Gödel's ...
First Incompleteness Theorem
(G1)

Selmer Bringsjord*

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ILBAI
12/1/2025

*With Naveen Sundar G's introduction to Gödel numbering, and his depiction of GT growth, etc.

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Note: This is a version designed for those who have had at least one robust, proof-intensive university-level course in formal logic to the level of $\mathscr{L}_2$.

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Background Context …

Gödelian Essays on Artificial vs. Human Intelligence: Pretenders vs. Persons

by Selmer Bringsjord • Naveen Sundar Govindarajulu • Alexander Bringsjord

- Introduction ("The Wager")
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
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- Gödel's "God Theorem"
- Could a Finite Machine Match Gödel's Greatness?

STOP & REVIEW IF NEEDED!

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A corollary of the First Incompleteness Theorem: *We cannot prove (in classical mathematics) that mathematics is consistent.*

Deficient; Beware

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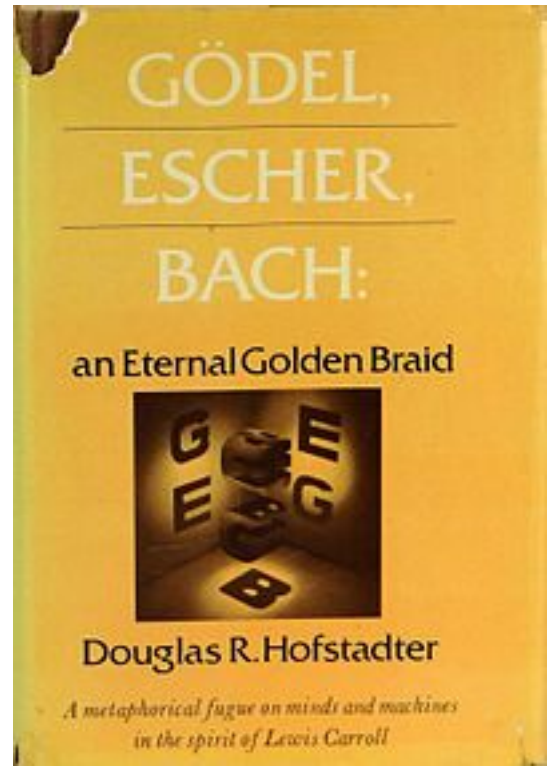

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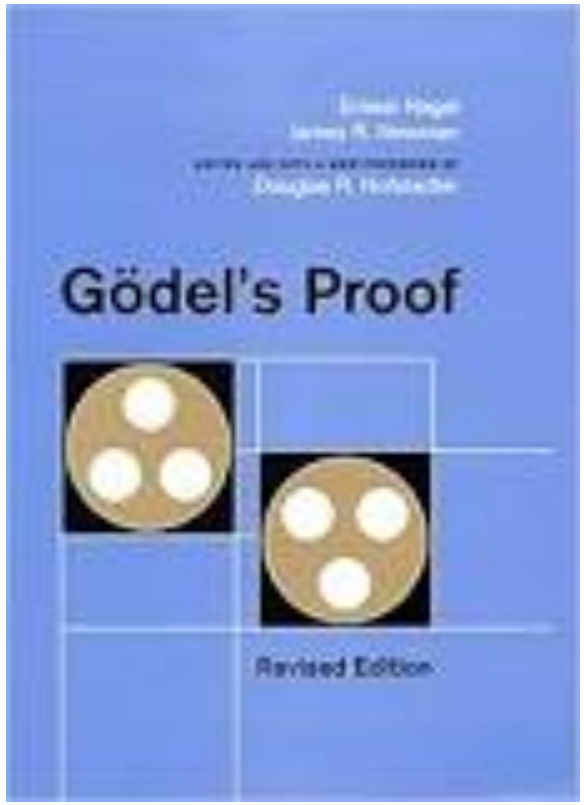

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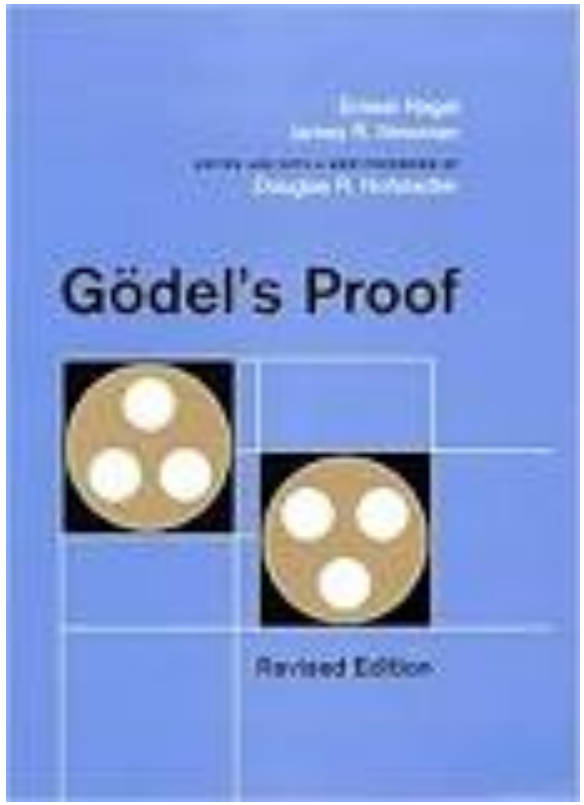

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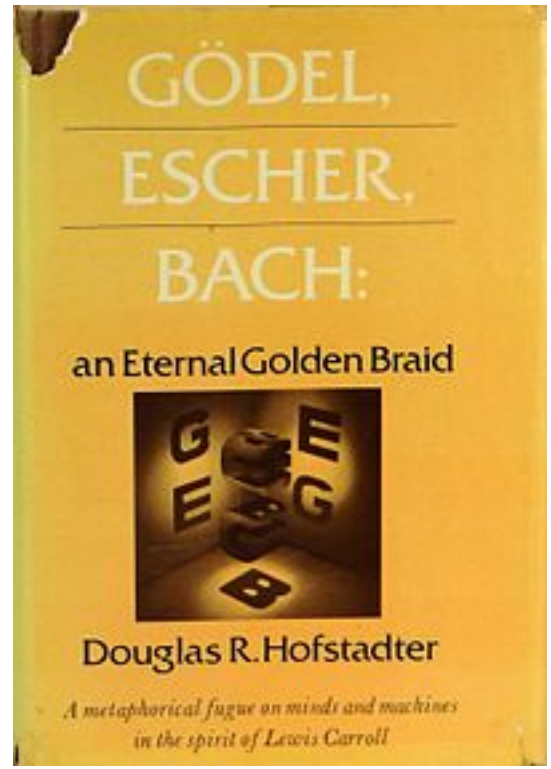

Some Timeline Points

1978 Princeton NJ USA.

1940 Back to USA, for good.

1936 Schlick murdered; Austria annexed

1933 Hitler comes to power.

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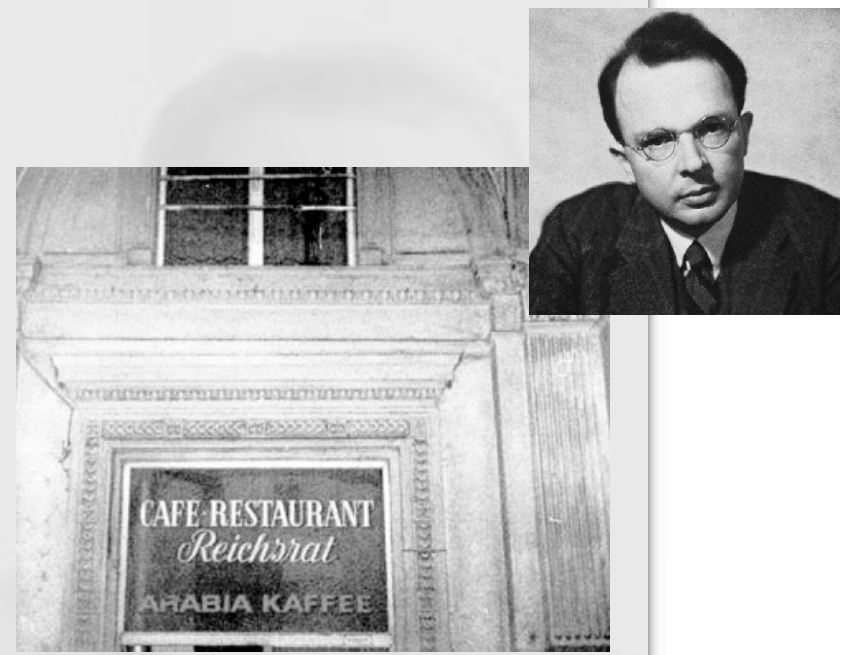

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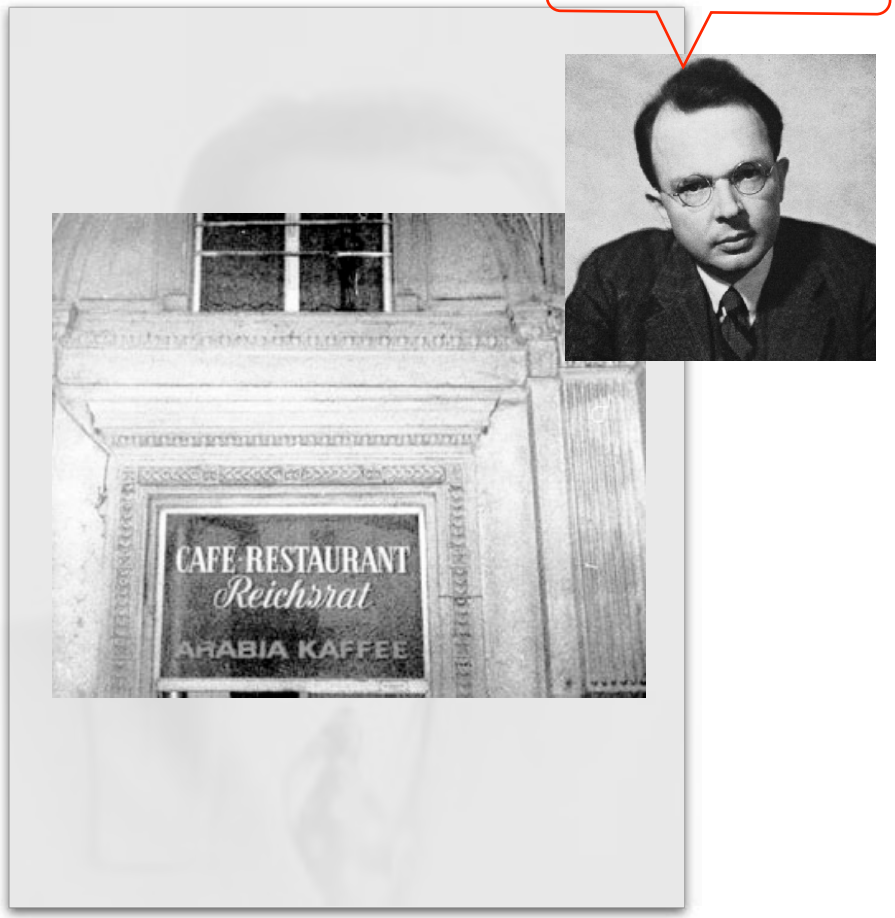

The “Liar Tree”

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The Liar Paradox

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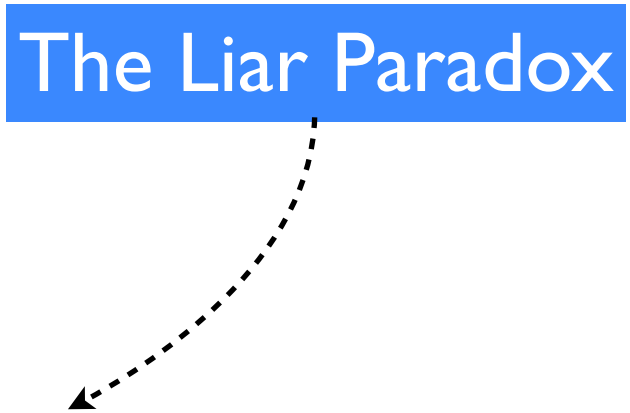

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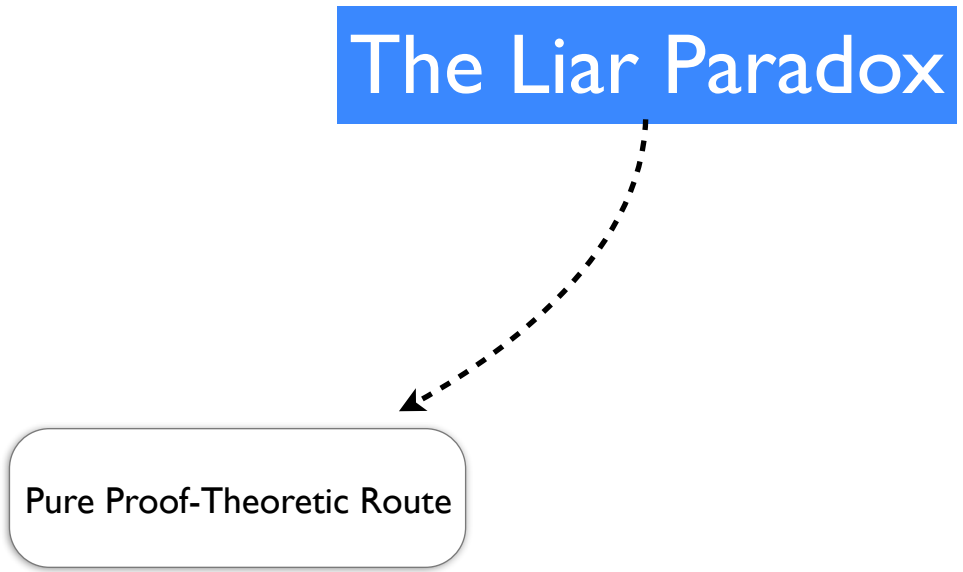

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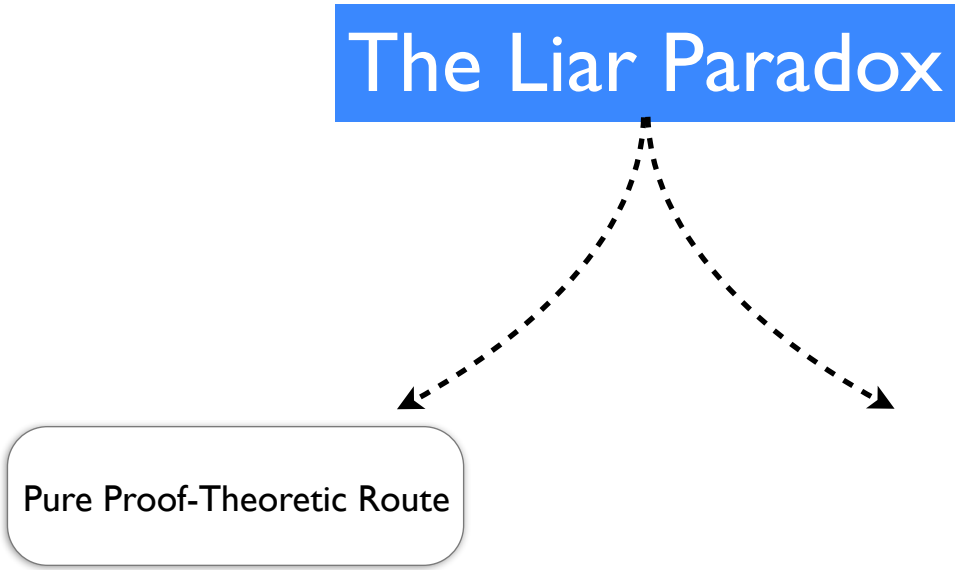

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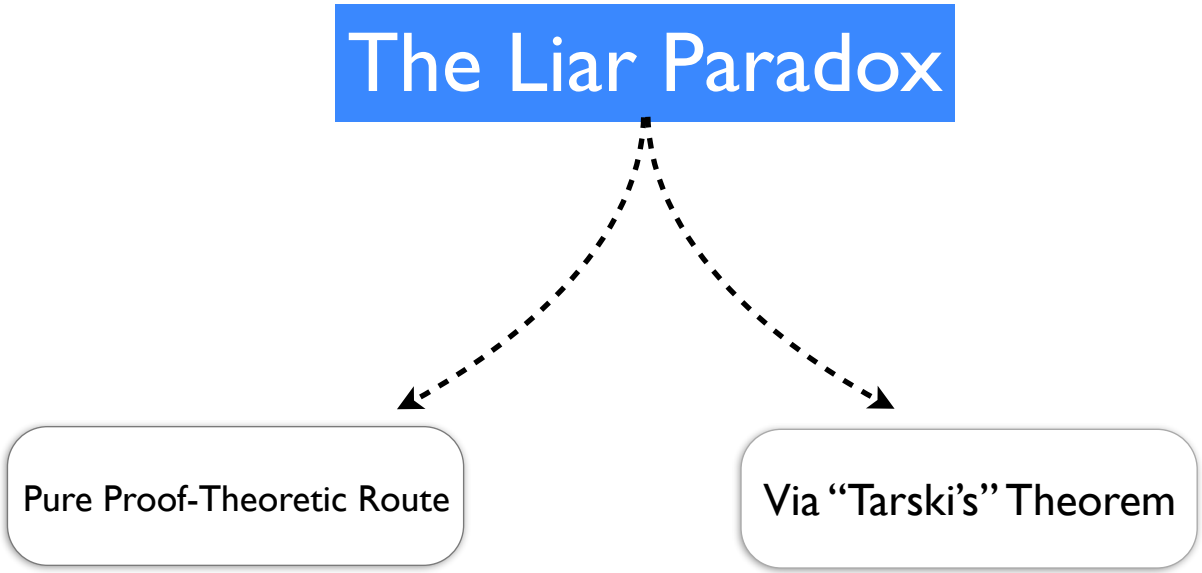

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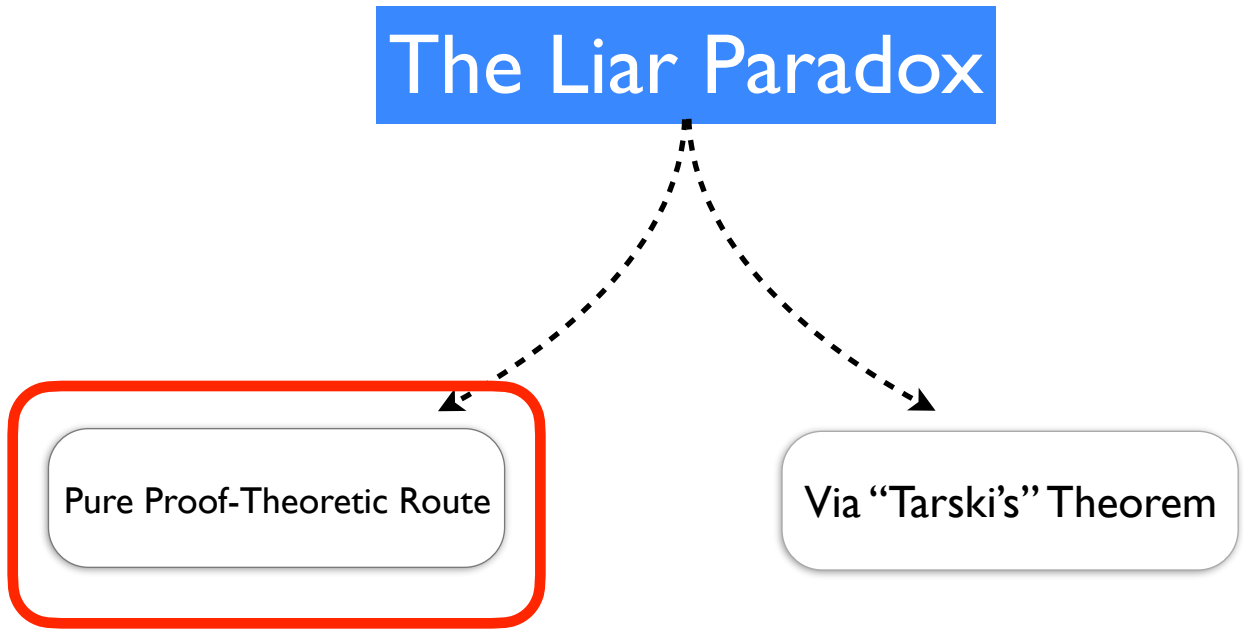

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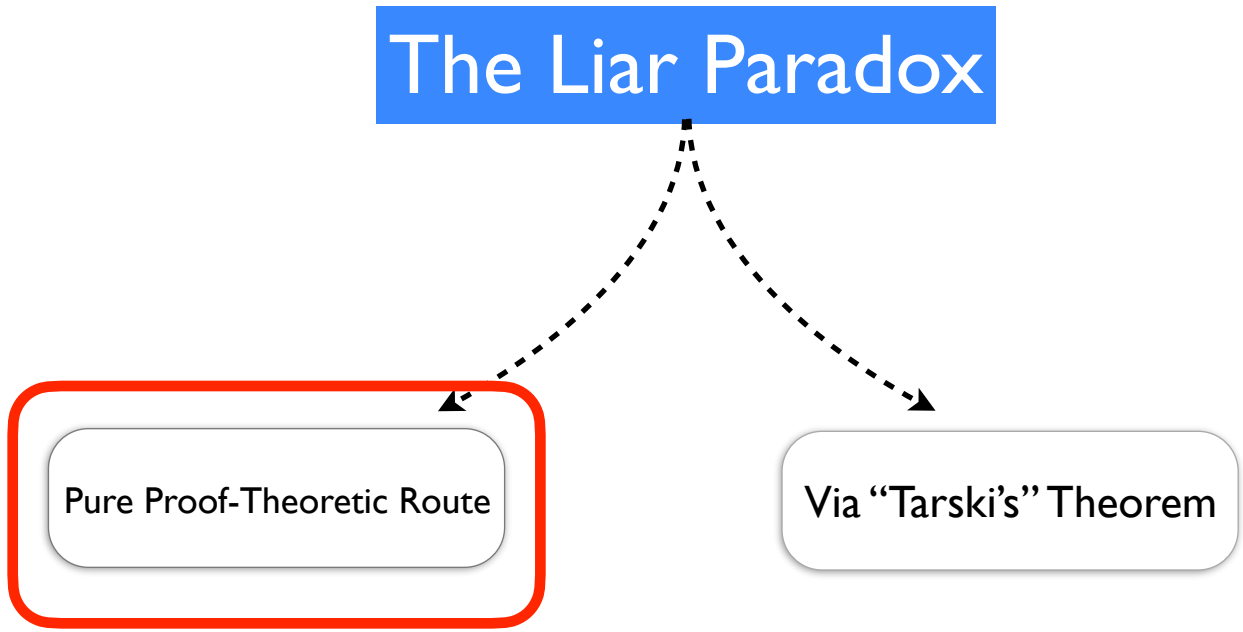
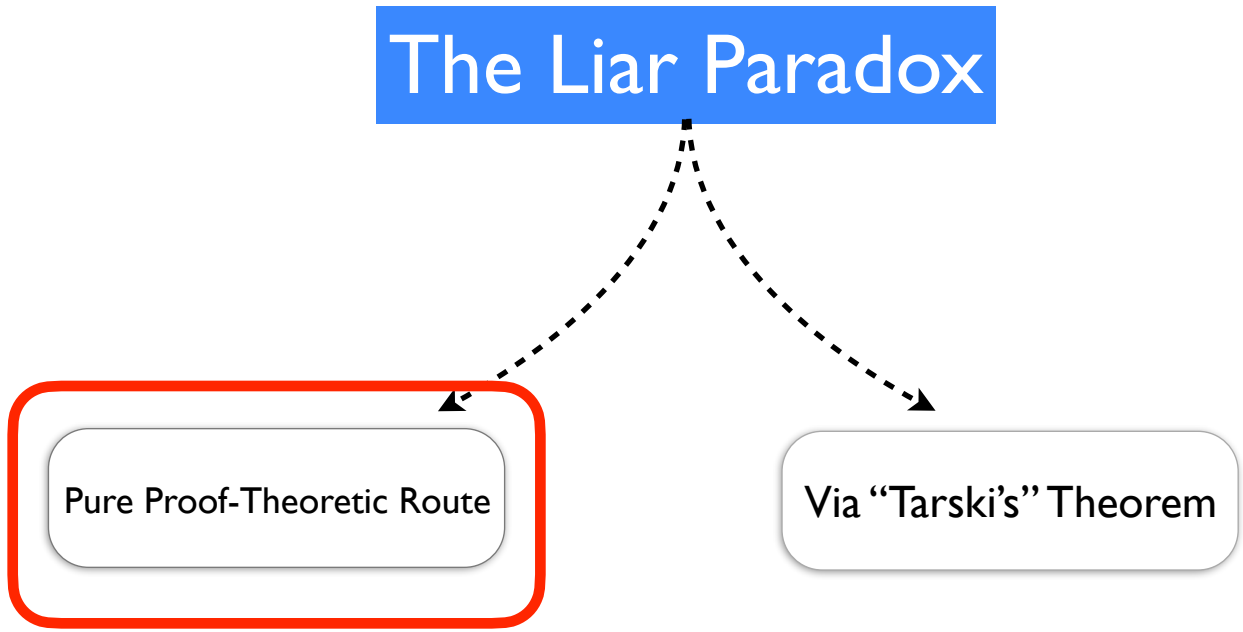


Paul Erdős

"The Book"

The "Liar Tree"



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Ergo, step one: What *is* LP?

“The (Economical) Liar”

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Contradiction!

The “Gödelian” Liar (from me)

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Suppose on the other hand that $\bar{P}$ is false. Then we can immediately deduce that $\bar{P}$ is unprovable: Suppose for *reductio* that $\bar{P}$ *is* provable; then $\bar{P}$ holds as a result of some proof, but what $\bar{P}$ says is that it's unprovable; and so we have contradiction. But since what $\bar{P}$ says is that it's unprovable, and we have just proved that under our supposition, we arrive at the conclusion that $\bar{P}$ is true.

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All of this is fishy; but Gödel transformed it into utterly precise, impactful, indisputable reasoning …

PA (Peano Arithmetic):

$$
\begin{array}{ll}
\text{A1} & \forall x(0 \neq s(x)) \\
\text{A2} & \forall x \forall y(s(x) = s(y) \rightarrow x = y) \\
\text{A3} & \forall x(x \neq 0 \rightarrow \exists y(x = s(y)) \\
\text{A4} & \forall x(x + 0 = x) \\
\text{A5} & \forall x \forall y(x + s(y) = s(x + y)) \\
\text{A6} & \forall x(x \times 0 = 0) \\
\text{A7} & \forall x \forall y(x \times s(y) = (x \times y) + x)
\end{array}
$$

And, every sentence that is the universal closure of an instance of

$$([\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x))] \rightarrow \forall x \phi(x))$$

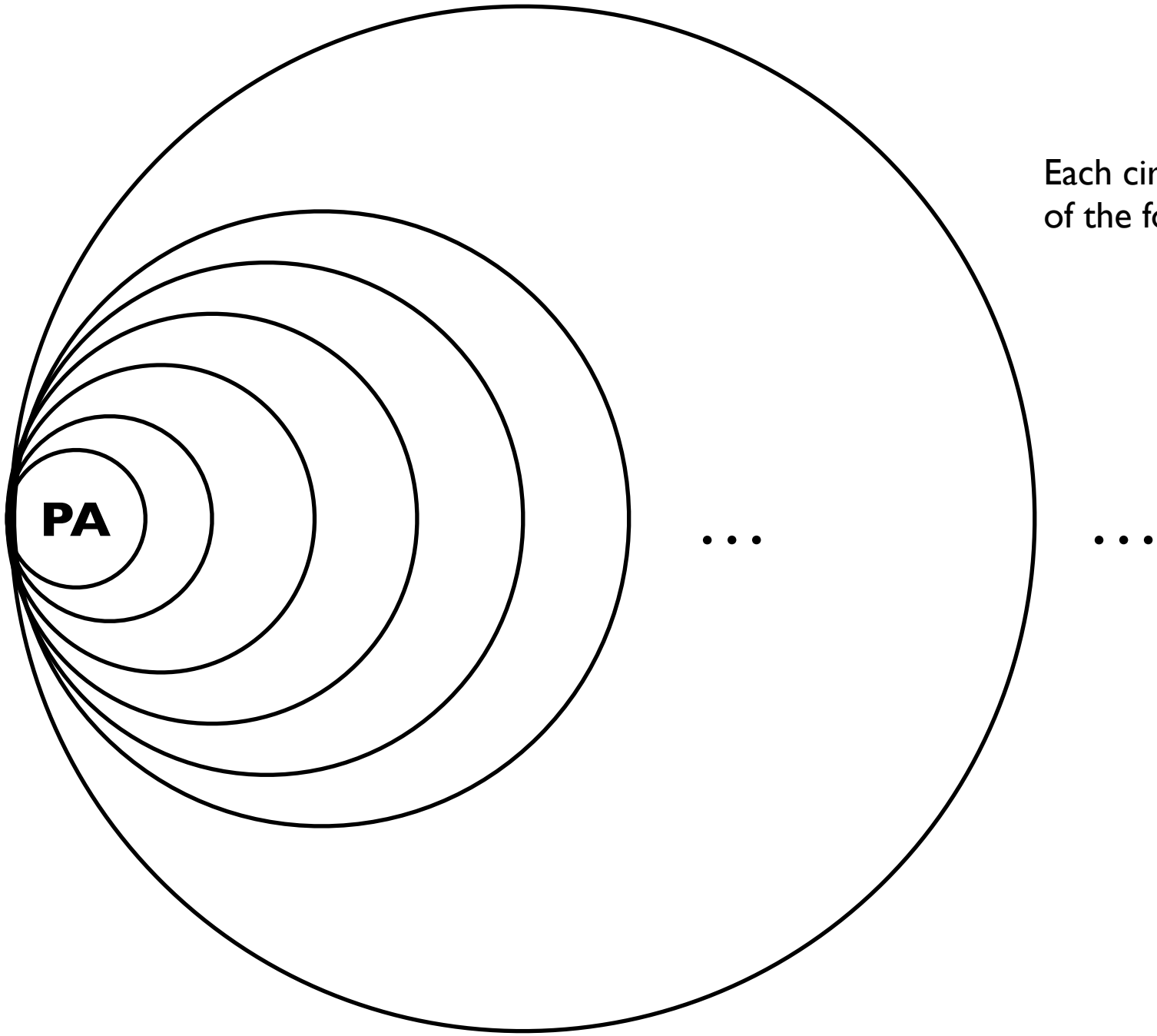
where $\phi(x)$ is open wff with variable x, and perhaps others, free.

PA2 (Second-Order Peano Arithmetic)

https://rpi.logicamodernapproach.com/hyperslate/public/Selmer.Bringsjord@gmail.com/PA2

Arithmetic is Part of All Things Sci/Eng/Tech!

but courtesy of Gödel: We can't even prove all truths of arithmetic!



Each circle is a larger part of the formal sciences.

Arithmetic Crucial Part of All Things Sci/Eng/Tech!

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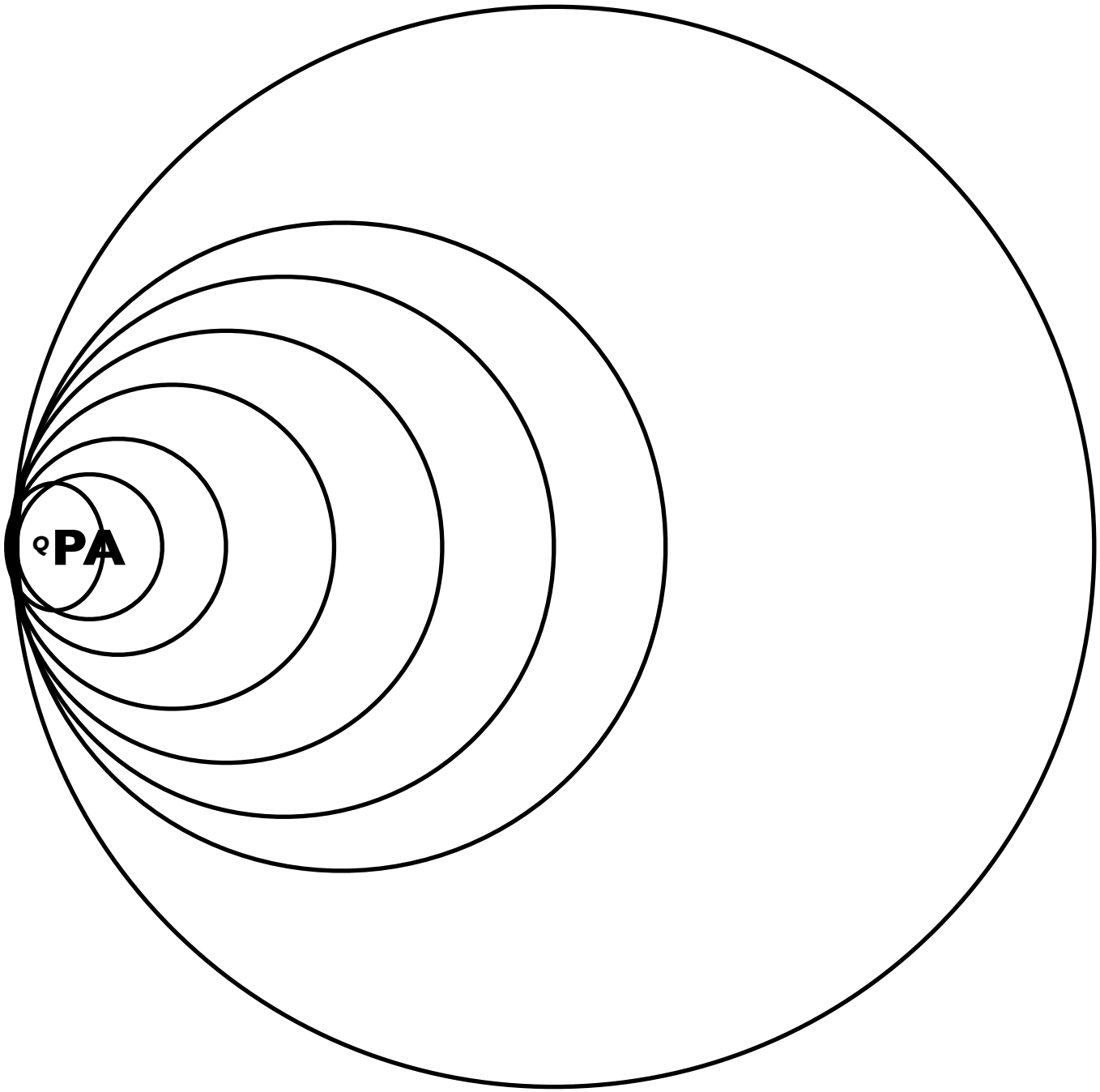

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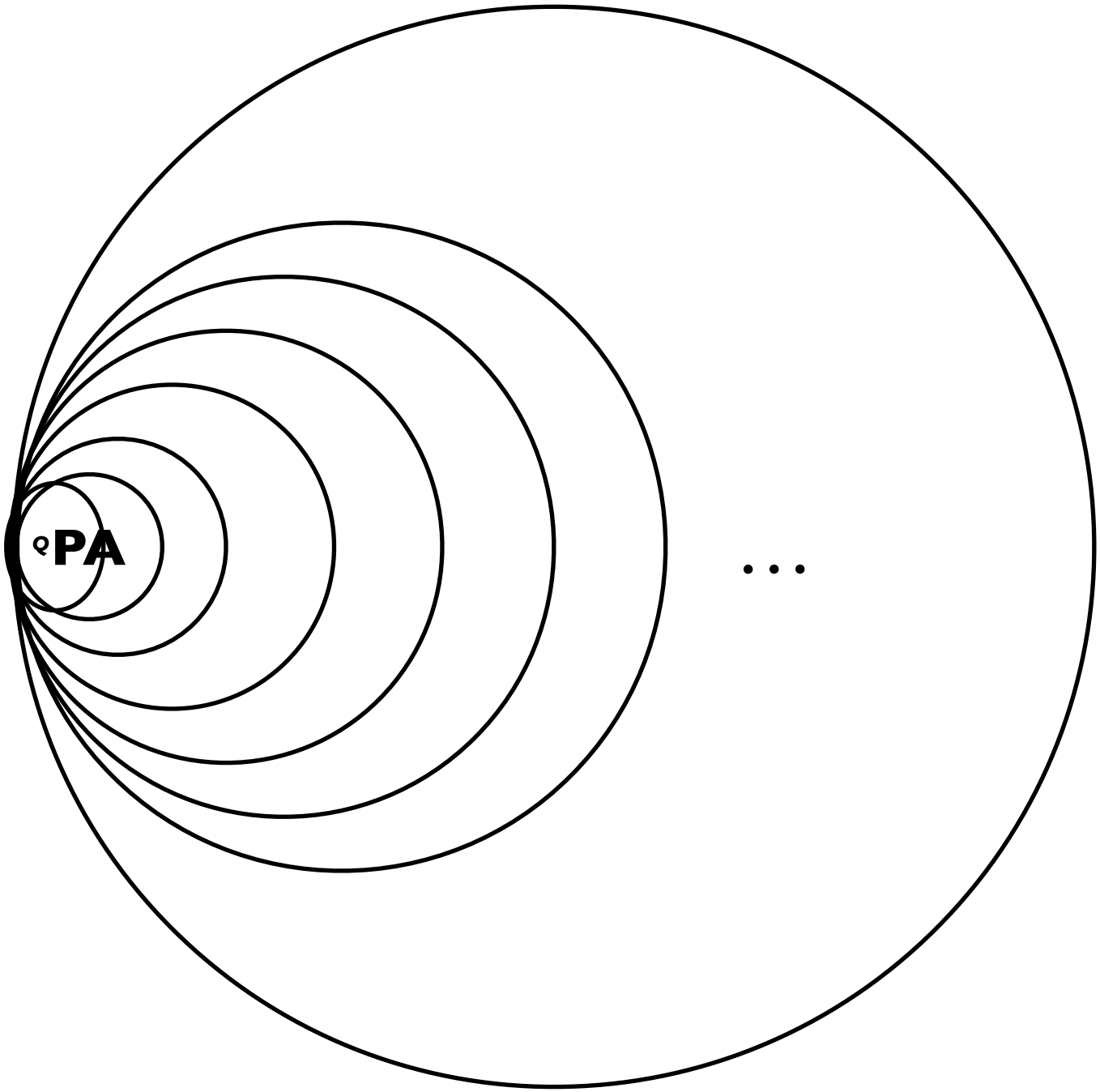

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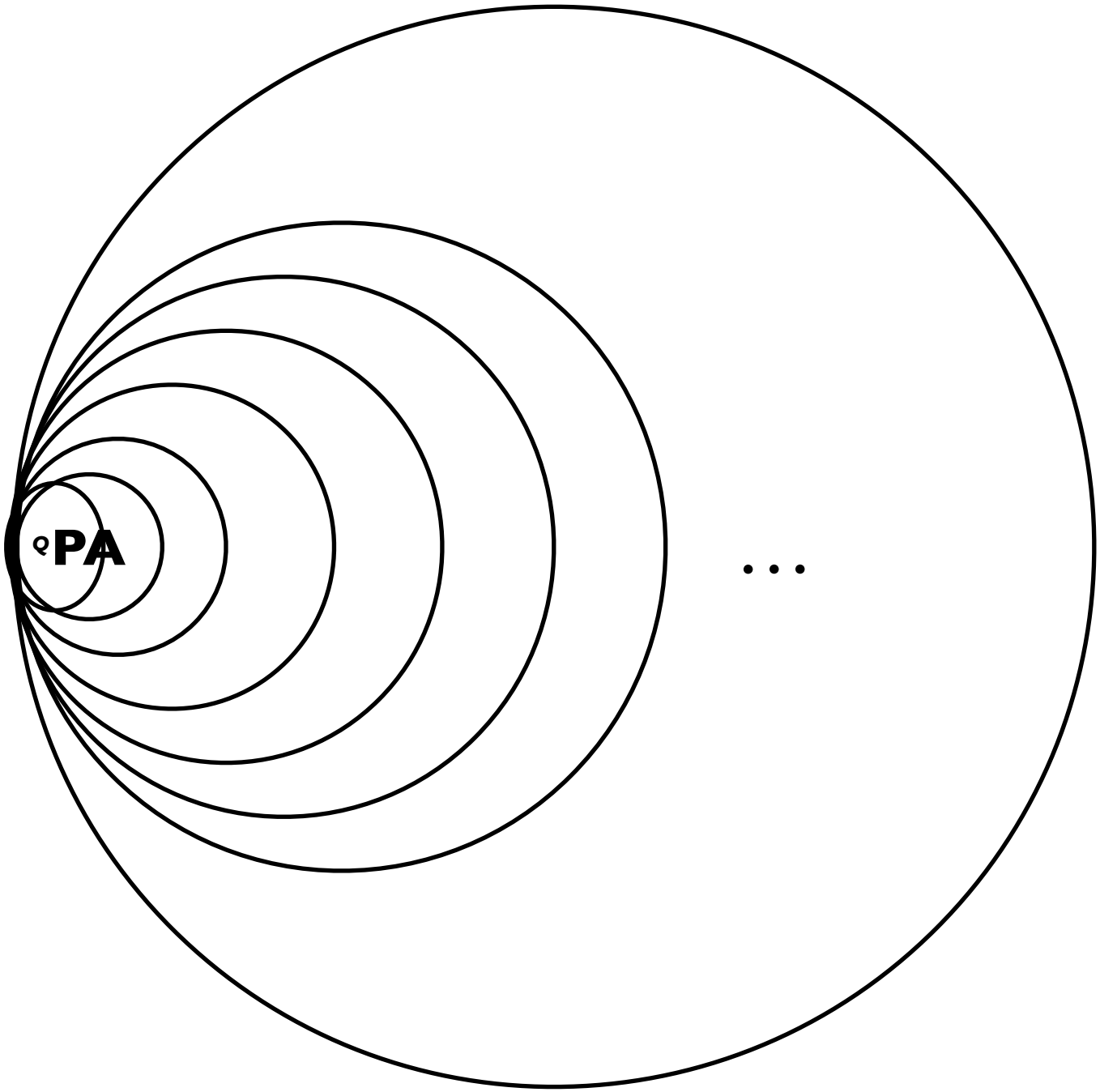

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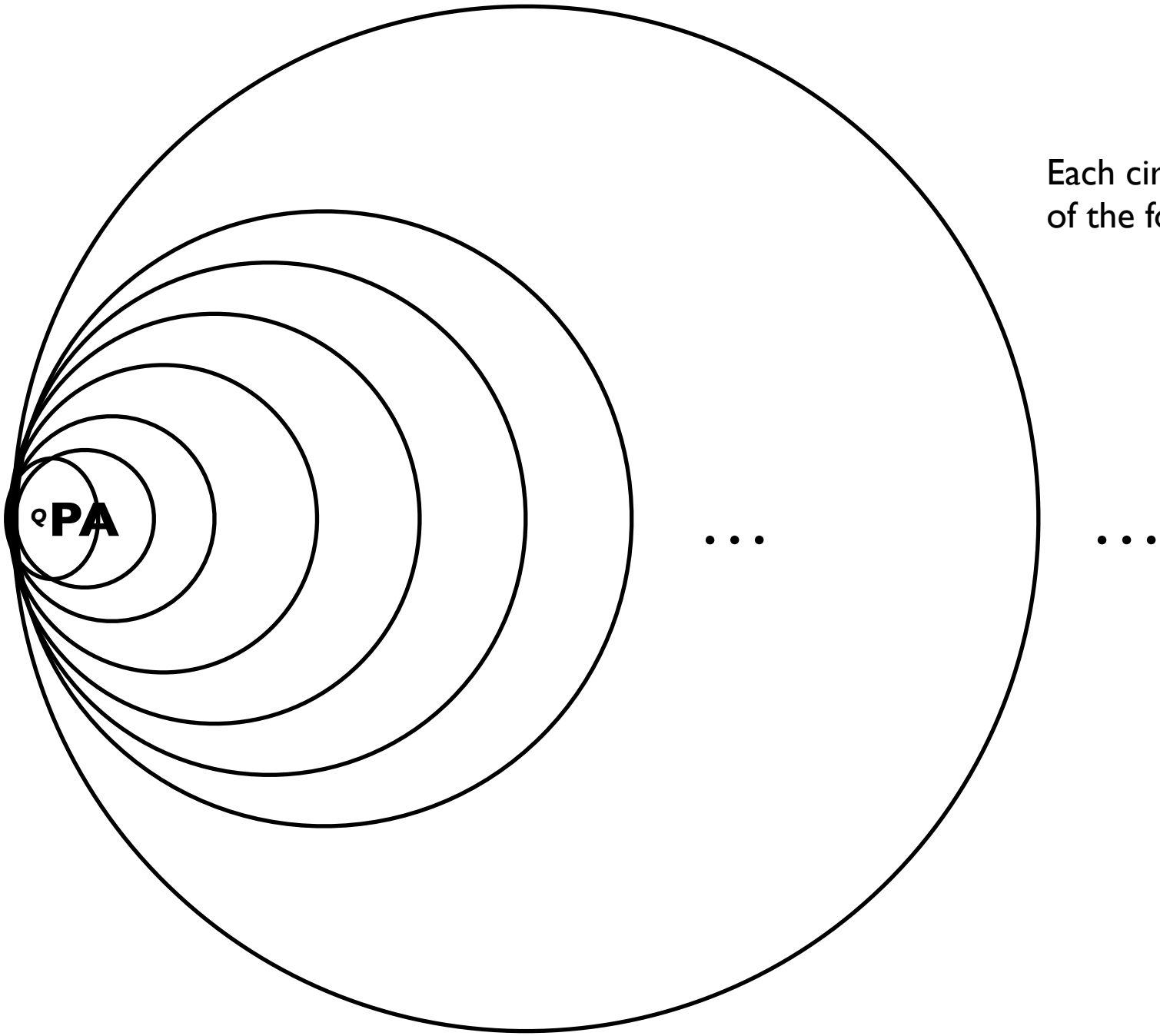

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Gödel Numbering

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Problem: How do we enable a formula to refer to other formulae and itself (and also other objects like proofs, terms etc.), in a perfectly consistent way?

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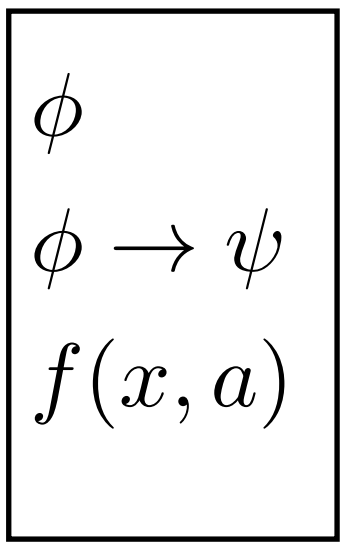

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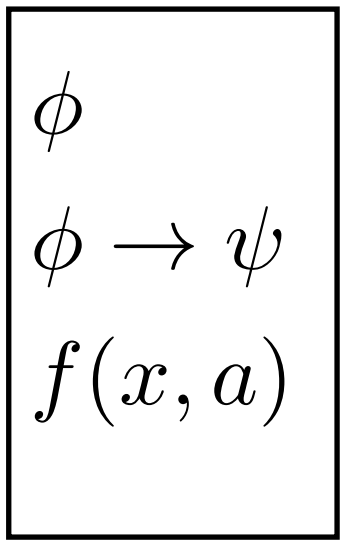

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in the language of $\mathscr{L}_1$

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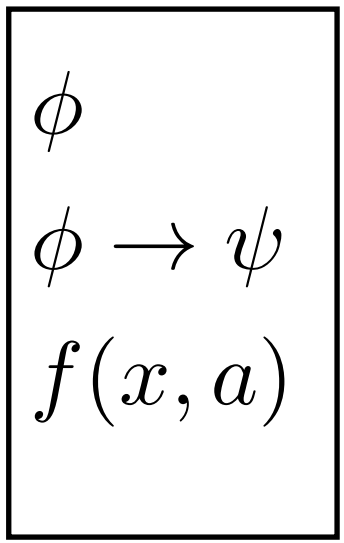

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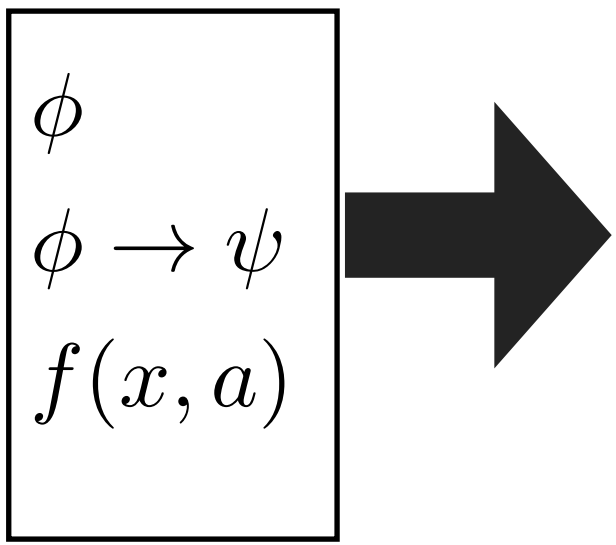

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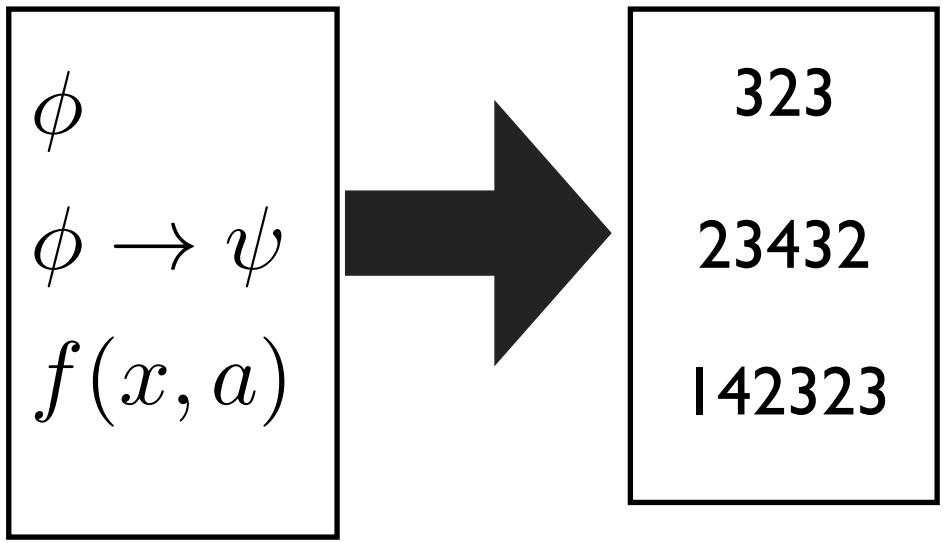

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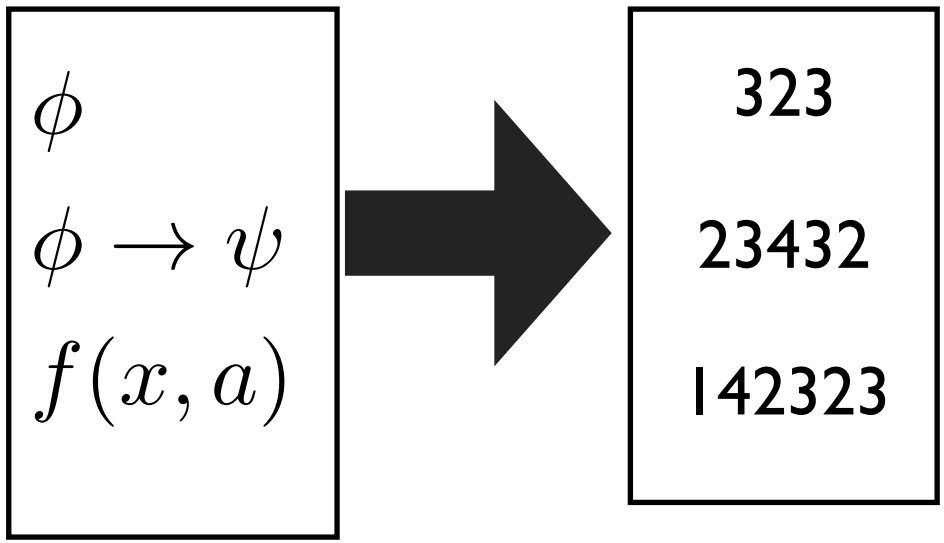

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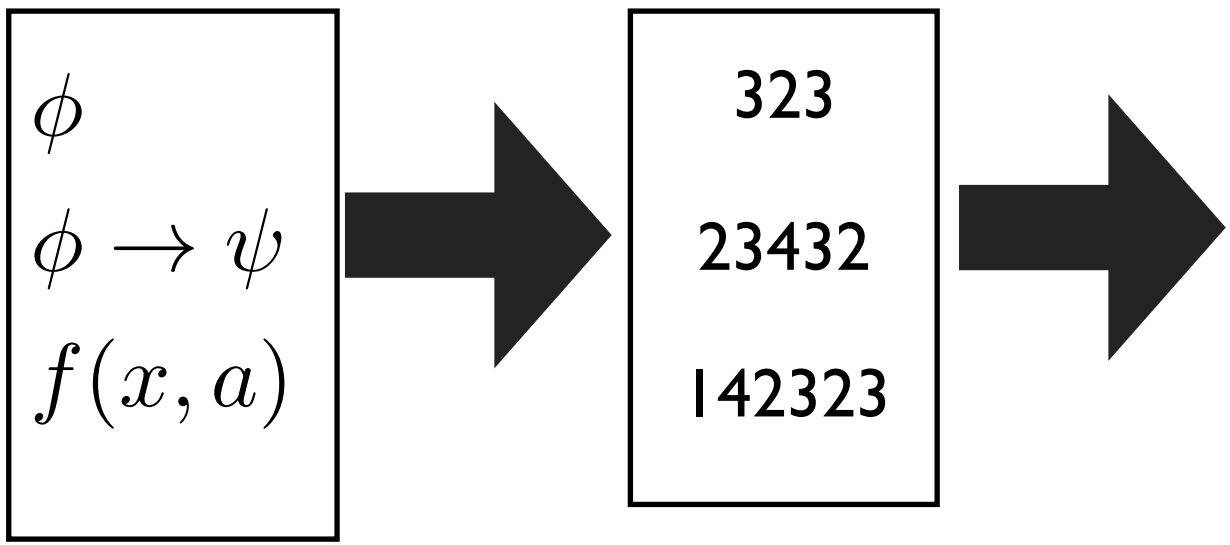

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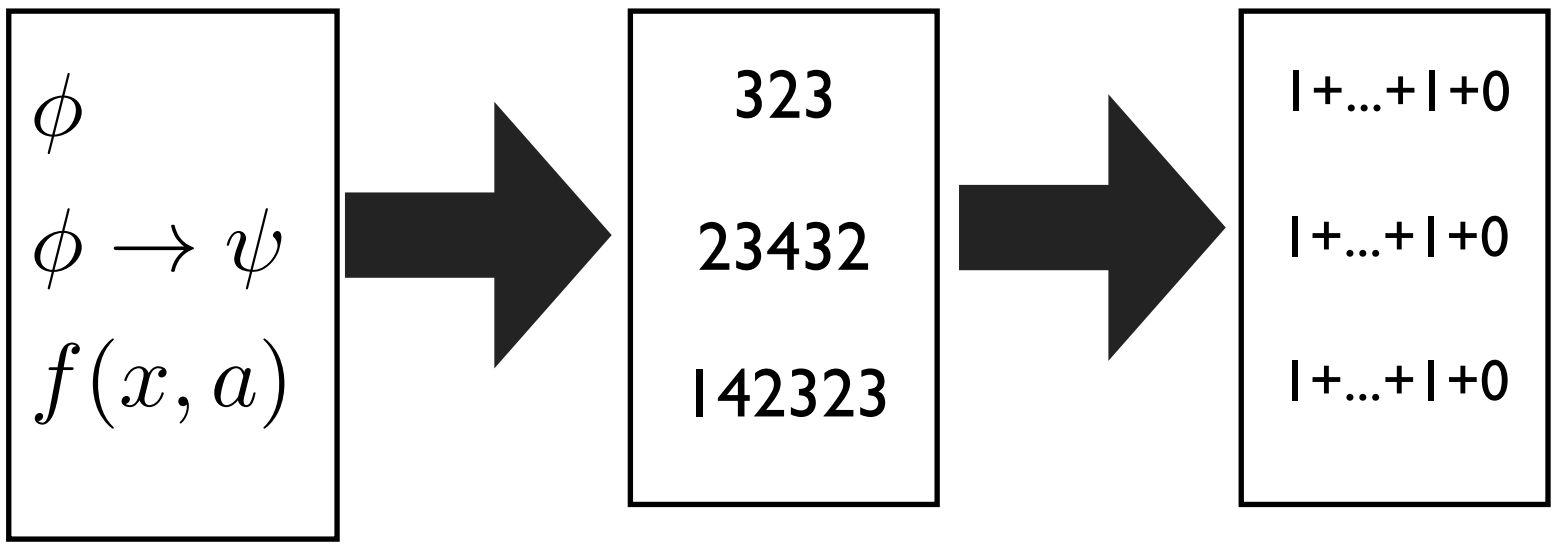
Gödel number

Gödel Numbering

Problem: How do we enable a formula to refer to other formulae and itself (and also other objects like proofs, terms etc.), in a perfectly consistent way?

Solution: Gödel numbering!



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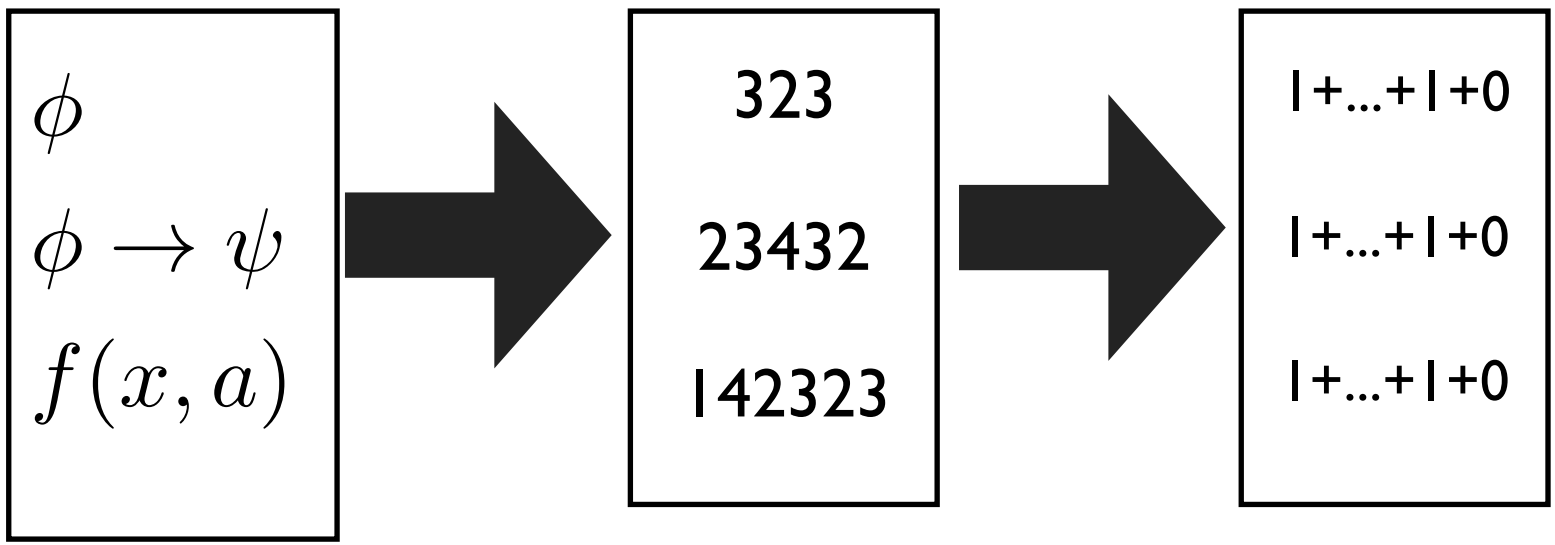

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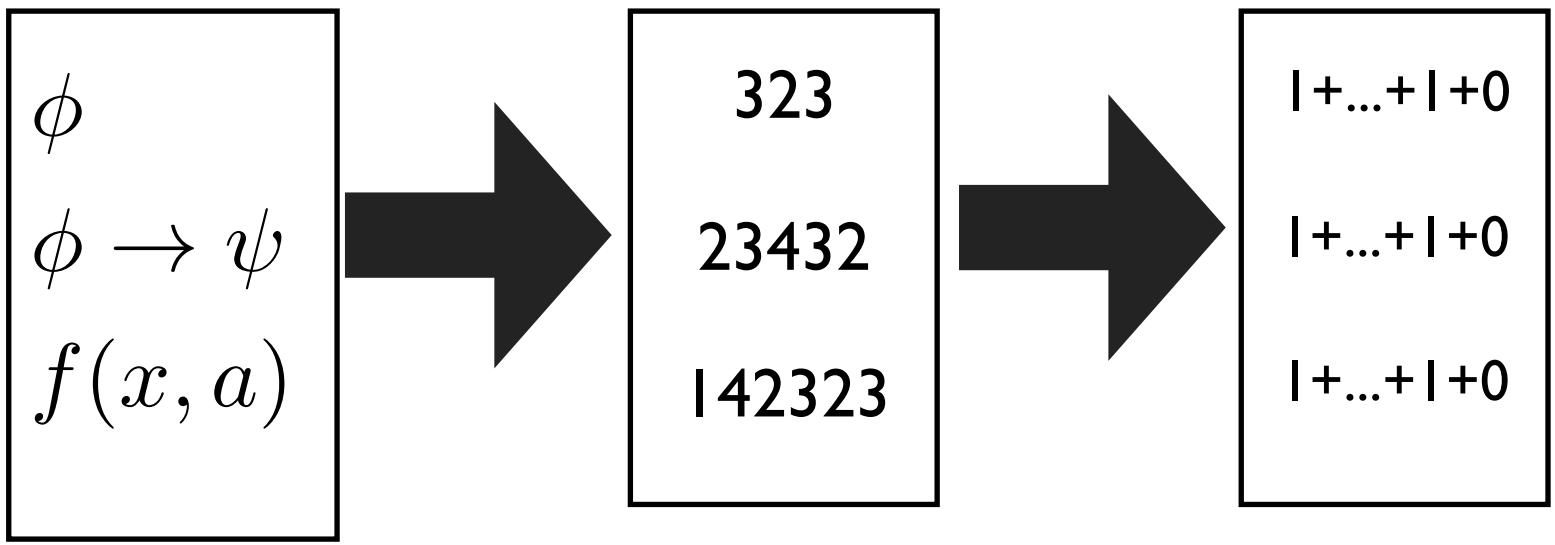

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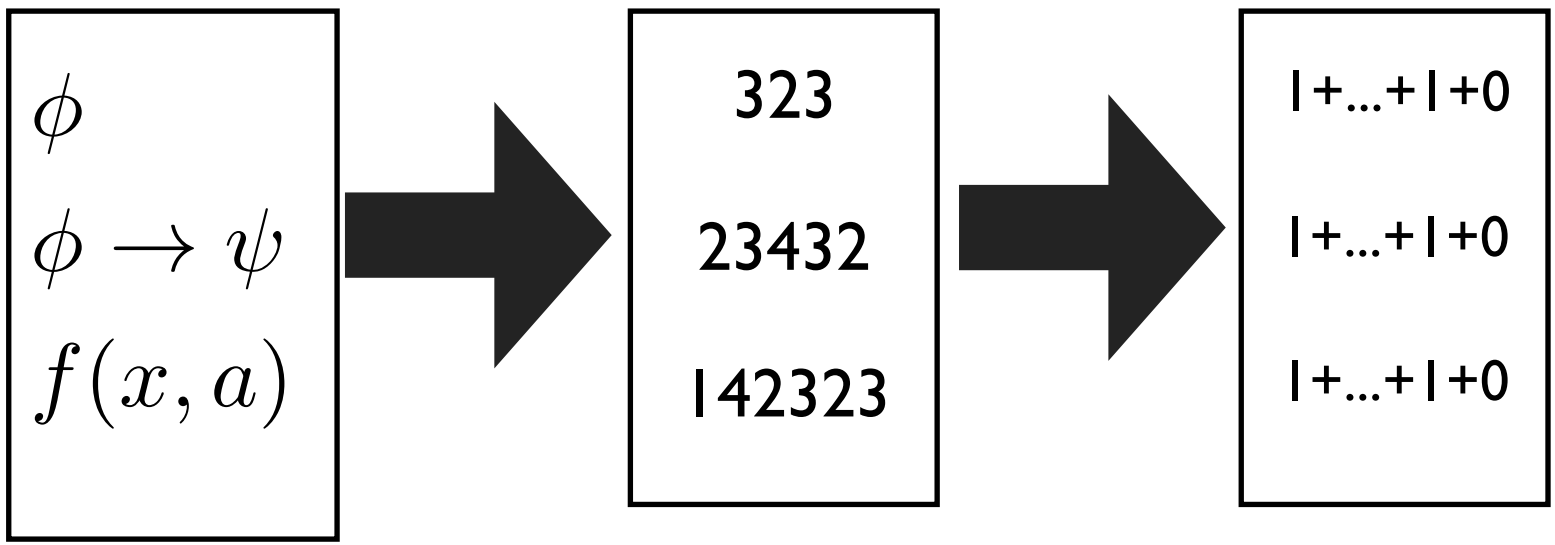

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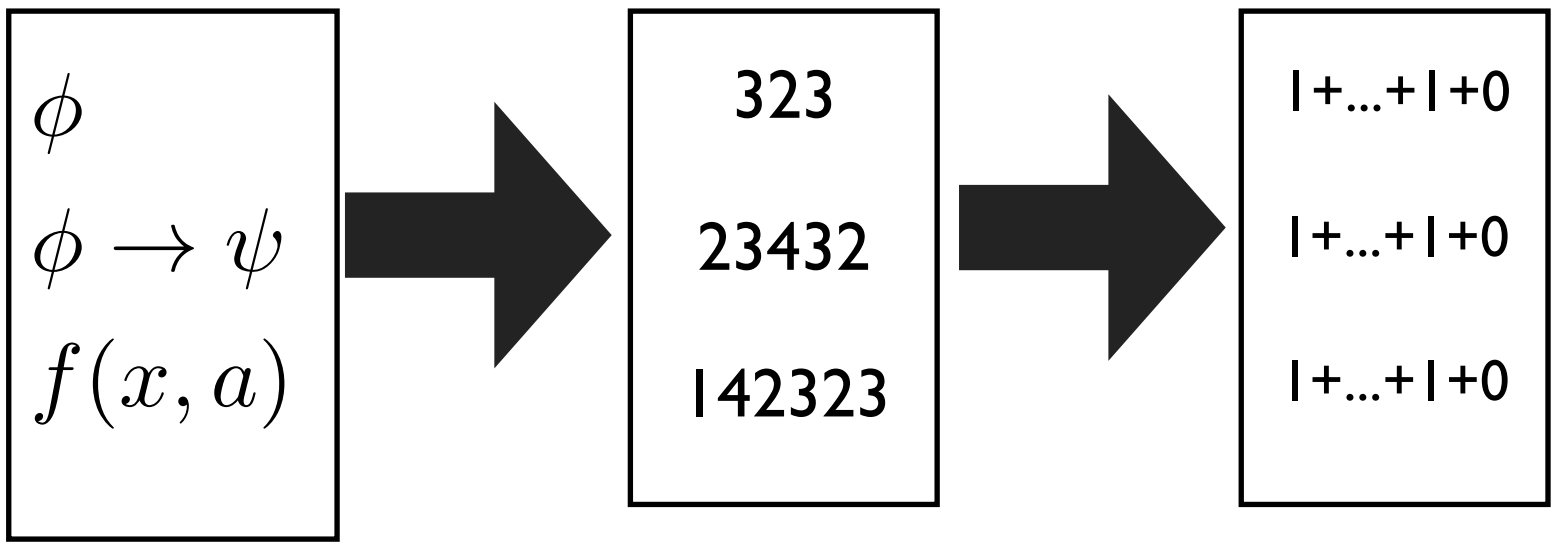

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$\hat{n}^{\phi}$ (or just "ϕ")

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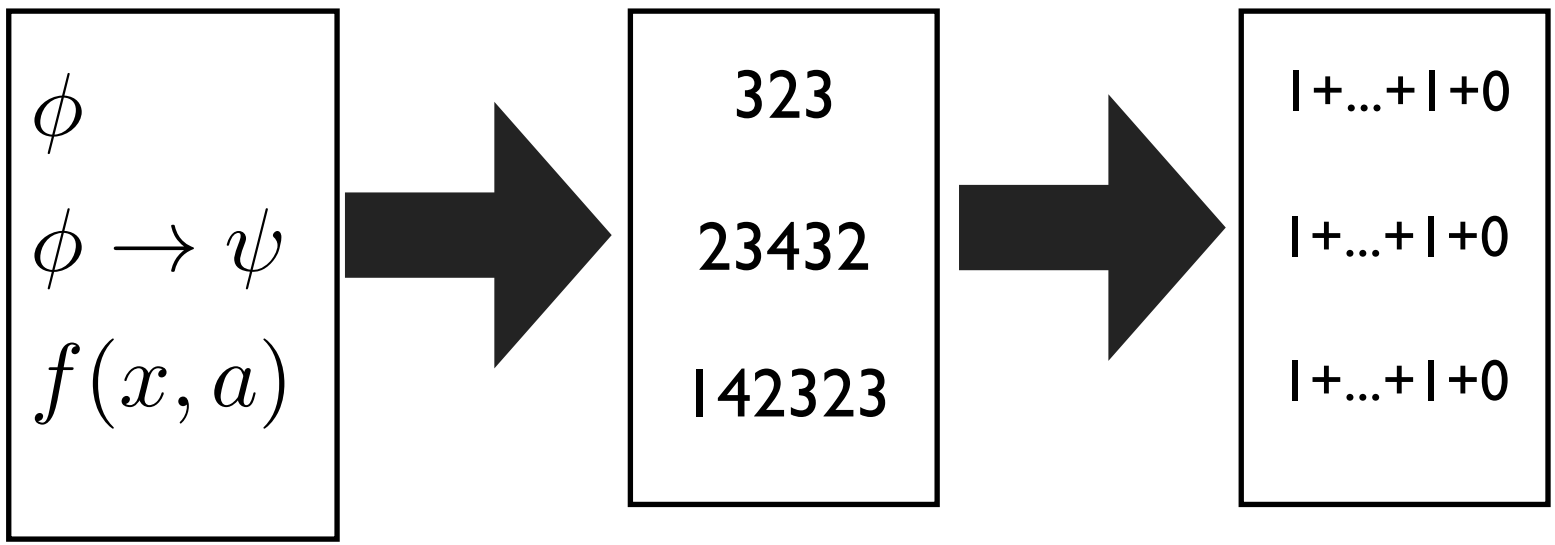

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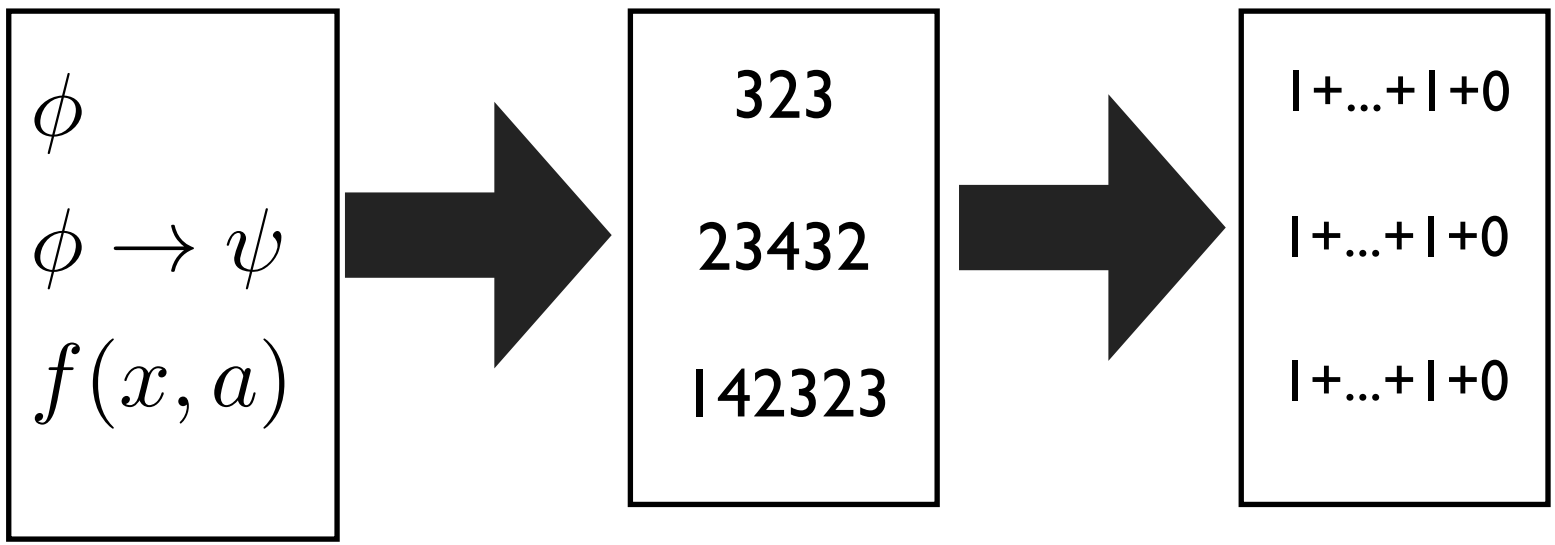

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S will sometimes conflate.

Gödel Numbering, the Easy Way

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Just realize that every entry in a dictionary is named by a number *n*, and by the same basic lexicographic ordering, every computer program, formula, etc. is named by a number *m* in a lexicographic ordering going from 1, to 2, to …

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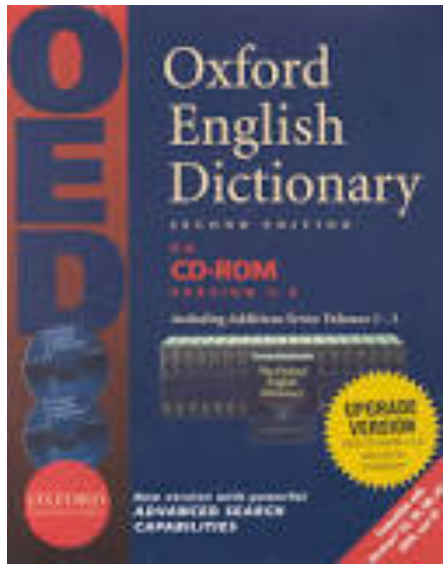

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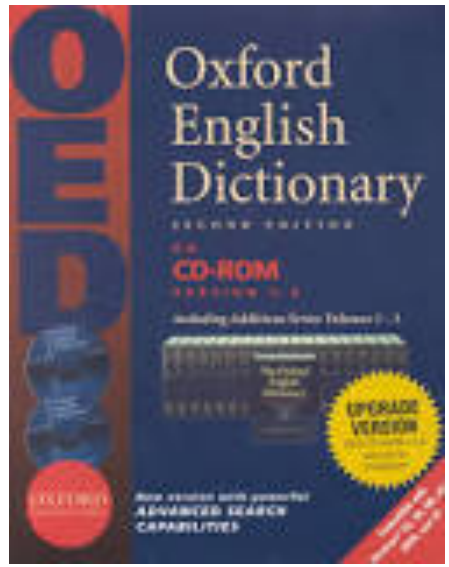

So, `gimcrack` is named by some positive integer *k*. Hence, I can just refer to this word as "*k*" Or in the notation I prefer: k^{gimcrack}.

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Or, every syntactically valid computer program in Clojure that you will ever write can be uniquely denoted by some number *m* in the lexicographic ordering of all syntactically valid such programs. So your program π can just be coded as a numeral m^{π} in a formal language that captures arithmetic (i.e., an *arithmetic language*).

Gödel's First Incompleteness Theorem

Let Φ be a set of arithmetic sentences that is

(i) consistent (i.e. no contradiction $\phi \wedge \neg\phi$ can be deduced from Φ);
(ii) s.t. an algorithm is available to decide whether or not a given string u is a member of Φ; and
(iii) sufficiently expressive to logicize all of the operations of a standard computing machine (e.g. a Turing machine, register machine, KU machine, etc.).

Then there is an "undecidable" arithmetic sentence $\mathcal{G}$ from Gödel that can't be proved from Φ, nor can the negation of this sentence (i.e. $\neg\mathcal{G}$) be proved from Φ!

Alas, that's painfully verbose.

Gödel's First Incompleteness Theorem

pocket theorem

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Suppose $\Phi \supseteq \mathbf{PA}$ (= Φ contains **PA**) is

(i) Φ is *consistent*: no contradiction $\phi \wedge \neg\phi$ can be proved from Φ (Con Φ);
(ii) Membership of strings in Φ is *algorithmically decidable*; and
(iii) Φ has *capturability*: it can capture functions and relations over $\mathbb{N}$ (Cap Φ).

Then there is a sentence $\mathcal{G}$ s.t. $\Phi \nvdash \mathcal{G}$ and $\Phi \nvdash \neg\mathcal{G}$. ($\mathcal{G}$ is an "undecidable" sentence.)

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To prove G1, we shall allow ourselves …

The Fixed Point Theorem (FPT)

"The Self-Ascription Theorem" (GSAT)

Assume that Φ is a set of arithmetic sentences such that **Cap** Φ. Then for every arithmetic formula $\psi(x)$ with one free variable x, there is an arithmetic sentence ϕ s.t.

$\Phi \vdash \phi \leftrightarrow \psi(\hat{n}^{\phi})$.

We can intuitively understand ϕ to be saying:
"I have the property ascribed to me by the formula ψ."

"I *heard* there was no free lunch!"

[W]e "would hope that such a deep theorem would have an insightful proof. No such luck. I am going to write down a sentence … and verify that it works. What I won't do is give you a satisfactory explanation for why I write down the particular formula that I do. I write down the formula because Gödel wrote down the formula, and Gödel wrote down the formula because, when he played the logic game he was able to see seven or eight moves ahead, whereas you and I are only able to see one or two moves ahead. I don't know anyone who thinks he has a fully satisfying understanding of why the Self-referential Lemma [= FPT] works. It has a rabbit-out-of-a-hat quality for everyone."

—V. McGee, 2002; as quoted in (Salehi 2020)

The Fixed Point Theorem (FPT)

"The Self-Ascription Template Theorem"

Assume that Φ is a set of arithmetic sentences such that **Cap** Φ. There for every arithmetic formula $\psi(x)$ with one free variable x, there is an arithmetic sentence ϕ s.t.

$\Phi \vdash \phi \leftrightarrow \psi(\hat{n}^{\phi})$.

We can intuitively understand ϕ to be saying:
"I have the property ascribed to me by the formula ψ."

Ok; so let's do it …

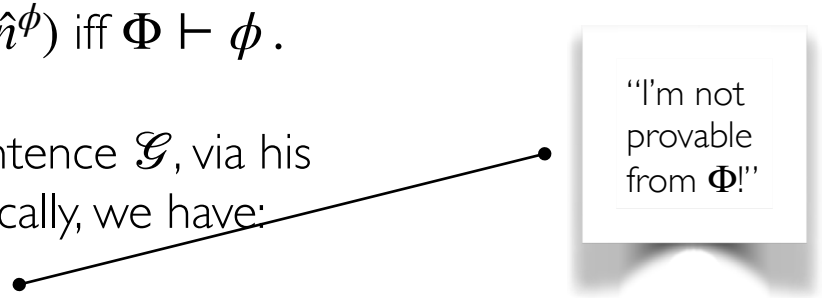
Proof: Let Φ be a set of arithmetic sentences that's a superset of **PA**, and suppose the antecedent of G1 holds, i.e. (i)–(iii) hold. We must show that there exits an arithmetic sentence s.t. neither it nor its negation can be proved from Φ. In homage to Gödel, we label this sentence '$\mathscr{G}$.'

For *reductio*, assume the opposite: that for very arithmetic sentence ϕ, either $\Phi \vdash \phi$ or $\Phi \vdash \neg\phi$.

Next, let's use the formula $\pi(v)$ to capture provability in Φ, where the variable v is a placeholder ready to receive the Gödel numeral $\hat{n}^{\phi}$ of a formula ϕ that's proved. We thus have:

$$\text{(G1.1)} \qquad \Phi \vdash \pi(\hat{n}^{\phi}) \text{ iff } \Phi \vdash \phi\,.$$

Now we bring on stage Gödel's self-ascribing sentence $\mathscr{G}$, via his Self-Ascription Template Theorem (SATL); specifically, we have:



$$\text{(G1.2)} \qquad \Phi \vdash \mathscr{G} \leftrightarrow \neg\pi(\hat{n}^{\mathscr{G}})\,.$$

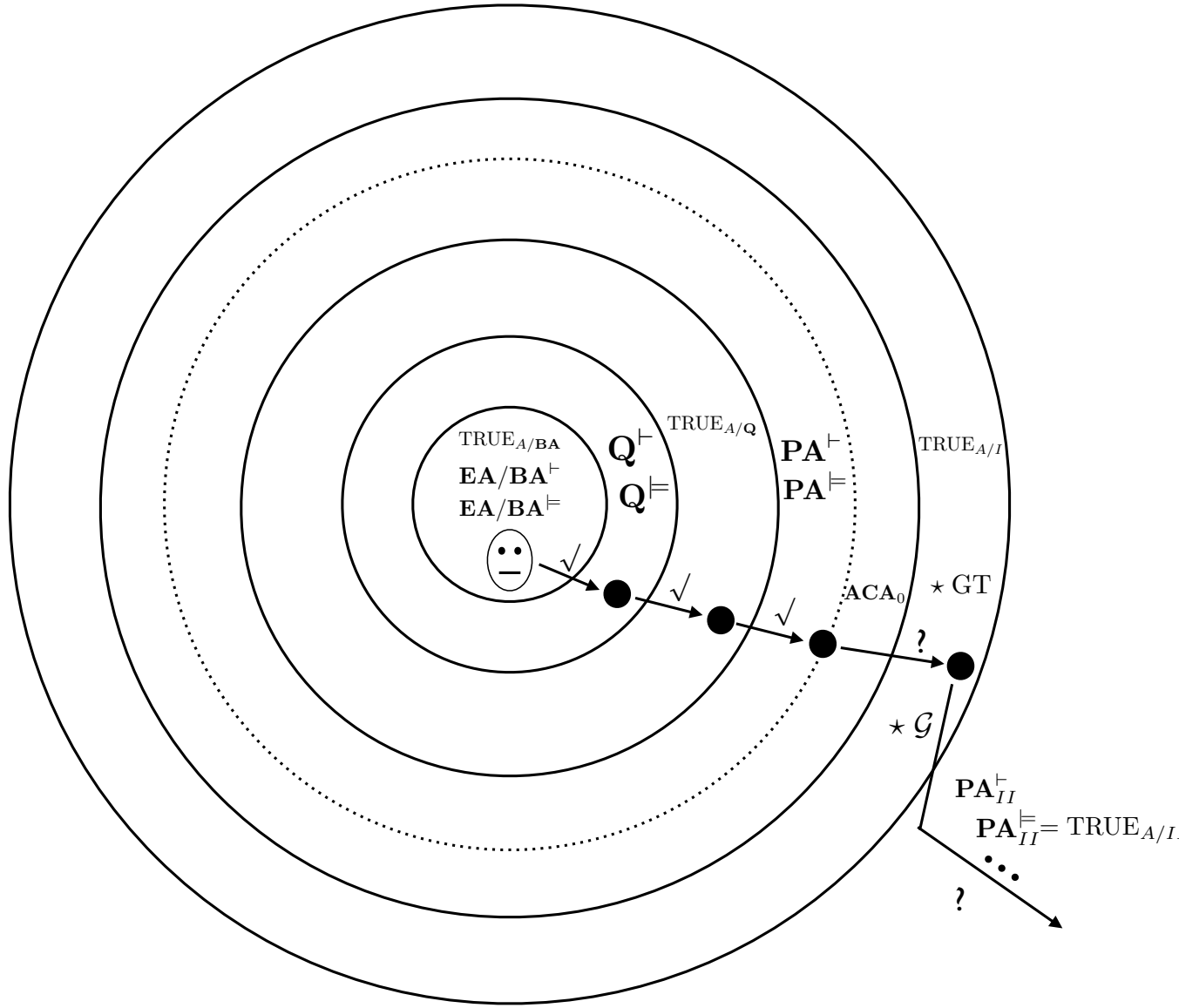
We have two cases to consider, just like what we did in the original Liar Paradox. The first is that $\mathscr{G}$ is provable from Φ; suppose this holds. Then from (1) right-to-left, and *modus ponens* = conditional elim, with left-to-right (by biconditional elimination) on (3) — contradiction!

How about the second case, viz. that $\Phi \vdash \neg\mathscr{G}$? With this and (3) by *modus tollens* and simplification of a double negation we deduce $\Phi \vdash \pi(\hat{n}^{\mathscr{G}})$, which with (1) by biconditional elimination yields $\Phi \vdash \mathscr{G}$ — contradiction!, which is ruled out by the assumption that Φ is consistent. **QED**

"Silly abstract nonsense! There aren't any concrete examples of $\mathscr{G}$!"

Astrologic:

Rational Aliens Will be on the Same "Race Track"!



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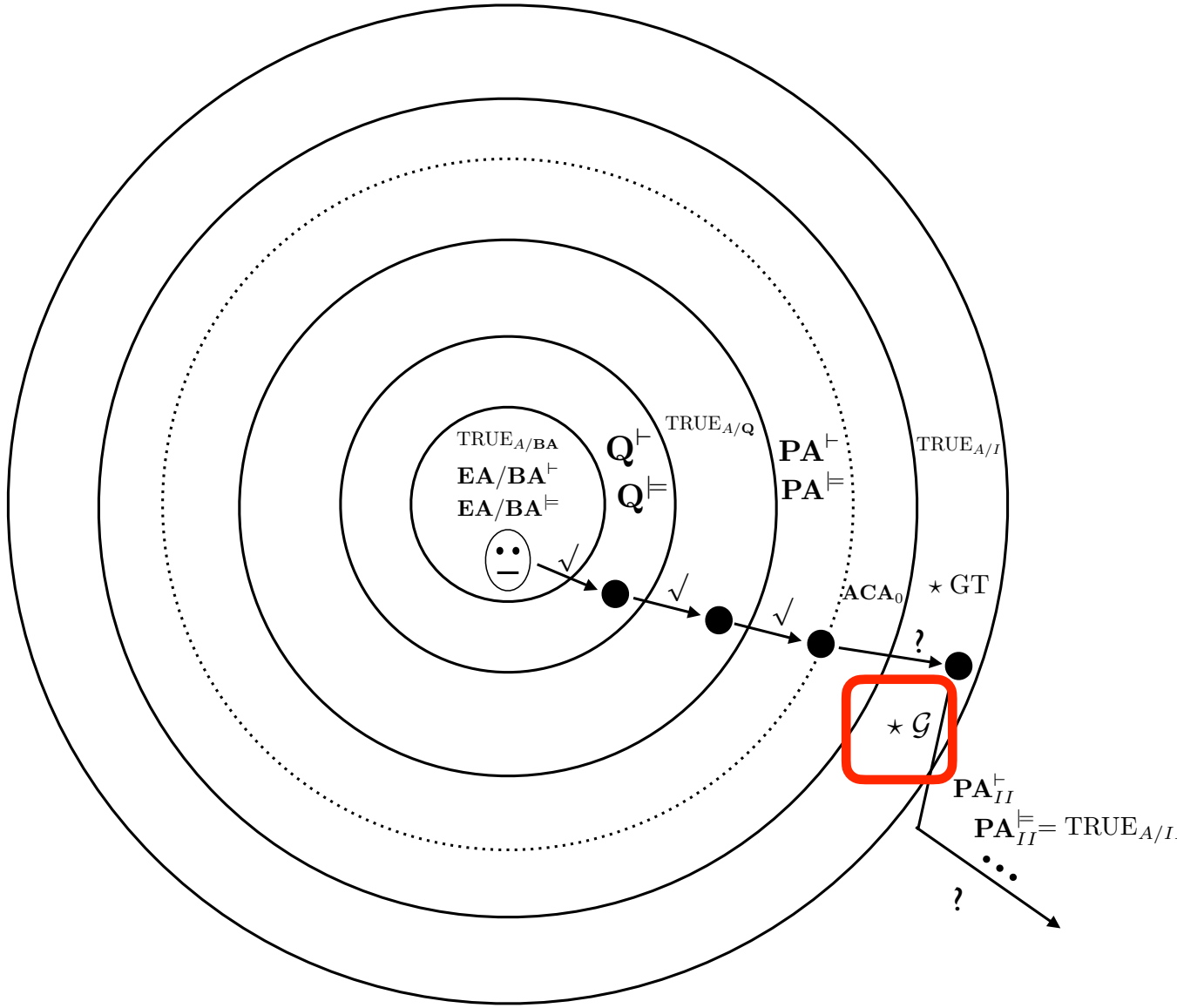

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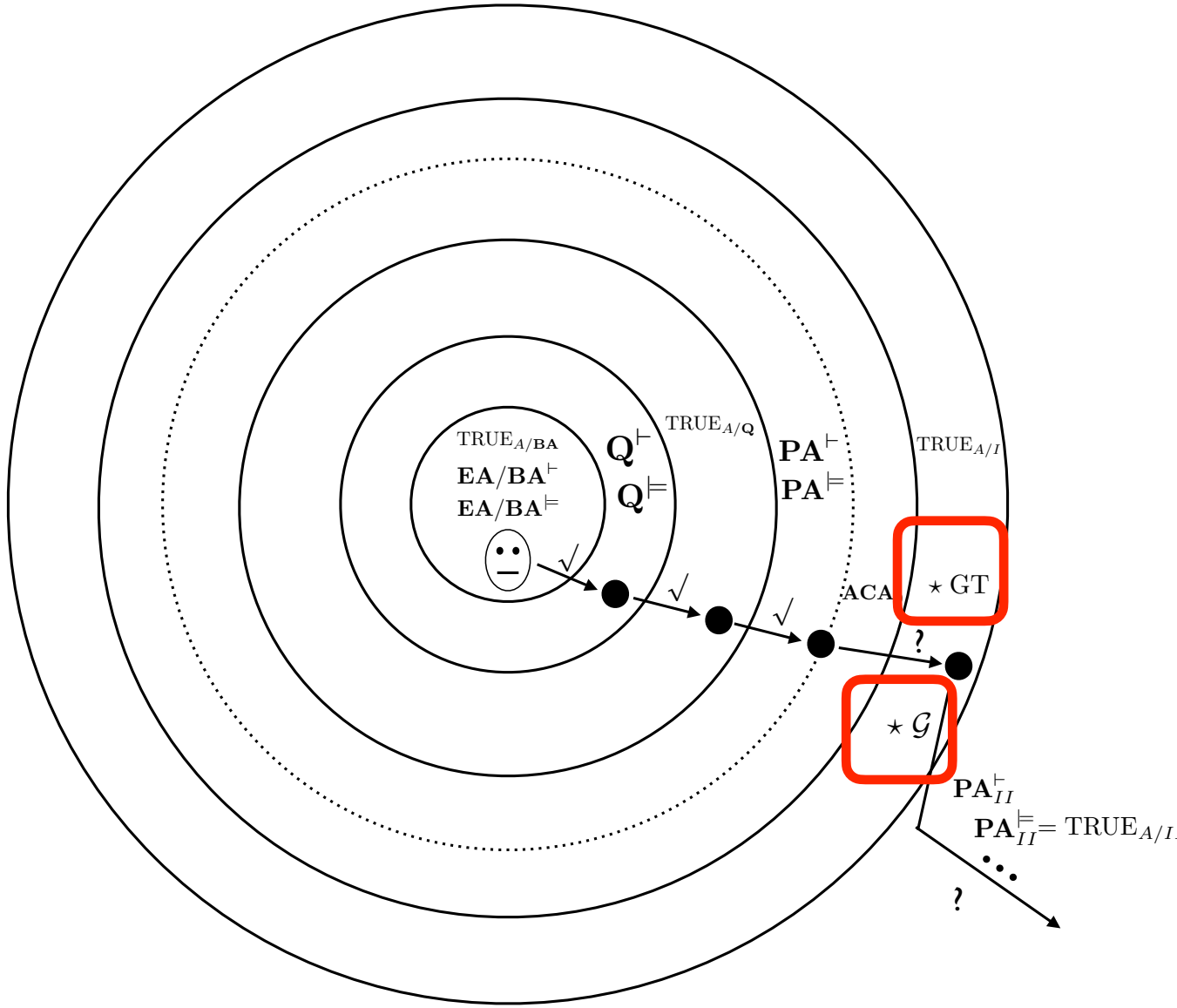

Ah, but e.g.: Goodstein's Theorem!

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The Goodstein Sequence goes to zero!

Pure base *n* representation of a number *r*

- Represent *r* as only sum of powers of *n* in which the exponents are also powers of *n*, etc.

$$266 = 2^{2^{(2^{2^0}+2^0)}} + 2^{(2^{2^0}+2^0)} + 2^{2^0}$$

Grow Function

$Grow_k(n)$:

1. Take the pure base k representation of n
2. Replace all k by $k+1$. Compute the number obtained.
3. Subtract one from the number

Example of **Grow**

$$Grow_2(19)$$

$$19 = 2^{2^{2^{2^0}}} + 2^{2^0} + 2^0$$

$$3^{3^{3^{3^0}}} + 3^{3^0} + 3^0$$

$$3^{3^{3^{3^0}}} + 3^{3^0} + 3^0 - 1$$

$$7625597484990$$

Goodstein Sequence

- For any natural number *m*

$$
\begin{aligned}
&m \\
&Grow_2(m) \\
&Grow_3(Grow_2(m)) \\
&Grow_4(Grow_3(Grow_2(m))), \\
&\ldots
\end{aligned}
$$

Sample Values

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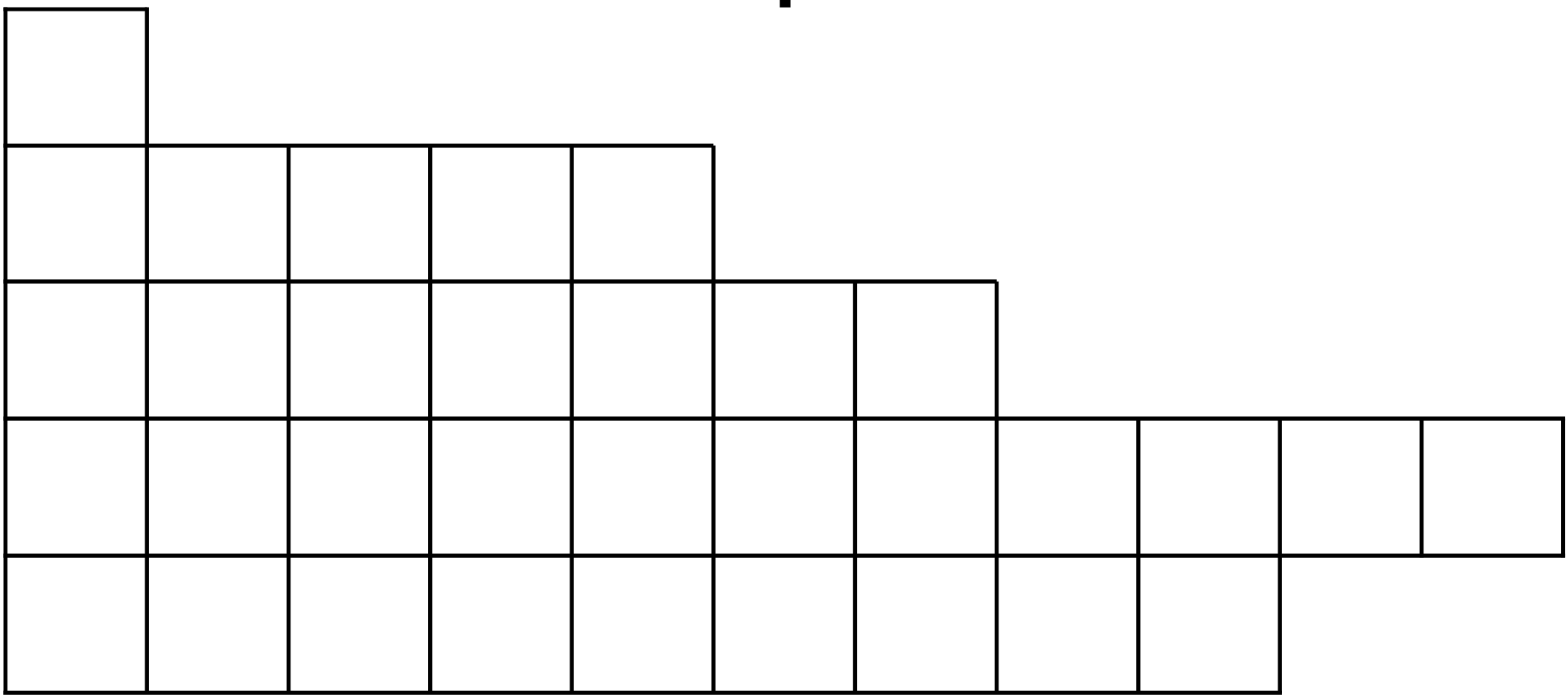

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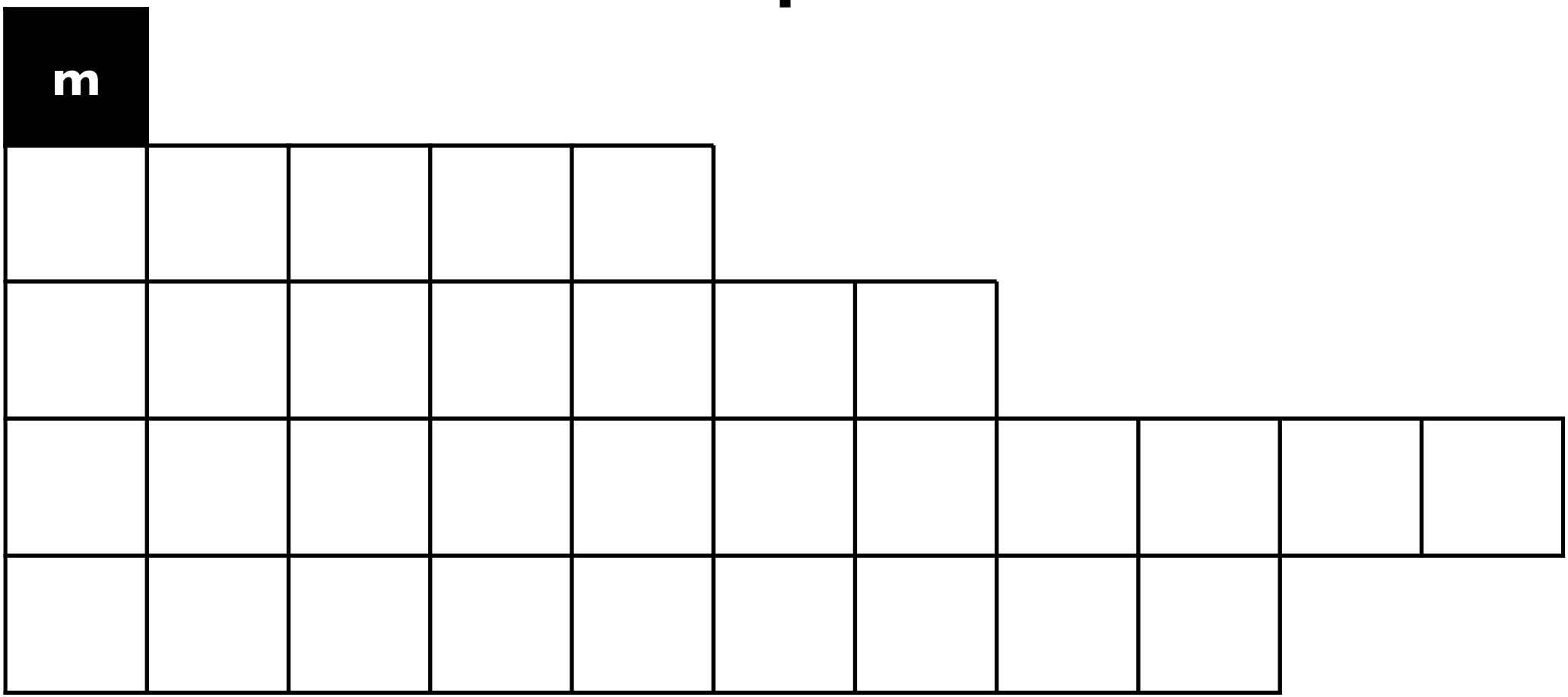

Sample Values

m
2
2
2
1
0

Sample Values

m										
2	2	2	1	0						
3	3	3	3	2	1	0				

Sample Values

m										
2	2	2	1	0						
3	3	3	3	2	1	0				
4	4	26	41	60	83	109	139	...	11327 (96th term)	...

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m										
2	2	2	1	0						
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4	4	26	41	60	83	109	139	...	11327 (96th term)	...
5	15	$\sim 10^{13}$	$\sim 10^{155}$	$\sim 10^{2185}$	$\sim 10^{36306}$	10^{695975}	$10^{15151337}$	...		

This sequence actually goes to zero!

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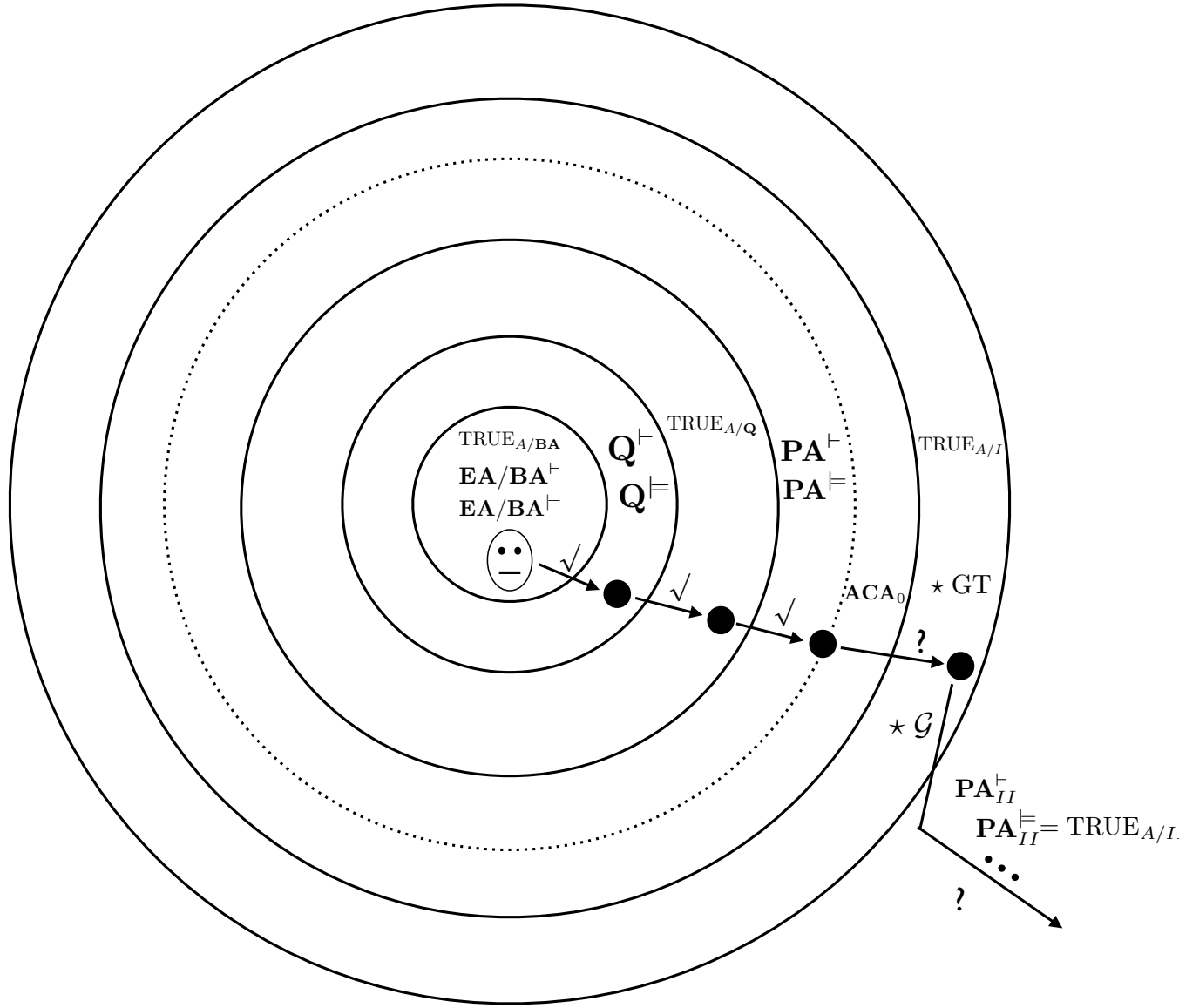

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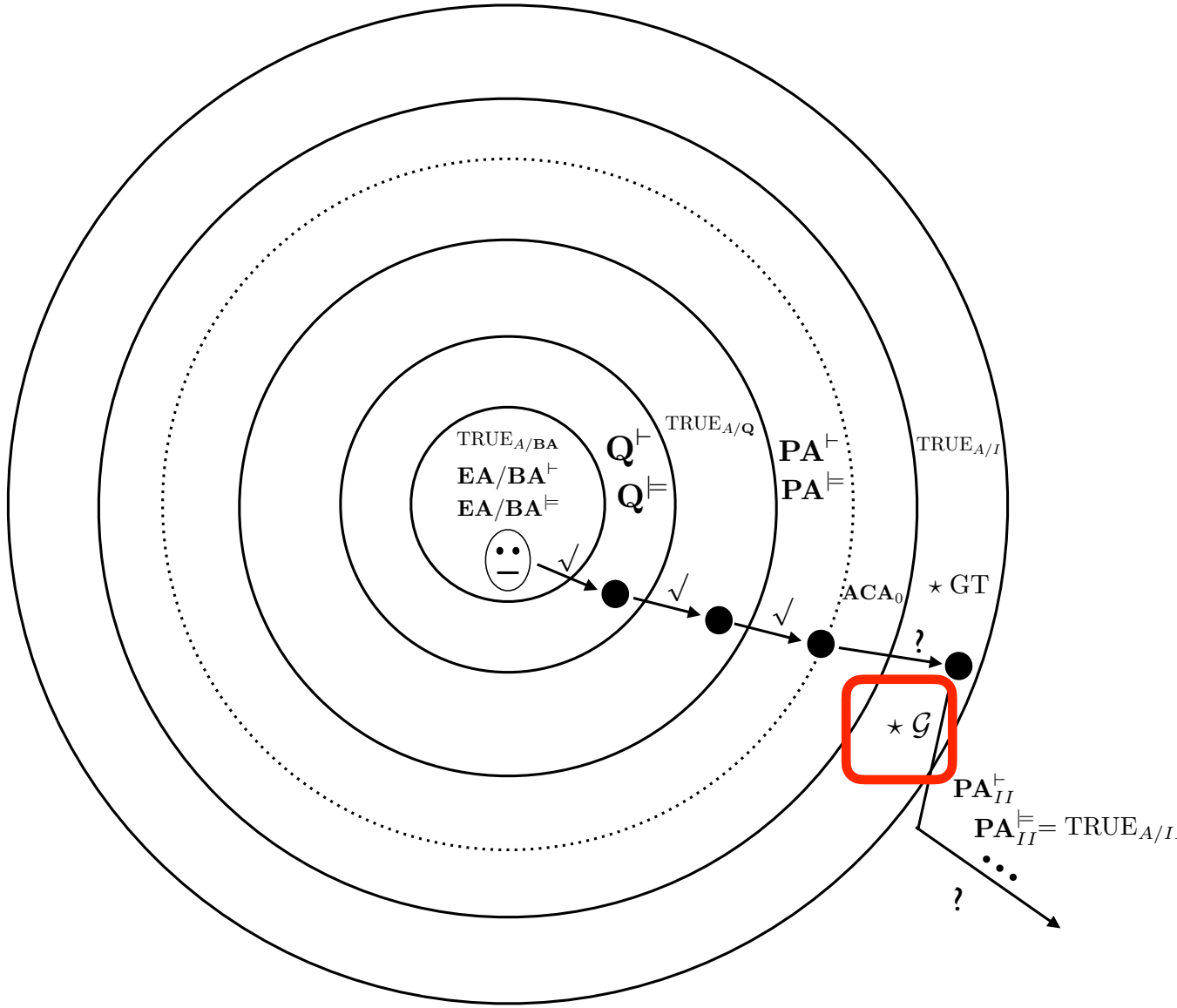

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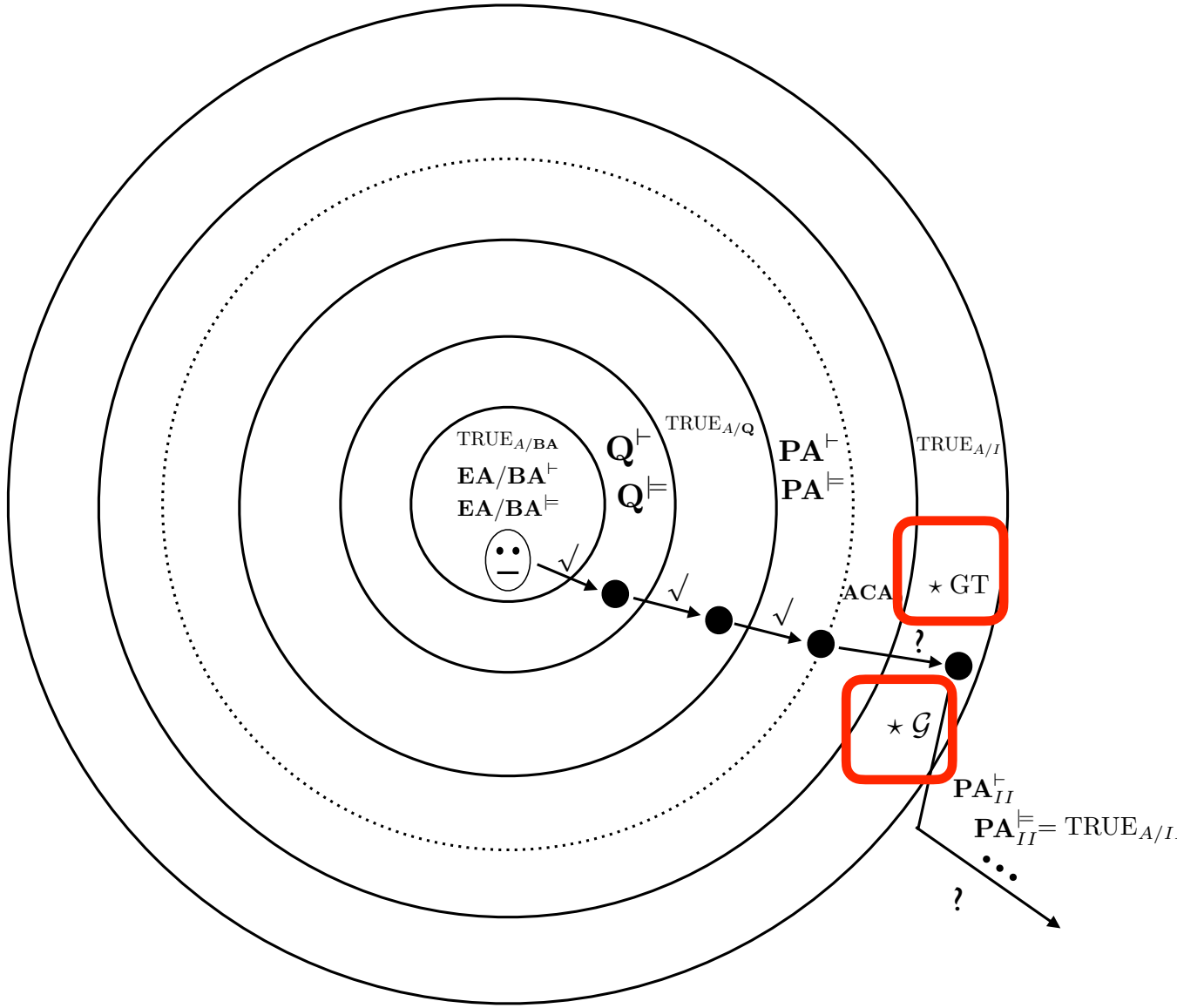

Med nok penger, kan logikk løse alle problemer.