \to \Box B);$

T: $\Box A \to A,$

5: $\Diamond A \to \Box\Diamond A;$

Note that these should all be “from: {}”



Link to the slides:

Beyond Necessity and Possibility

Some Example Linguistic Modals

	Box	Diamond (Dual)
Alethic	“Necessary”	“Possible”
Epistemic (Knowledge)	“Knows”	(epistemic consistency)
Doxastic (Belief)	“Believes”	(epistemic consistency)
Deontic	“Obligatory” (Ought to)	“Permissible”
Temporal	“Always”	“Eventually”
Provability	“Provable”	“Consistent”

…

Readings Of K

$$\mathbf{K}\text{: } \Box(A \to B) \to (\Box A \to \Box B)$$

The K axiom merely states that box distributes over material implication.

Some readings in various modalities:

- Alethic: “If it is necessary that A implies B, then if A is necessary, B is necessary.”
- Epistemic: “If it is known that A implies B, then if A is known, B is known.”
- Deonic: “If it ought to be that A implies B, then if it ought to be that A, it ought to be that B.”
- Temporal: “If A always implies B, then if A is always true, B is always true.”
- Provability: “If it is provable that A implies B, then if A is provable, B is provable.”

Readings Of T

T: $\Box A \to A$

Some readings in various modalities for S5:

- Alethic: “If it is necessary that A then A is true.”
- Epistemic: “If A is known then A is true.” (JTB interpretation of knowledge)

Why can’t we use this for the following modalities?

- Deonic: “If it ought to be that A, then A.”
- Epistemic Belief. “If it is believed that A then A.”
- Temporal: “If always A, then A” (Future facing always operator, does not consider the past)
- Provability: “If it is provable that A then A” or “everything provable is true”. (Contradicts Godel's 2nd Incompleteness theorem, see: Many Dimensional Modal Logics: Theory and Applications, page 8)

Readings Of 5

5: $\Diamond A \to \Box\Diamond A$ $\qquad\qquad$ $\Diamond\Box A \to \Box A$

Some readings in various modalities for S5:

- Alethic: “If it is possible that A then it necessary that A is possible.”
- Epistemic: (Expand Diamond and read as)

  ”If it is not known that A is not known, then it is known that A.”

  In epistemic logic this is called the the negative introspection axiom.

Other Logics In the Modal Subsumption Cube: KD45

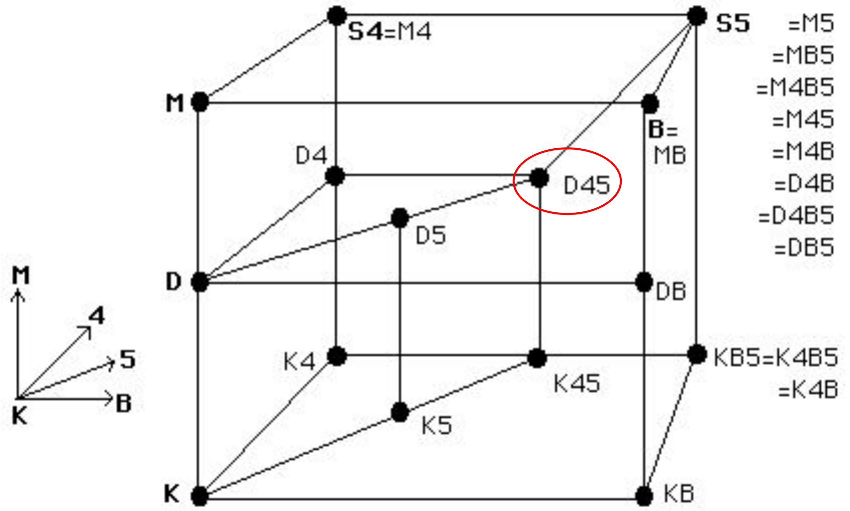
D45 (KD45) is the foundation of modal logic dealing with belief. (interpret the box as “believe”)

Why throw out T? **T**: $\Box A \to A$

“If it is believed that A then A.” NO!

D is still ok tho. $\overline{\Box A \to \Diamond A}$

“If it is believed that A then it is not believed that not A”

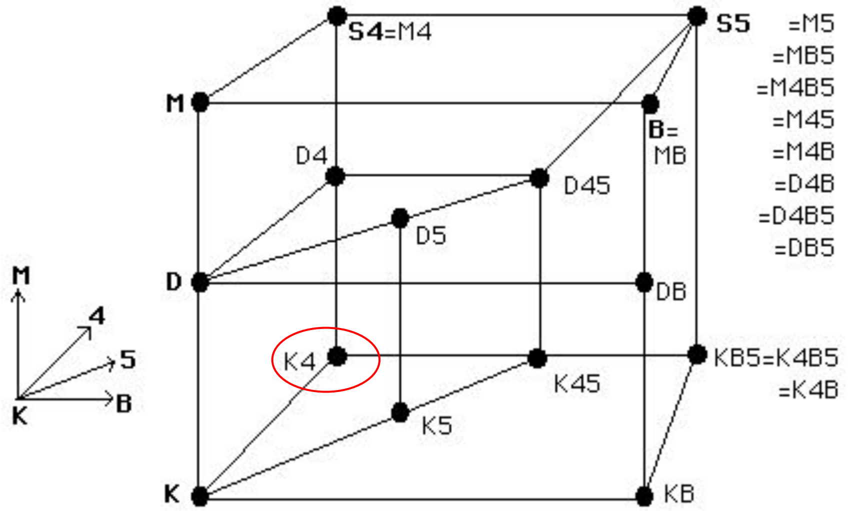


Other Logics In the Modal Subsumption Cube: K4

K4 is used as the base for many temporal logics. (where box is interpreted as "always in the future")

Only admit 4. $\mathbf{4}: \Box A \rightarrow \Box\Box A$

"If always A, then always always A"



An even larger universe…

The modal cube is based off basic axioms from 1934, but this cube is just a small portion of an even larger modal logic lattice.

GL (Godel-Lob) Logic is used for reasoning about provability. Where box is read as “It is provable that”

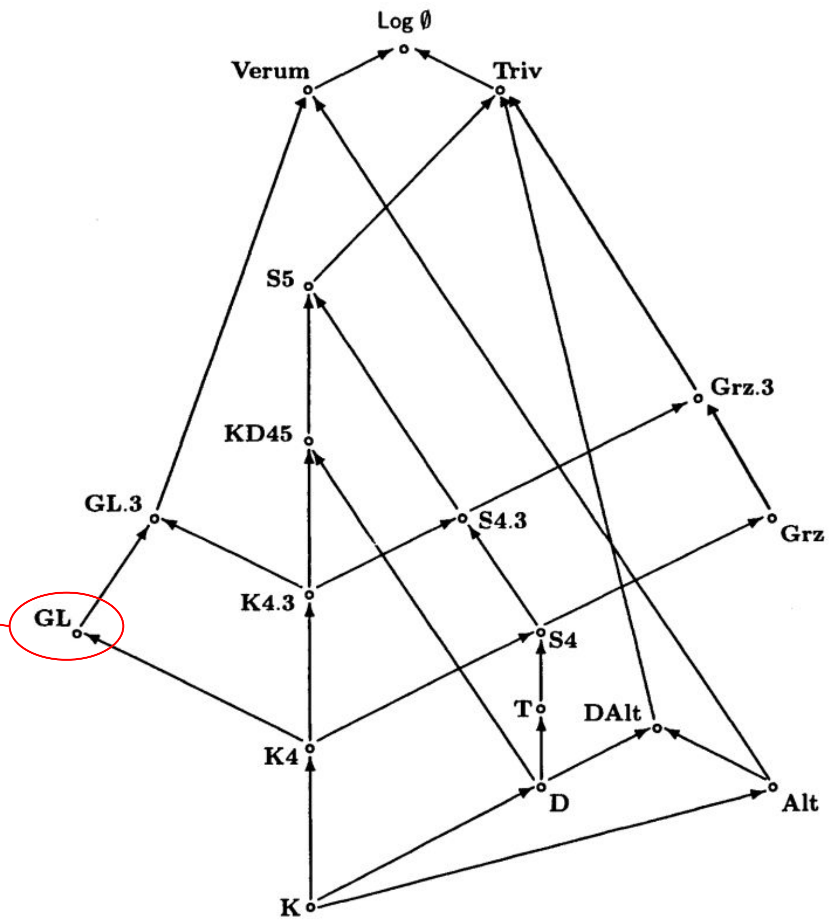


Figure From “Many Dimensional Modal Logics: Theory and Applications” (2003) Page 14.

Final Activity: Your Own Modality

Think up your own linguistic modal, (there are hundreds).

Recall a *linguistic modal* modifies The truth value of a propositional sentence in a non-truth functional way.

Does your modal have a natural dual? What axioms hold for it?

Email me your response at oswalj@rpi.edu

K: $\Box(A \to B) \to (\Box A \to \Box B)$

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D: $\Box A \to \Diamond A$

5: $\Diamond A \to \Box\Diamond A;$

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B: $A \to \Box\Diamond A.$

Link to the slides:

The Thirty-Ninth AAAI Conference on Artificial Intelligence (AAAI-25)

A Modal Logic of Optimality (Student Abstract)

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Abstract

We present our work on a new modal logic of optimality, $\mathcal{OPT}$, whose semantics are modeled in terms of optimal paths through reward-weighted transition systems. We prove some basic properties of $\mathcal{OPT}$, including its status as a *normal* modal logic, as well as its relation to some of the standard modal axioms. We end with a discussion of applications to AI and future research directions and extensions.

Introduction

The notion of optimality plays a key role in diverse areas of computer science, particularly in AI. Despite this, optimality has yet to be given a formal-logic treatment, even with its clear status as a linguistic modal: the statement "It is optimal that φ" has an intensional truth value. In our semantics we define a formula φ to be optimal with respect to a reward-weighted transition system and a state s iff on all optimal paths p from s (paths which maximize reward through a reward-weighted transition system), φ holds in all states on

are defined compositionally. Given a set of atomic propositions $\mathbf{At}$, we define formulae φ as follows, where $a \in \mathbf{At}$.

$$\varphi ::= a \mid \neg\varphi \mid \varphi \land \varphi \mid \Box\varphi$$

Statements of the form $\Box\varphi$ are read as "It is optimal that φ."

Semantics

$\mathcal{OPT}$ formulae are interpreted over reward-weighted transition systems $\mathcal{M} = (S, T, v, r)$ where:

- S is a finite[1] set of states (possible worlds);
- $T \subseteq S \times S$ is the transition (accessibility) relation;
- $v : \mathbf{At} \to \mathcal{P}(S)$ is a valuation function that assigns atomic propositions to the set of states they hold in; and
- $r : T \to \mathbb{R}$ is a reward function assigning transitions a value.

To define the notion of optimality, we need some additional machinery. First, the notion of a path through a transition system: A path $p : S^{\leq\omega} := (p_0, p_1, p_2, \ldots)$ is a fi-