Second-Order Logic and the *k*-order Ladder; Second-Order Axiomatized Arithmetic; Gödel's "God Theorem" & Speedup Theorem

Selmer Bringsjord

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Intro to Logic
3/28/2019

RAIR
Rensselaer AI and Reasoning Lab

FOL

√ FOL

√ FOL

Epistemic + FOL

$\mathbf{B}_d \mathbf{B}_v \mathbf{B}_d V v$

√ FOL

Epistemic + FOL (for coverage of "killer" robots)

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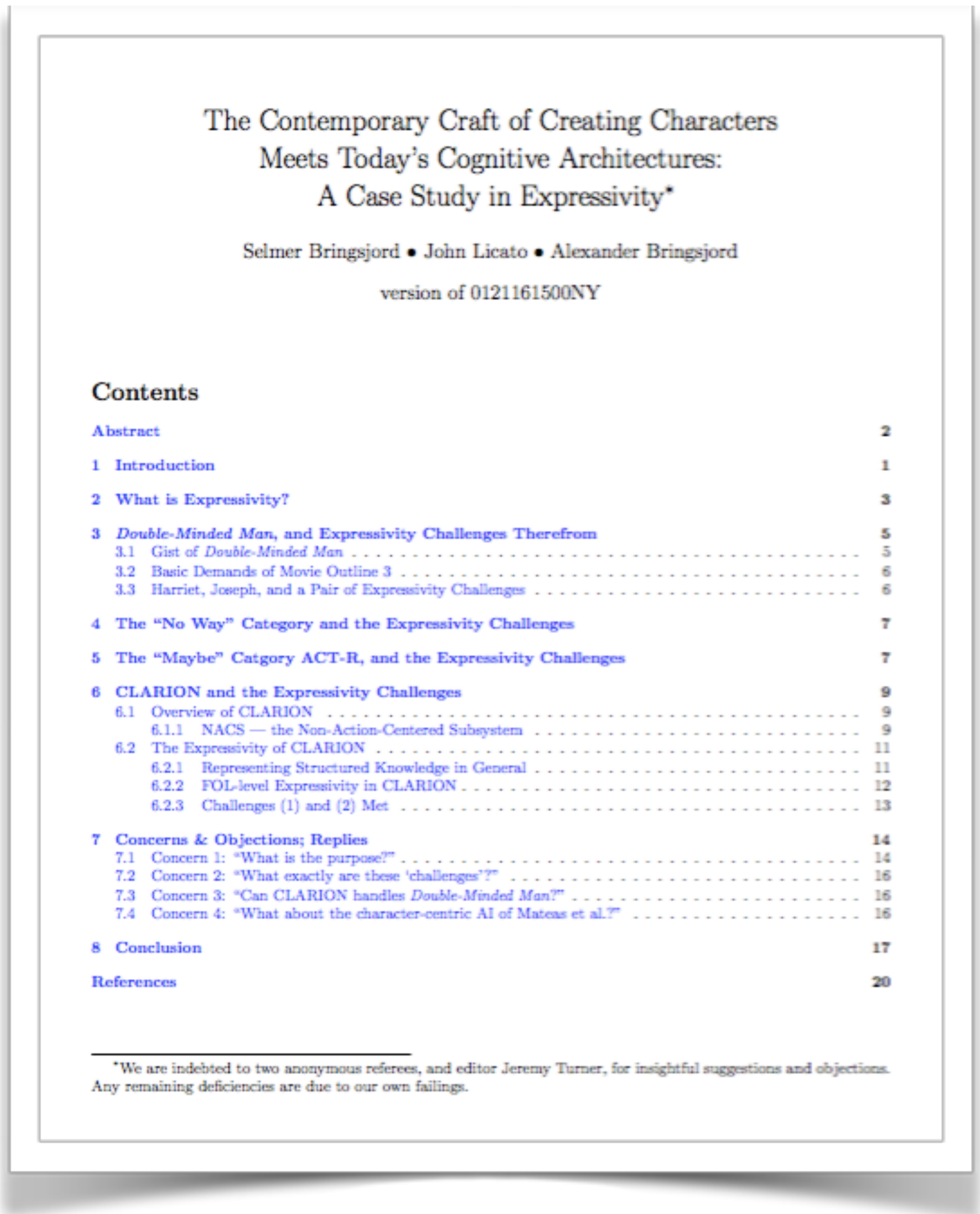
Double-Minded Man



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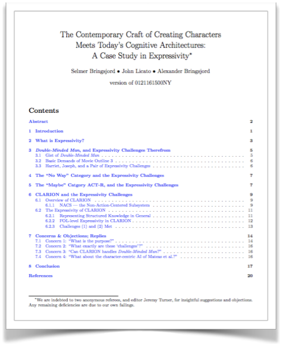

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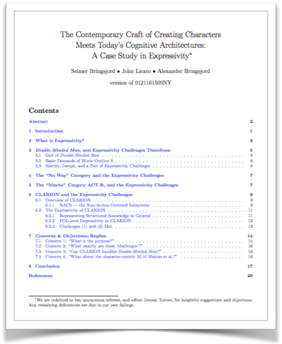
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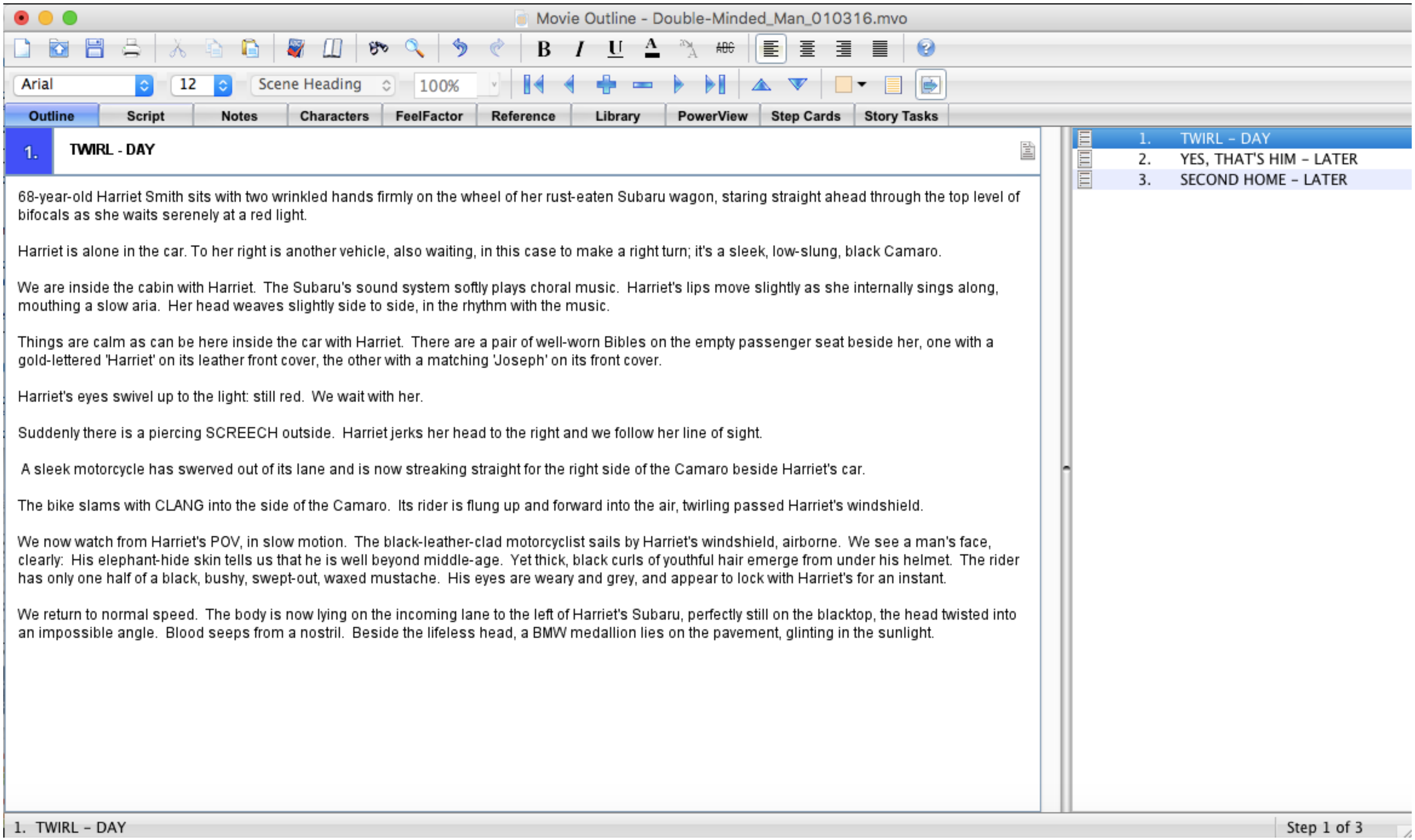

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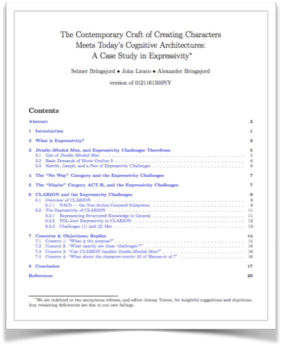

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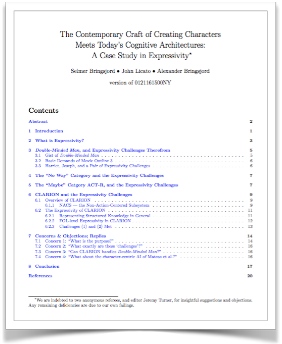

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by

S Bringsjord & A Bringsjord

DRAFT #5

Selmer.Bringsjord@gmail.com

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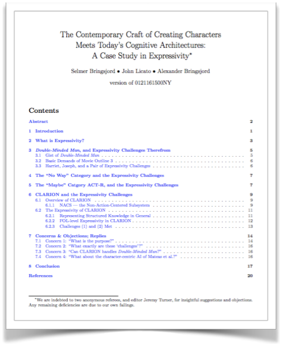
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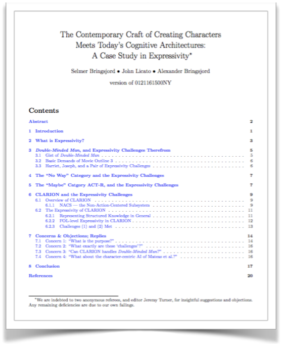

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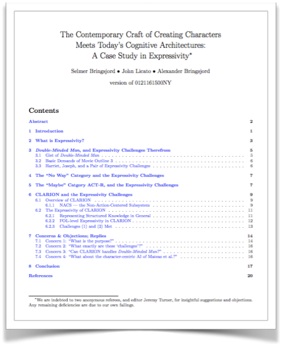
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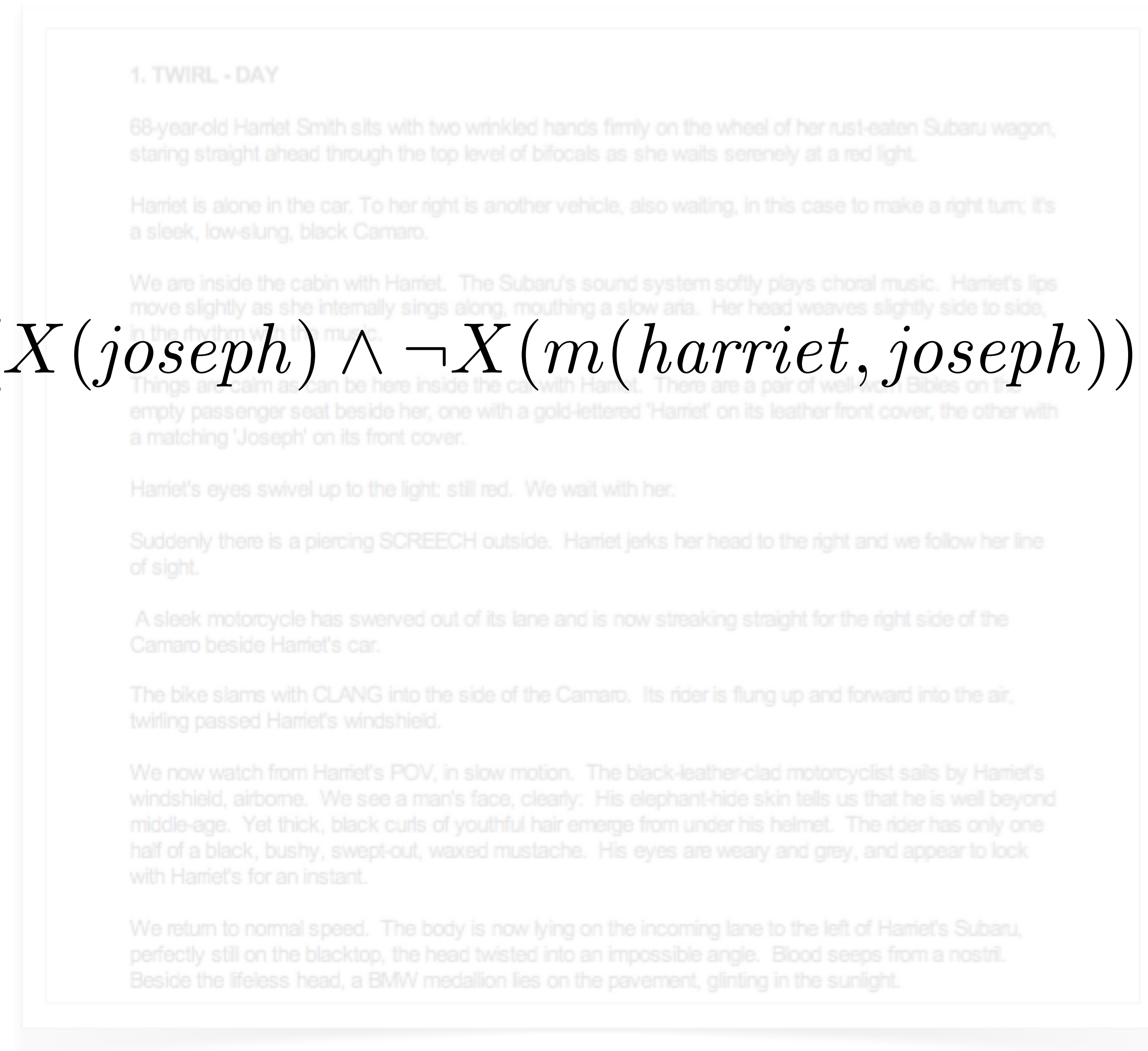
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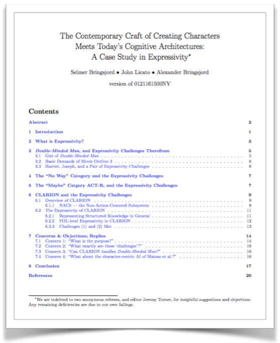

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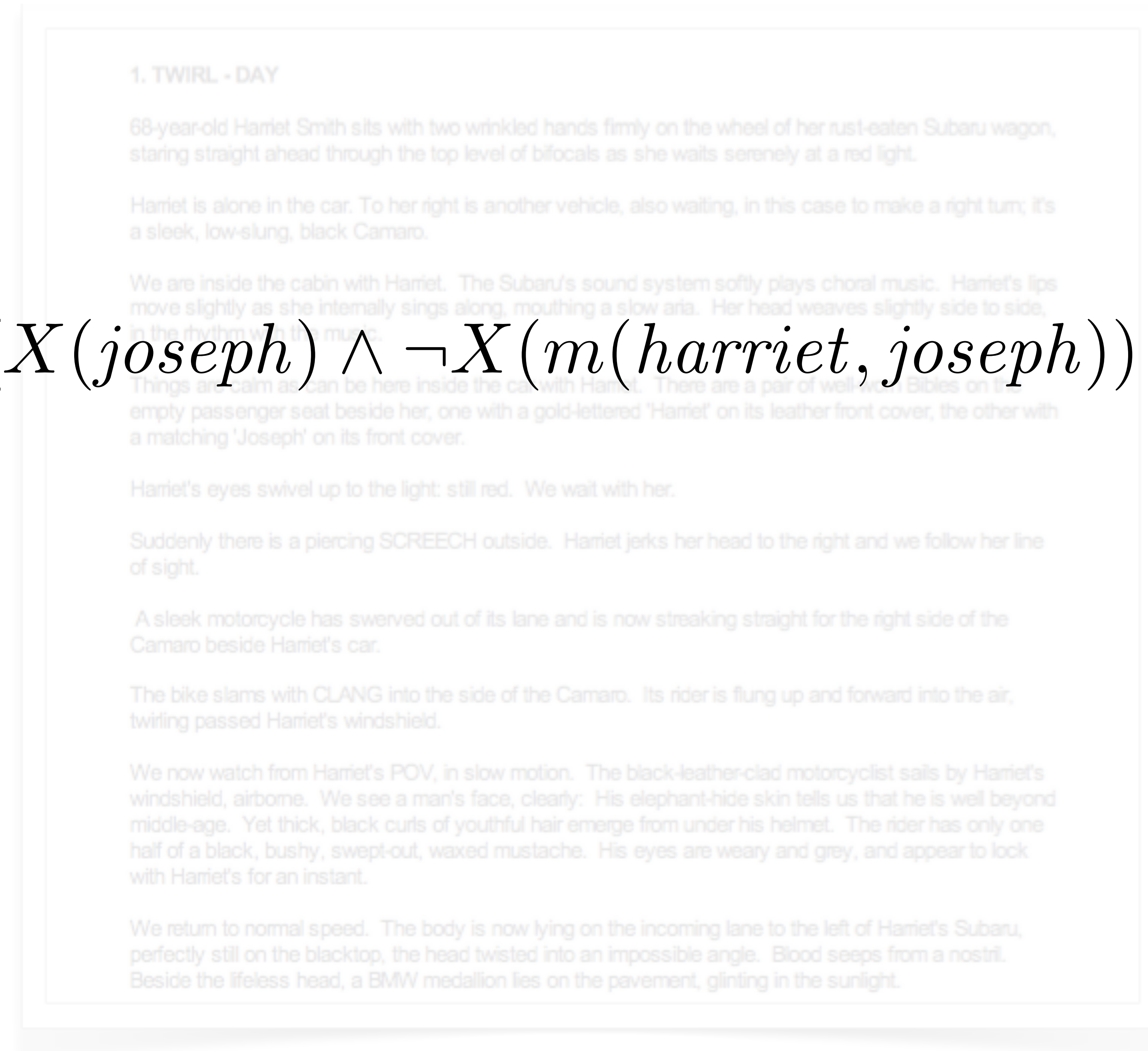

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Climbing the *k*-order Ladder

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a is a llama, as is *b*, *a* likes *b*, and the father of *a* is a llama as well.

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$$Llama(a) \wedge Llama(b) \wedge Likes(a, b) \wedge Llama(fatherOf(a))$$

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$$\exists x \exists y \exists R[R(x) \land R(y) \land Likes(x, y) \land R(fatherOf(x))]$$

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“Leibniz was right that Descartes was right that … : God exists, necessarily.”

Gödel's "God Theorem"

		Part I	
	(1′)	The absence of a positive property is not positive.	premise
	(1)	The absence of a positive property is not positive; and if a property isn't positive, the absence of that property is.	premise
	(2)	Any property entailed by a positive property is itself positive.	premise
∴	(3)	Every positive property P is possibly possessed by something.	(1), (2)
		Part II	
	(4)	Anything that is God has all positive properties.	definition
	(5)	The property of being God is itself a positive property.	premise
∴	(6)	It's possible that God exists.	(3), (5)
		Part III	
	(7)	Positive properties are necessarily positive.	premise
	(8)	A thing x has an essence E if and only if (i) E is a property x has; and (ii) for any property P that x has, x's having this property P is necessarily implied by x's having essence E.	definition
∴	(9)	The property of being God is an essence of any thing that has this property.	(8), (7), (4), (1)
∴	(9)	The property of *being God* ($= G$) is an essential property of any thing that has G.	(8), (7), (4), (1)
		Part IV	
	(10)	A thing has necessary existence if and only if all the essences that thing has imply that something exists and has all those essences.	definition
	(10)	A thing has necessary existence if and only if all the essential properties that thing has imply that something exists and has all those essential properties.	definition
	(11)	Necessary existence is a positive property.	premise
∴	(12)	Necessarily, God exists.	(6), (9), (10), (11) **QED**

Gödel's "God Theorem" (formalized, machine verified)

	(1)	$\forall P\ [Pos(\neg P) \leftrightarrow \neg Pos(P)]$	premise
	(2)	$\forall P_1\ \forall P_2\ \{Pos(P_1) \wedge \Box\forall x\ [P_1(x) \rightarrow P_2(x)] \rightarrow Pos(P_2)\}$	premise
$\therefore$	(3)	$\forall P\ [Pos(P) \rightarrow \Diamond\exists x\ P(x)]$	theorem
	(4)	$\forall x\ [G(x) \leftrightarrow \forall P\ [Pos(P) \rightarrow P(x)]$	definition
	(5)	$Pos(G)$	premise
$\therefore$	(6)	$\Diamond\exists x\ G(x)$	corollary
	(7)	$\forall P\ [Pos(P) \rightarrow \Box Pos(P)]$	premise
	(8)	$\forall x \forall P\ \{Ess(P,x) \leftrightarrow [P(x) \wedge \forall P'\ (P'(x) \rightarrow \Box\forall y(P(y) \rightarrow P'(y))\}$	definition
$\therefore$	(9)	$\forall x\ [G(x) \rightarrow Ess(G,x)]$	theorem
	(10)	$\forall x\ \{NE(x) \leftrightarrow \forall P\ [Ess(P,x) \rightarrow \Box\exists y\ P(y)]\}$	definition
	(11)	$Pos(NE)$	premise
$\therefore$	(12)	$\Box\exists x\ G(x)$ (a.k.a. "Necessarily, God exists.)	theorem

PA = Z_1

$$
\begin{array}{ll}
\text{A1} & \forall x(0 \neq s(x)) \\
\text{A2} & \forall x \forall y(s(x) = s(y) \rightarrow x = y) \\
\text{A3} & \forall x(x \neq 0 \rightarrow \exists y(x = s(y)) \\
\text{A4} & \forall x(x + 0 = x) \\
\text{A5} & \forall x \forall y(x + s(y) = s(x + y)) \\
\text{A6} & \forall x(x \times 0 = 0) \\
\text{A7} & \forall x \forall y(x \times s(y) = (x \times y) + x)
\end{array}
$$

And, every sentence that is the universal closure of an instance of

$$([\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x))] \rightarrow \forall x \phi(x))$$

where $\phi(x)$ is open wff with variable x, and perhaps others, free.

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New!

PA = Z_2

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A2	$\forall x \forall y(s(x) = s(y) \to x = y)$
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Comprehension Schema $\exists X(\forall x X(x) \leftrightarrow \phi(x))$ where $\phi(x) \in C$

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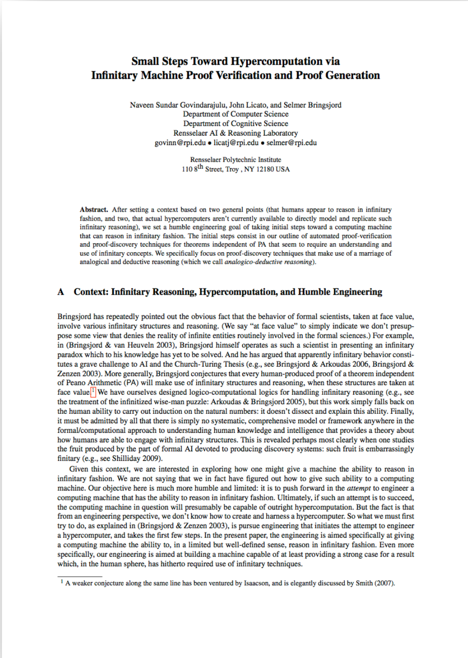

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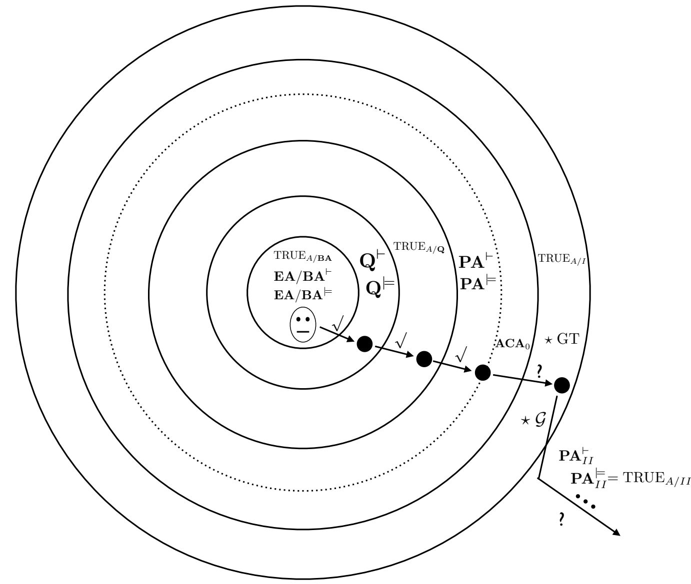
G1



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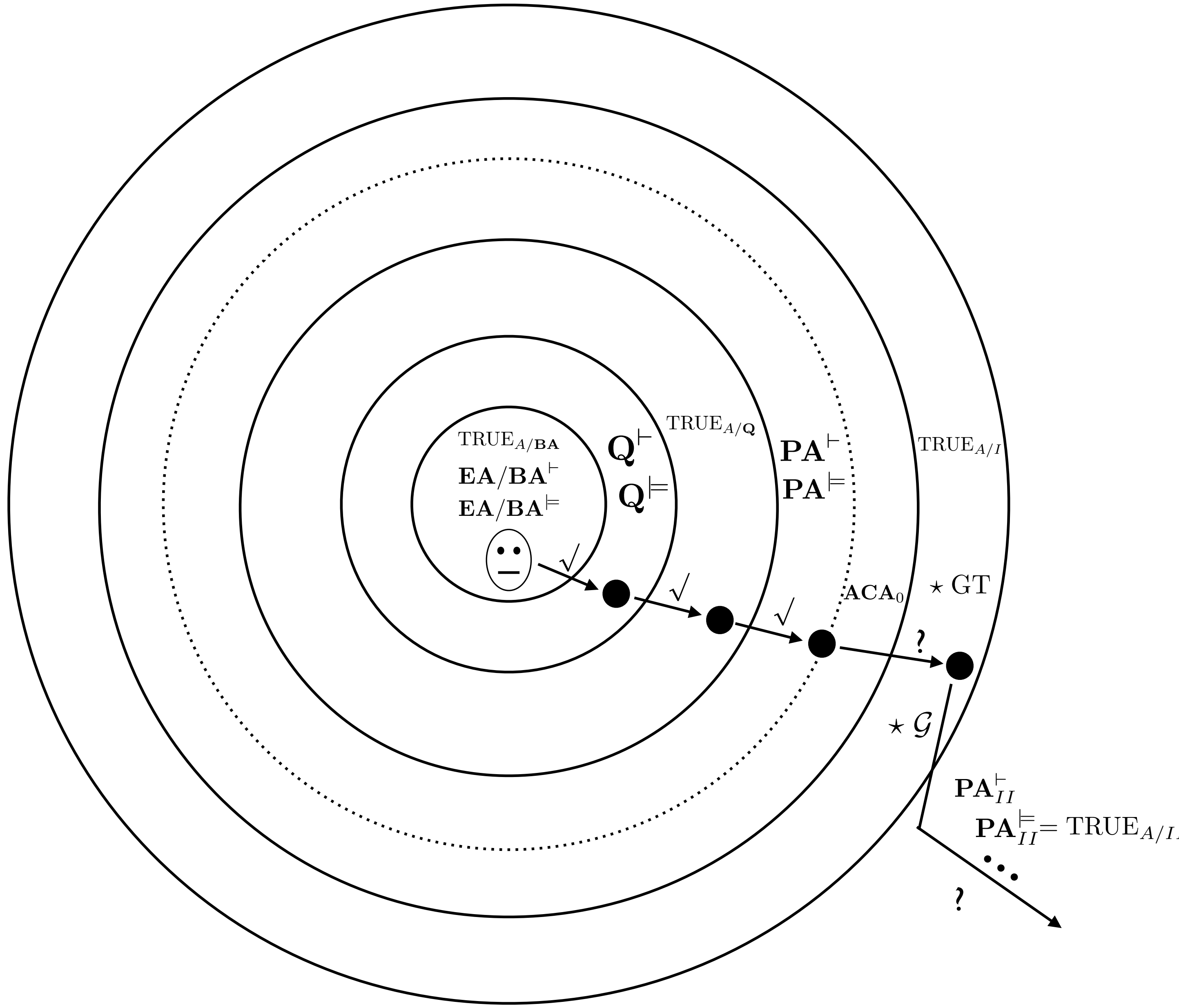

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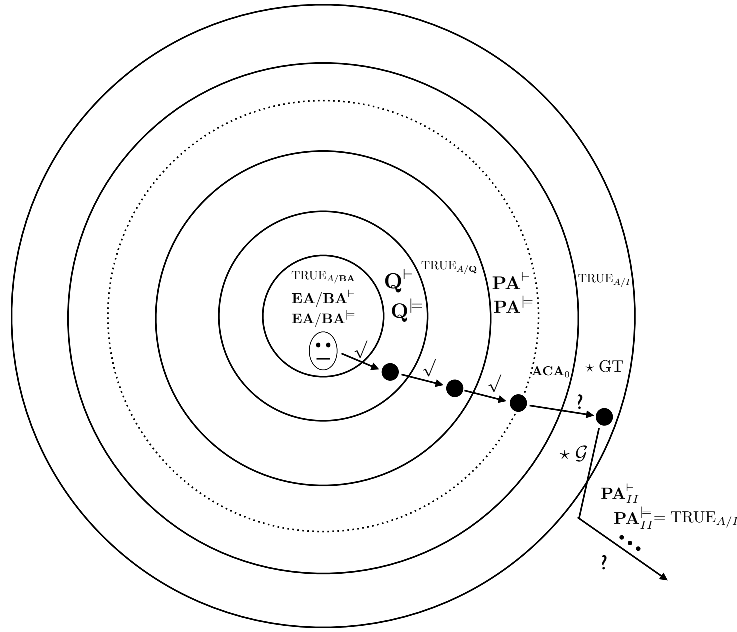

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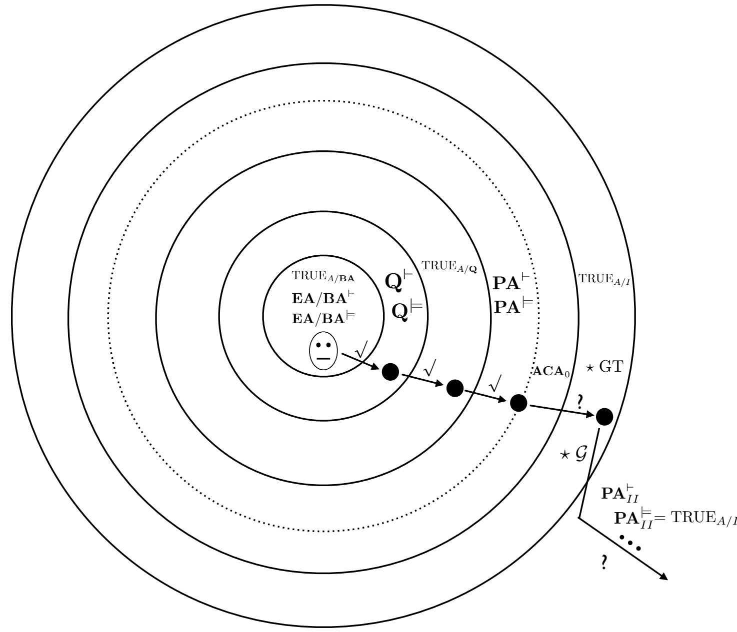

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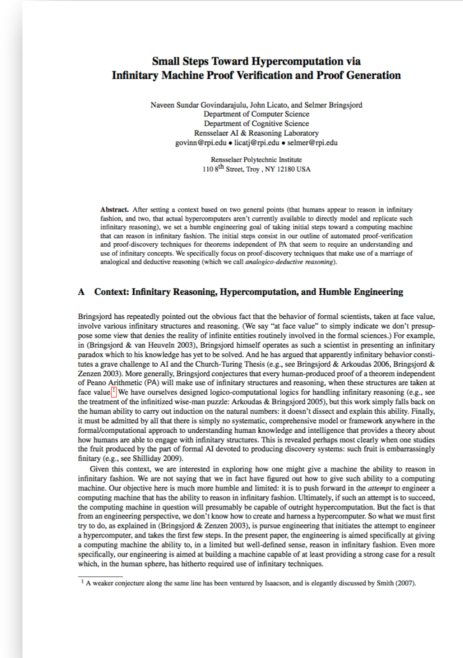

Small Steps Toward Hypercomputation via Infinitary Machine Proof Verification and Proof Generation

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Abstract. After setting a context based on two general points (that humans appear to reason in infinitary fashion, and two, that actual hypercomputers aren't currently available to directly model and replicate such infinitary reasoning), we set a humble engineering goal of taking initial steps toward a computing machine that can reason in infinitary fashion. The initial steps consist in our outline of automated proof-verification and proof-discovery techniques for theorems independent of PA that seem to require an understanding and use of infinitary concepts. We specifically focus on proof-discovery techniques that make use of a marriage of analogical and deductive reasoning (which we call *analogico-deductive reasoning*).

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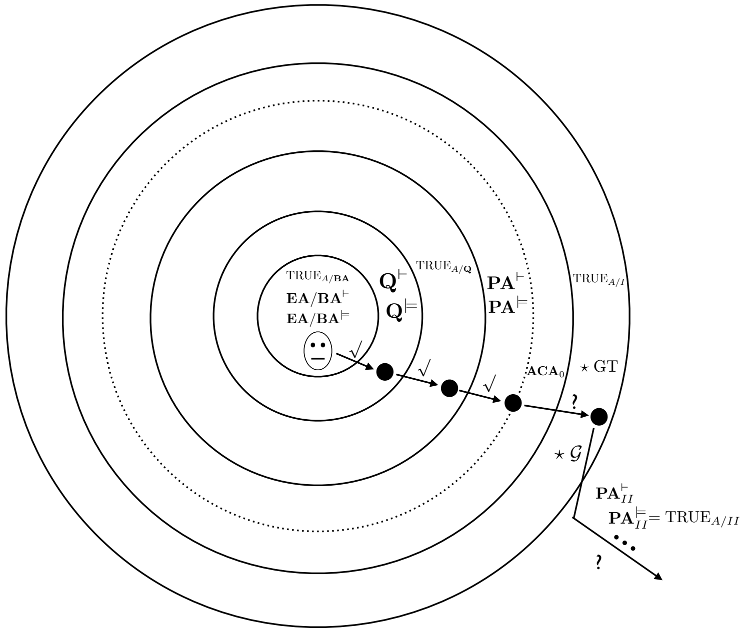
2 The Analogico-Deductive Approach

2.1 Conjecture Generation

G1



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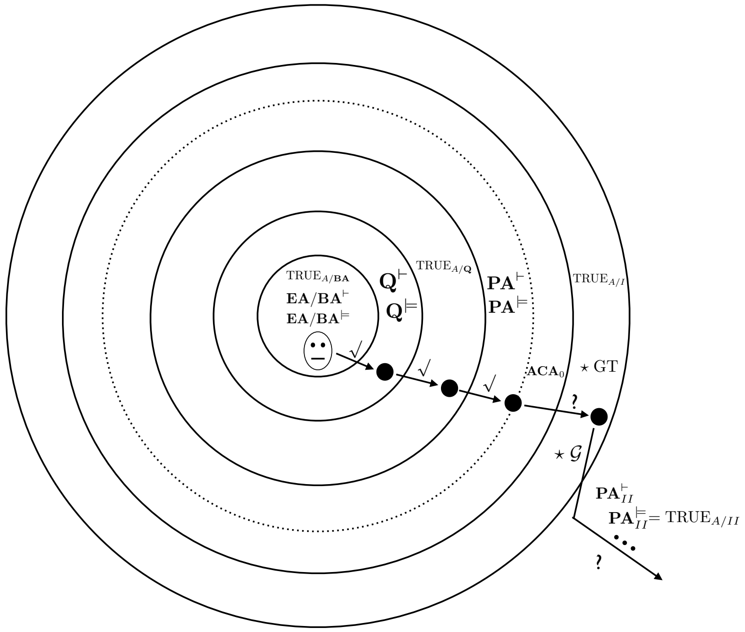

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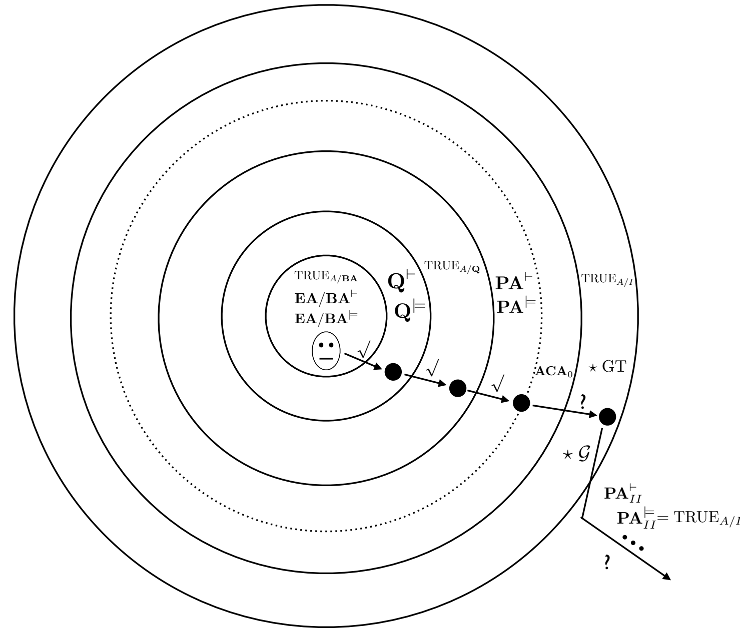

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Likewise, [Amnon, 1993] is an exact parallel of the human-devised proof given by [Kleene, 1996]. Finally, in much more recent and truly impressive work by [Sieg and Field, 2005], there is a move to natural-deduction formats, which we applaud—but the machine essentially begins its processing at a point exceedingly close to where it needs to end up. As Sieg and Field concede: "As axioms we take for granted the representability and derivability conditions for the central syntactic notions as well as the diagonal lemma for constructing self-referential sentences." If one takes for granted such things, finding a proof of **G1** is effortless for a computing machine.[2] In sum, while a lot of commendable work has been done to build the foundation for our prospective work, the daunting formal and engineering challenge of producing a computational system able to produce **G1** without clever seeding from a human remains entirely unmet.

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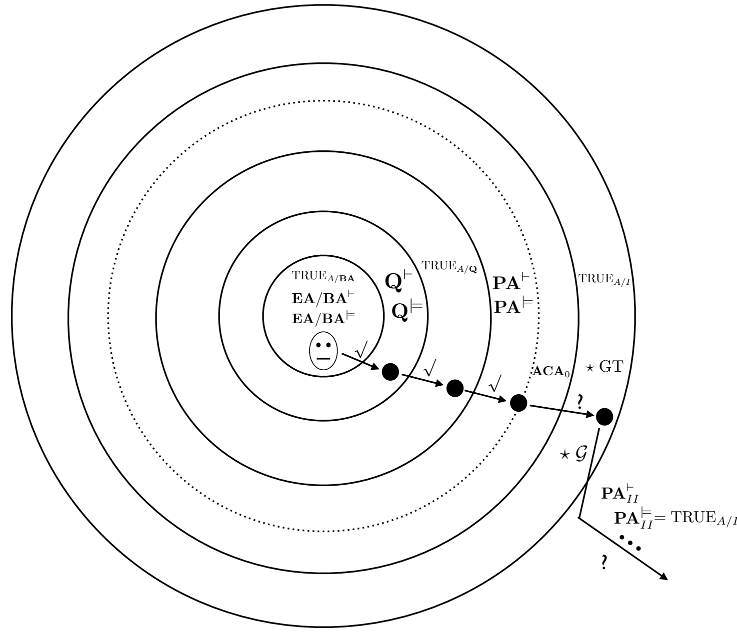
The problem with the purely deductive method is simply that it does not allow us to come close to the type of model-based reasoning that great thinkers are known to have used. Gödel himself has been described as having a "line of thought [which] seems to move from conjecture to conjecture" [Wang, 1995]. Reasoners in general are known to conjecture through analogy when a straightforward answer

[1] **G1** of course applies to any axiom system meeting the standard conditions (Turing-decidability, representability, consistency), but we tend to refer to **PA** for economization.

[2] A video demonstration of the small-distance process can be found at http://kryten.mm.rpi.edu/GodelI_abstract_in_Slate.mov.

G1



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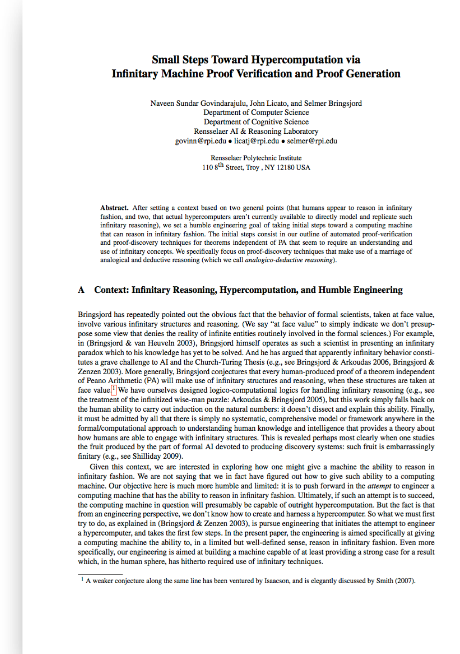
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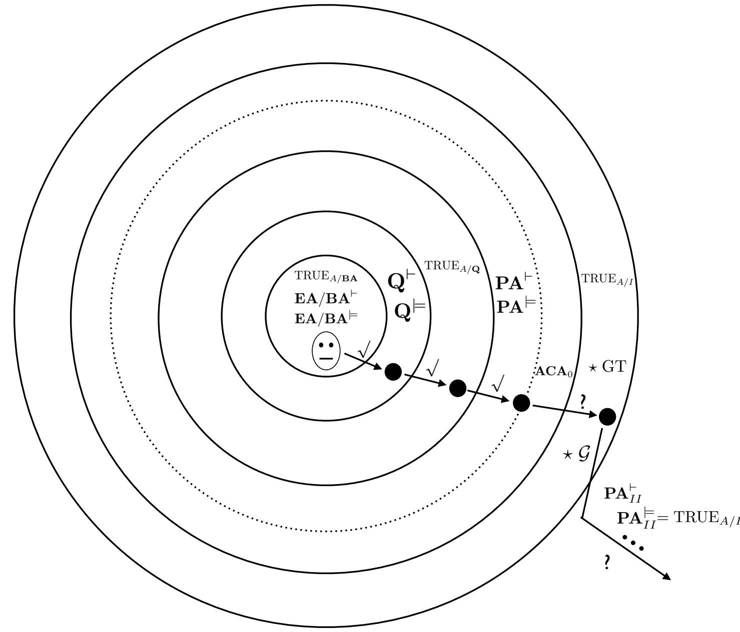

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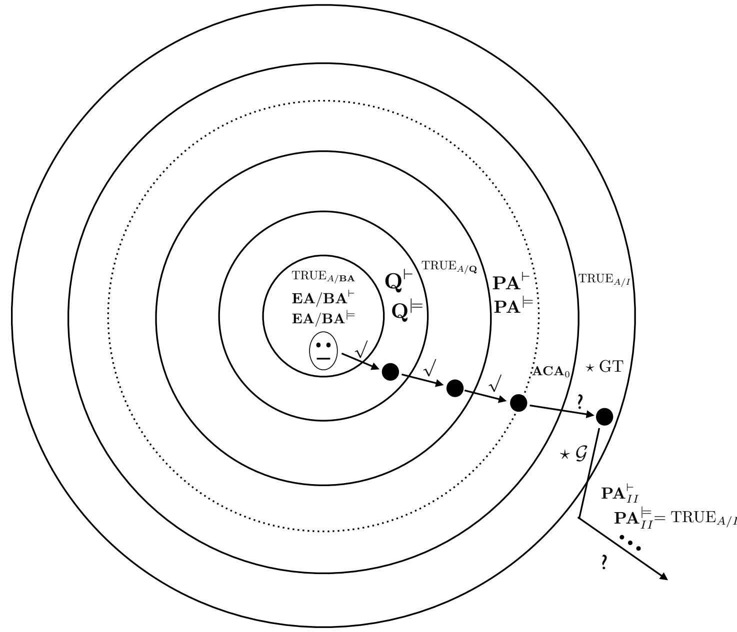

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