Propositional Calculus III:
Reductio ad Absurdum

Selmer Bringsjord

Rensselaer AI & Reasoning (RAIR) Lab
Department of Cognitive Science
Department of Computer Science
Lally School of Management & Technology
Rensselaer Polytechnic Institute (RPI)
Troy, New York 12180 USA

Intro to Logic
2/4/2019

R A I R
Rensselaer AI and Reasoning Lab

Logistics …

Logistics …

Any questions?

E-Housekeeping Pts, Again

E-Housekeeping Pts, Again

- Make sure OS fully up-to-date.

E-Housekeeping Pts, Again

- Make sure OS fully up-to-date.
- Make sure browser fully up-to-date.

E-Housekeeping Pts, Again

- Make sure OS fully up-to-date.
- Make sure browser fully up-to-date.
- Chrome best (but I use Safari).

E-Housekeeping Pts, Again

- Make sure OS fully up-to-date.
- Make sure browser fully up-to-date.
- Chrome best (but I use Safari).
- Always work in the same browser window with multiple tabs; must do this with email and HyperGrader & HyperSlate.

Couple of HyperSlate Pts

Couple of HyperSlate Pts

- Consider using right-clicking on nodes.

Couple of HyperSlate Pts

- Consider using right-clicking on nodes.
- “Where are my files?”

Couple of HyperSlate Pts

- Consider using right-clicking on nodes.
- “Where are my files?”
- We are not covering all inference rules/schemata in class.

Couple of HyperSlate Pts

- Consider using right-clicking on nodes.
- "Where are my files?"
- We are not covering all inference rules/schemata in class.
  - `conditional intro`, briefly

Schedule …

Schedule …

How many have taken
a programming course?

Schedule …

How many have taken
a programming course?

How many have learned a
logic-programming language?

Schedule …

How many have taken
a programming course?

How many have learned a
logic-programming language?

Thursday …

Propositional Calculus III:
Reductio ad Absurdum

Selmer Bringsjord

Rensselaer AI & Reasoning (RAIR) Lab
Department of Cognitive Science
Department of Computer Science
Lally School of Management & Technology
Rensselaer Polytechnic Institute (RPI)
Troy, New York 12180 USA

Intro to Logic
3/4/2019

RAIR
Rensselaer AI and Reasoning Lab

Reductio …

“Reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.”

–G. H. Hardy

A Greek-shocking Example …

$$\frac{p}{q}$$

What are rational numbers?

$$\frac{p}{q}$$

What are rational numbers?

- Any number that can be expressed in the form of $\frac{p}{q}$ such that we have a numerator p and a non-zero denominator.

- Rational numbers are a subset of real numbers!

- Examples of *ir*rational numbers?

What are rational numbers?

- Any number that can be expressed in the form of $\frac{p}{q}$ such that we have a numerator p and a non-zero denominator.

- Rational numbers are a subset of real numbers!

- Examples of *ir*rational numbers?

$$\sqrt{2}$$

What are rational numbers?

- Any number that can be expressed in the form of $\frac{p}{q}$ such that we have a numerator p and a non-zero denominator.

- Rational numbers are a subset of real numbers!

- Examples of *ir*rational numbers?

$$\sqrt{2} \quad e$$

Prove that:

Prove that:

$$\sqrt{2}$$

Prove that:

$\sqrt{2}$ is irrational

Suppose $\sqrt{2}$ is **rational.** That means it can be written as the ratio of two integers p and q

$$\sqrt{2} = \frac{p}{q} \tag{1}$$

where we may assume that p **and** q **have no common factors**. (If there are any common factors we cancel them in the numerator and denominator.) Squaring in (1) on both sides gives

$$2 = \frac{p^2}{q^2} \tag{2}$$

which implies

$$p^2 = 2q^2 \tag{3}$$

Thus p^2 is even. The only way this can be true is that p itself is even. But then p^2 is actually divisible by 4. Hence q^2 and therefore q must be even. So p and q are both even which is a contradiction to our assumption that they have no common factors. The square root of 2 cannot be rational!

And now, what are *prime* numbers?

And now, what are *prime* numbers?

- A number that can be divided by only two numbers, one and itself.
- Must be a whole number.
- Example: 2,3,5,7……

And recall: Euclidean "Magic"

Theorem: There are infinitely many primes.

Proof: We take an indirect route. Let $\Pi = p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_k$ be a finite, exhaustive consecutive sequence of prime numbers. Next, let $\mathbf{M}_\Pi$ be $p_1 \times p_2 \times \cdots \times p_k$, and set $\mathbf{M}'_\Pi$ to $\mathbf{M}_\Pi + 1$. Either $\mathbf{M}'_\Pi$ is prime, or not; we thus have two (exhaustive) cases to consider.

- C1 Suppose $\mathbf{M}'_\Pi$ is prime. In this case we immediately have a prime number beyond any in Π — contradiction!
- C2 Suppose on the other hand that $\mathbf{M}'_\Pi$ is *not* prime. Then some prime p divides $\mathbf{M}'_\Pi$. (Why?) Now, p itself is either in Π, or not; we hence have two sub-cases. Supposing that p is in Π entails that p divides $\mathbf{M}_\Pi$. But we are operating under the supposition that p divides $\mathbf{M}'_\Pi$ as well. This implies that p divides 1, which is absurd (a contradiction). Hence the prime p is outside Π.

Hence for *any* such list Π, there is a prime outside the list. That is, there are infinitely many primes. **QED**

And recall: Euclidean "Magic"

Theorem: There are infinitely many primes.

Proof: We take an indirect route. Let $\Pi = p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_k$ be a finite, exhaustive consecutive sequence of prime numbers. Next, let $\mathbf{M}_\Pi$ be $p_1 \times p_2 \times \cdots \times p_k$, and set $\mathbf{M}'_\Pi$ to $\mathbf{M}_\Pi + 1$. Either $\mathbf{M}'_\Pi$ is prime, or not; we thus have two (exhaustive) cases to consider.

- C1 Suppose $\mathbf{M}'_\Pi$ is prime. In this case we immediately have a prime number beyond any in Π — contradiction!

- C2 Suppose on the other hand that $\mathbf{M}'_\Pi$ is *not* prime. Then some prime p divides $\mathbf{M}'_\Pi$. (Why?) Now, p itself is either in Π, or not; we hence have two sub-cases. Supposing that p is in Π entails that p divides $\mathbf{M}_\Pi$. But we are operating under the supposition that p divides $\mathbf{M}'_\Pi$ as well. This implies that p divides 1, which is absurd (a contradiction). Hence the prime p is outside Π.

Hence for *any* such list Π, there is a prime outside the list. That is, there are infinitely many primes. **QED**

And recall: Euclidean "Magic"

Theorem: There are infinitely many primes.

Proof: We take an indirect route. Let $\Pi = p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_k$ be a finite, exhaustive consecutive sequence of prime numbers. Next, let $\mathbf{M}_\Pi$ be $p_1 \times p_2 \times \cdots \times p_k$, and set $\mathbf{M}'_\Pi$ to $\mathbf{M}_\Pi + 1$. Either $\mathbf{M}'_\Pi$ is prime, or not; we thus have two (exhaustive) cases to consider.

- C1 Suppose $\mathbf{M}'_\Pi$ is prime. In this case we immediately have a prime number beyond any in Π — contradiction!
- C2 Suppose on the other hand that $\mathbf{M}'_\Pi$ is *not* prime. Then some prime p divides $\mathbf{M}'_\Pi$. (Why?) Now, p itself is either in Π, or not; we hence have two sub-cases. Supposing that p is in Π entails that p divides $\mathbf{M}_\Pi$. But we are operating under the supposition that p divides $\mathbf{M}'_\Pi$ as well. This implies that p divides 1, which is absurd (a contradiction). Hence the prime p is outside Π.

Hence for *any* such list Π, there is a prime outside the list. That is, there are infinitely many primes. **QED**

And recall: Euclidean "Magic"

Theorem: There are infinitely many primes.

Proof: We take an indirect route. Let $\Pi = p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_k$ be a finite, exhaustive consecutive sequence of prime numbers. Next, let $\mathbf{M}_\Pi$ be $p_1 \times p_2 \times \cdots \times p_k$, and set $\mathbf{M}'_\Pi$ to $\mathbf{M}_\Pi + 1$. Either $\mathbf{M}'_\Pi$ is prime, or not; we thus have two (exhaustive) cases to consider.

- C1 Suppose $\mathbf{M}'_\Pi$ is prime. In this case we immediately have a prime number beyond any in Π — contradiction!
- C2 Suppose on the other hand that $\mathbf{M}'_\Pi$ is *not* prime. Then some prime p divides $\mathbf{M}'_\Pi$. (Why?) Now, p itself is either in Π, or not; we hence have two sub-cases. Supposing that p is in Π entails that p divides $\mathbf{M}_\Pi$. But we are operating under the supposition that p divides $\mathbf{M}'_\Pi$ as well. This implies that p divides 1, which is absurd (a contradiction). Hence the prime p is outside Π.

Hence for *any* such list Π, there is a prime outside the list. That is, there are infinitely many primes. **QED**

And recall: Euclidean "Magic"

Theorem: There are infinitely many primes.

Proof: We take an indirect route. Let $\Pi = p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_k$ be a finite, exhaustive consecutive sequence of prime numbers. Next, let M_Π be $p_1 \times p_2 \times \cdots \times p_k$, and set M'_Π to $\mathrm{M}_\Pi + 1$. Either M'_Π is prime, or not; we thus have two (exhaustive) cases to consider.

C1 Suppose M'_Π is prime. In this case we immediately have a prime number beyond any in Π — contradiction!

C2 Suppose on the other hand that M'_Π is *not* prime. Then some prime p divides M'_Π. (Why?) Now, p itself is either in Π, or not; we hence have two sub-cases. Supposing that p is in Π entails that p divides M_Π. But we are operating under the supposition that p divides M'_Π as well. This implies that p divides 1, which is absurd (a contradiction). Hence the prime p is outside Π.

Hence for *any* such list Π, there is a prime outside the list. That is, there are infinitely many primes. **QED**

And recall: Euclidean "Magic"

Theorem: There are infinitely many primes.

Proof: We take an indirect route. Let $\Pi = p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_k$ be a finite, exhaustive consecutive sequence of prime numbers. Next, let M_Π be $p_1 \times p_2 \times \cdots \times p_k$, and set M'_Π to $\mathrm{M}_\Pi + 1$. Either M'_Π is prime, or not; we thus have two (exhaustive) cases to consider.

Notice that this proof uses two applications of *indirect proof,* and one application of *proof by cases.*

- C1 Suppose M'_Π is prime. In this case we immediately have a prime number beyond any in Π — contradiction!

- C2 Suppose on the other hand that M'_Π is *not* prime. Then some prime p divides M'_Π. (Why?) Now, p itself is either in Π, or not; we hence have two sub-cases. Supposing that p is in Π entails that p divides M_Π. But we are operating under the supposition that p divides M'_Π as well. This implies that p divides 1, which is absurd (a contradiction). Hence the prime p is outside Π.

Hence for *any* such list Π, there is a prime outside the list. That is, there are infinitely many primes. **QED**

And recall: Euclidean "Magic"

Theorem: There are infinitely many primes.

Proof: We take an indirect route. Let $\Pi = p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_k$ be a finite, exhaustive consecutive sequence of prime numbers. Next, let M_Π be $p_1 \times p_2 \times \cdots \times p_k$, and set M'_Π to $M_\Pi + 1$. Either M'_Π is prime, or not; we thus have two (exhaustive) cases to consider.

Notice that this proof uses two applications of *indirect proof*, and one application of *proof by cases*.

C1 Suppose M'_Π is prime. [illegible] prime number beyond any in Π — contradiction!

C2 Suppose on the other hand that M'_Π is *not* prime. Then some prime p [illegible] two sub-cases. Supposing that p is in Π entails that p divides M_Π. But we are operating under the supposition that p divides M'_Π as well. This implies that p divides 1, which is absurd (a contradiction). Hence the prime p is outside Π.

Study it word by word until you endorse it with your very soul!

Hence for *any* such list Π, there is a prime outside the list. That is, there are infinitely many primes. **QED**

From Algebra 2 in High School …

(Pearson Common-Core Compliant Textbook)

A proof involving indirect reasoning is an **indirect proof.** Often in an indirect proof, a statement and its negation are the only possibilities. When you see that one of these possibilities leads to a conclusion that contradicts a fact you know to be true, you can eliminate that possibility. For this reason, indirect proof is sometimes called *proof by contradiction*.

TAKE NOTE Key Concept

Writing an Indirect Proof

Step 1 State as a temporary assumption the opposite (negation) of what you want to prove.

Step 2 Show that this temporary assumption leads to a contradiction.

Step 3 Conclude that the temporary assumption must be false and that what you want to prove must be true.

Problem 3 Writing an Indirect Proof

Proof

Given: $\triangle ABC$ is scalene.

Prove: $\angle A$, $\angle B$, and $\angle C$ all have different measures.

THINK	WRITE
Assume temporarily the opposite of what you want to prove.	Assume temporarily that two angles of $\triangle ABC$ have the same measure. Assume that $m\angle A = m\angle B$.
Show that this assumption leads to a contradiction.	By the Converse of the Isosceles Triangle Theorem, the sides opposite $\angle A$ and $\angle B$ are congruent. This contradicts the given information that $\triangle ABC$ is scalene.
Conclude that the temporary assumption must be false and that what you want to prove must be true.	The assumption that two angles of $\triangle ABC$ have the same measure must be false. Therefore, $\angle A$, $\angle B$, and $\angle C$ all have different measures.

SHOW SOLUTION

GOT IT?

Problem 3 Writing an Indirect Proof

Proof

Given: $\triangle ABC$ is scalene.

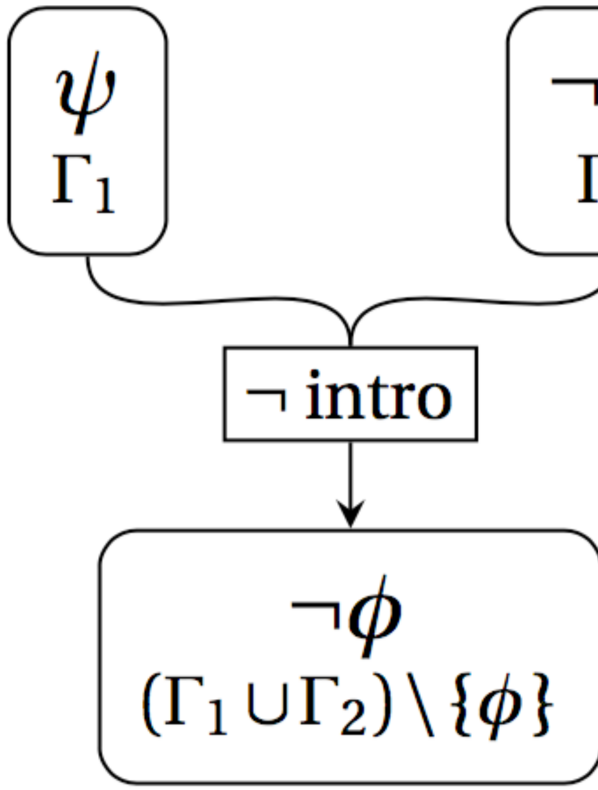
Prove: $\angle A$, $\angle B$, and $\angle C$ all have different measures.

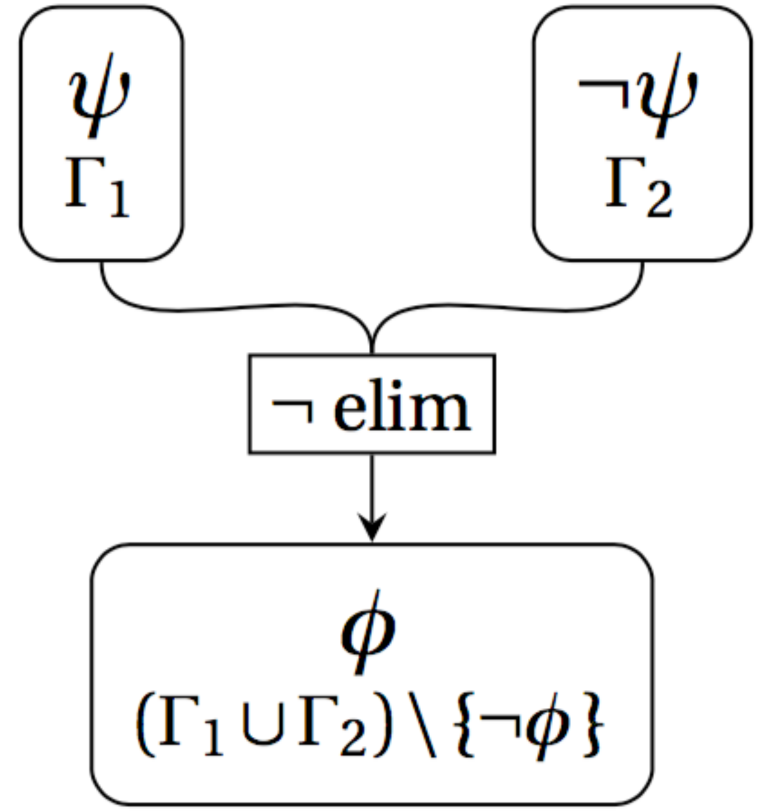
THINK	WRITE
Assume temporarily the opposite of what you want to prove.	Assume temporarily that two angles of $\triangle ABC$ have the same measure. Assume that $m\angle A = m\angle B$.
Show that this assumption leads to a contradiction.	By the Converse of the Isosceles Triangle Theorem, the sides opposite $\angle A$ and $\angle B$ are congruent. This contradicts the given information that $\triangle ABC$ is scalene.
Conclude that the temporary assumption must be false and that what you want to prove must be true.	The assumption that two angles of $\triangle ABC$ have the same measure must be false. Therefore, $\angle A$, $\angle B$, and $\angle C$ all have different measures.

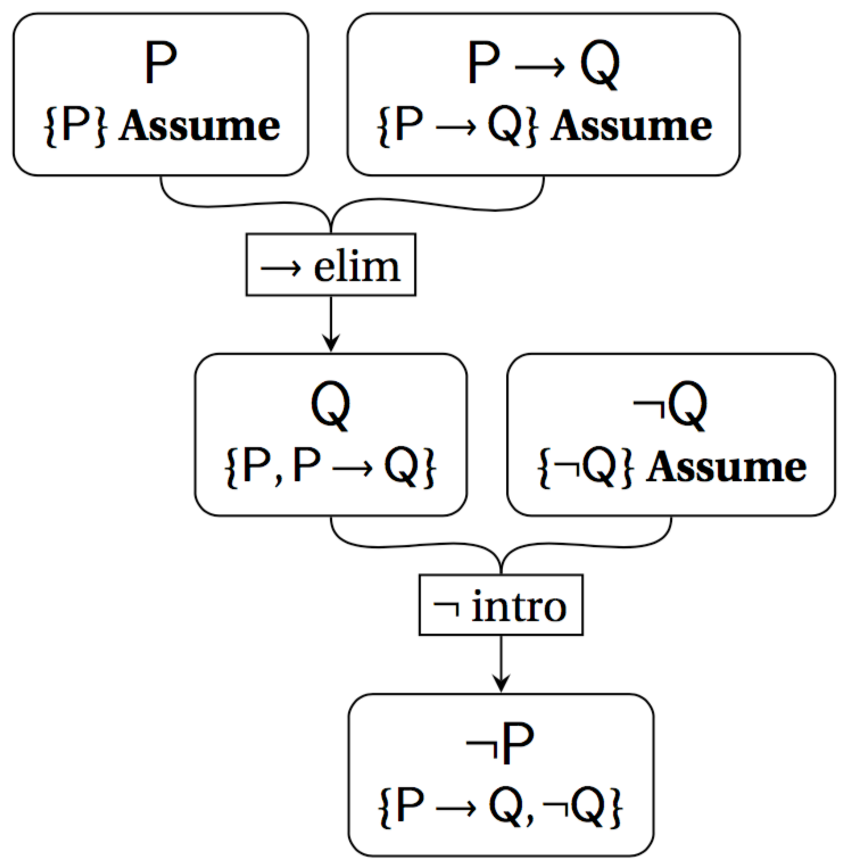
```
@article{moore.proof,
        Author = {R. C. Moore},
        Journal = {Educational Studies in Mathematics},
        Pages = {249–266},
        Title = {Making the Transition to Formal Proof},
        Volume = {27.3},
        Year = 1994}
```

SHOW SOLUTION

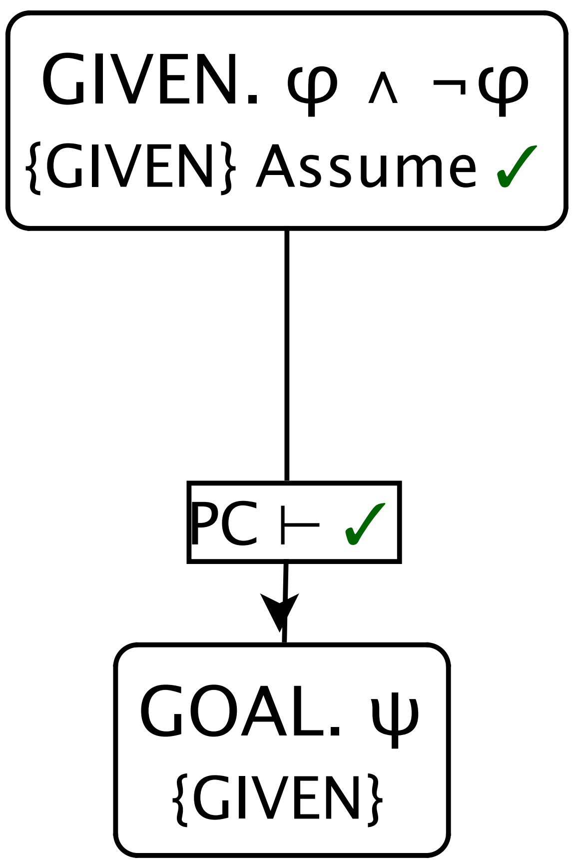
GOT IT?


ψ
Γ_1
$\neg\psi$
Γ_2
$\neg$ intro
$\neg\phi$
$(\Gamma_1 \cup \Gamma_2) \setminus \{\phi\}$

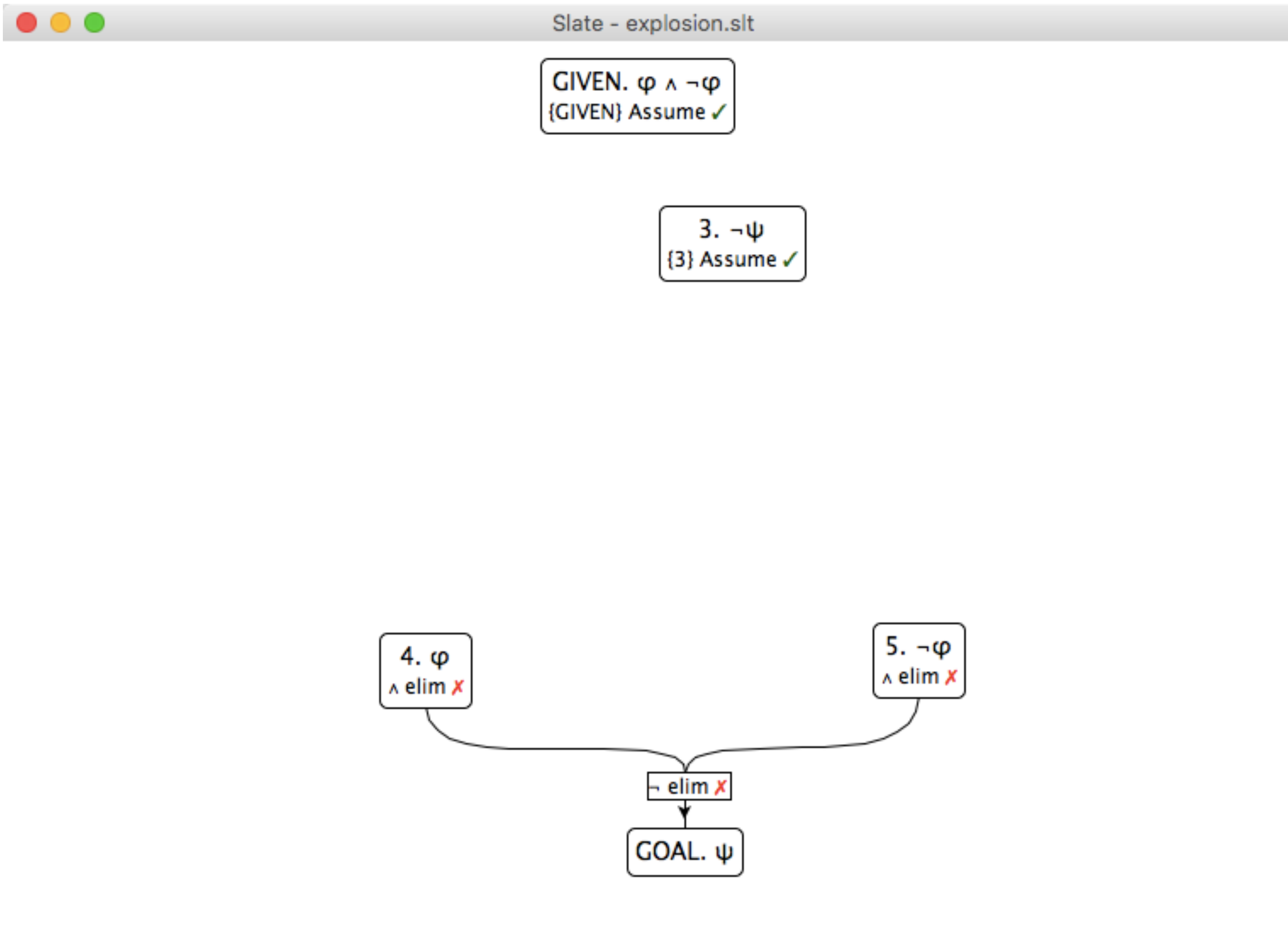

ψ
Γ_1
$\neg\psi$
Γ_2
$\neg$ elim
ϕ
$(\Gamma_1 \cup \Gamma_2) \setminus \{\neg\phi\}$


P
{P} Assume
P → Q
{P → Q} Assume
→ elim
Q
{P, P → Q}
¬Q
{¬Q} Assume
¬ intro
¬P
{P → Q, ¬Q}

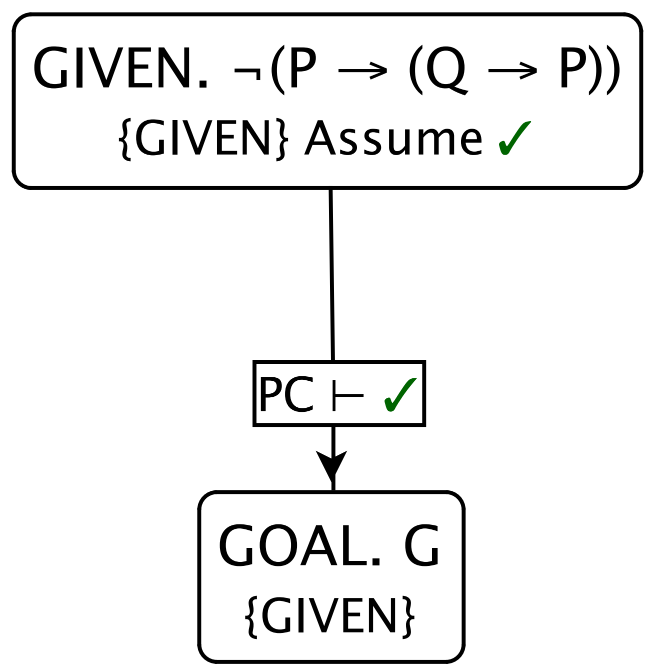
Explosion



Explosion: Partial Proof Plan



GreenCheeseMoon1



GreenCheeseMoon1: Partial Proof Plan