

# Propositional Calculus III:

## *Reductio ad Absurdum*

**Selmer Bringsjord**

Rensselaer AI & Reasoning (RAIR) Lab  
Department of Cognitive Science  
Department of Computer Science  
Lally School of Management & Technology  
Rensselaer Polytechnic Institute (RPI)  
Troy, New York 12180 USA

Intro to Logic  
2/6/2020



# Logistics ...

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**Any questions?**

# The Starting Code to Purchase in Bookstore

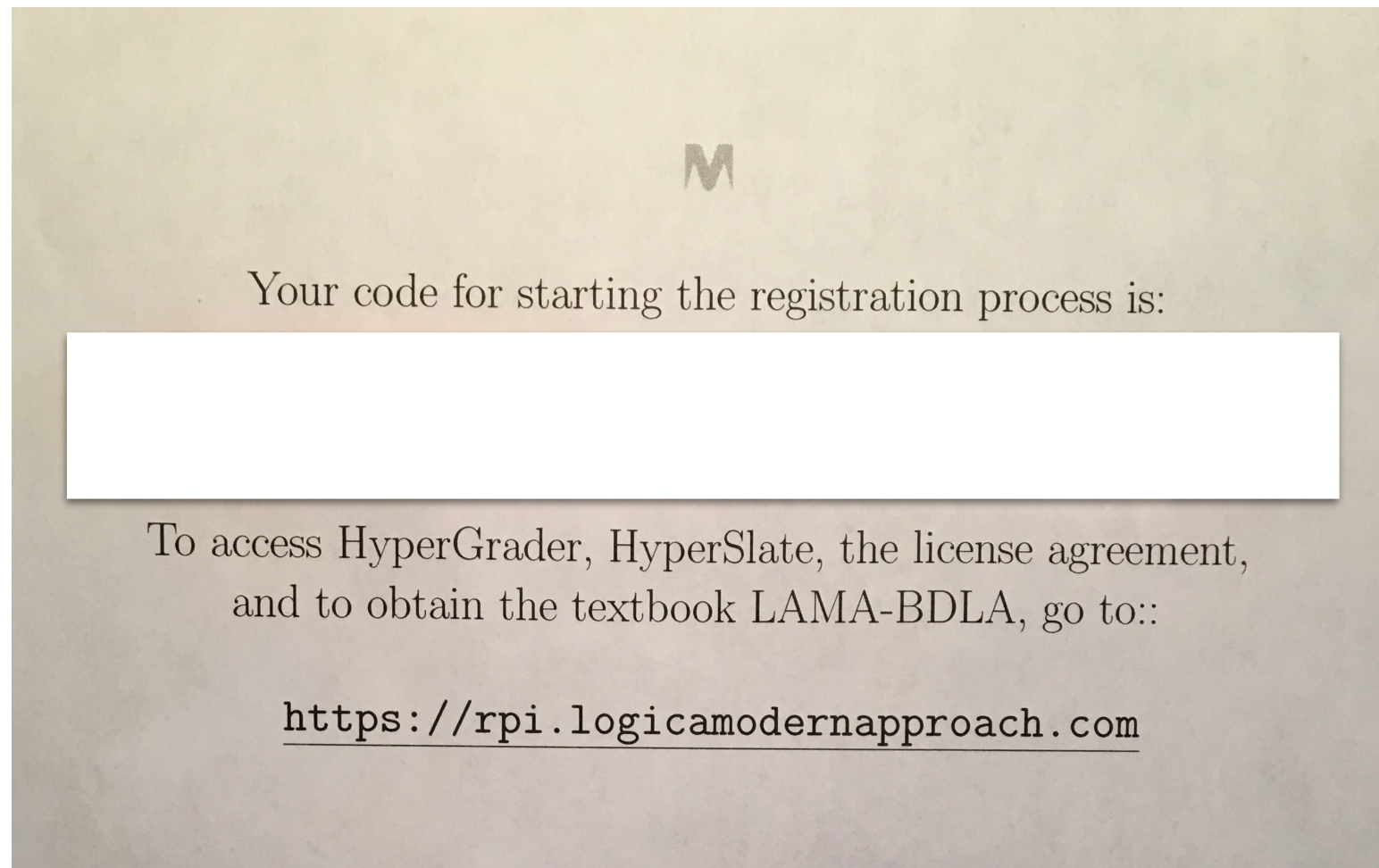
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Your code for starting the registration process is:

To access HyperGrader, HyperSlate, the license agreement,  
and to obtain the textbook LAMA-BDLA, go to::

<https://rpi.logicamodernapproach.com>

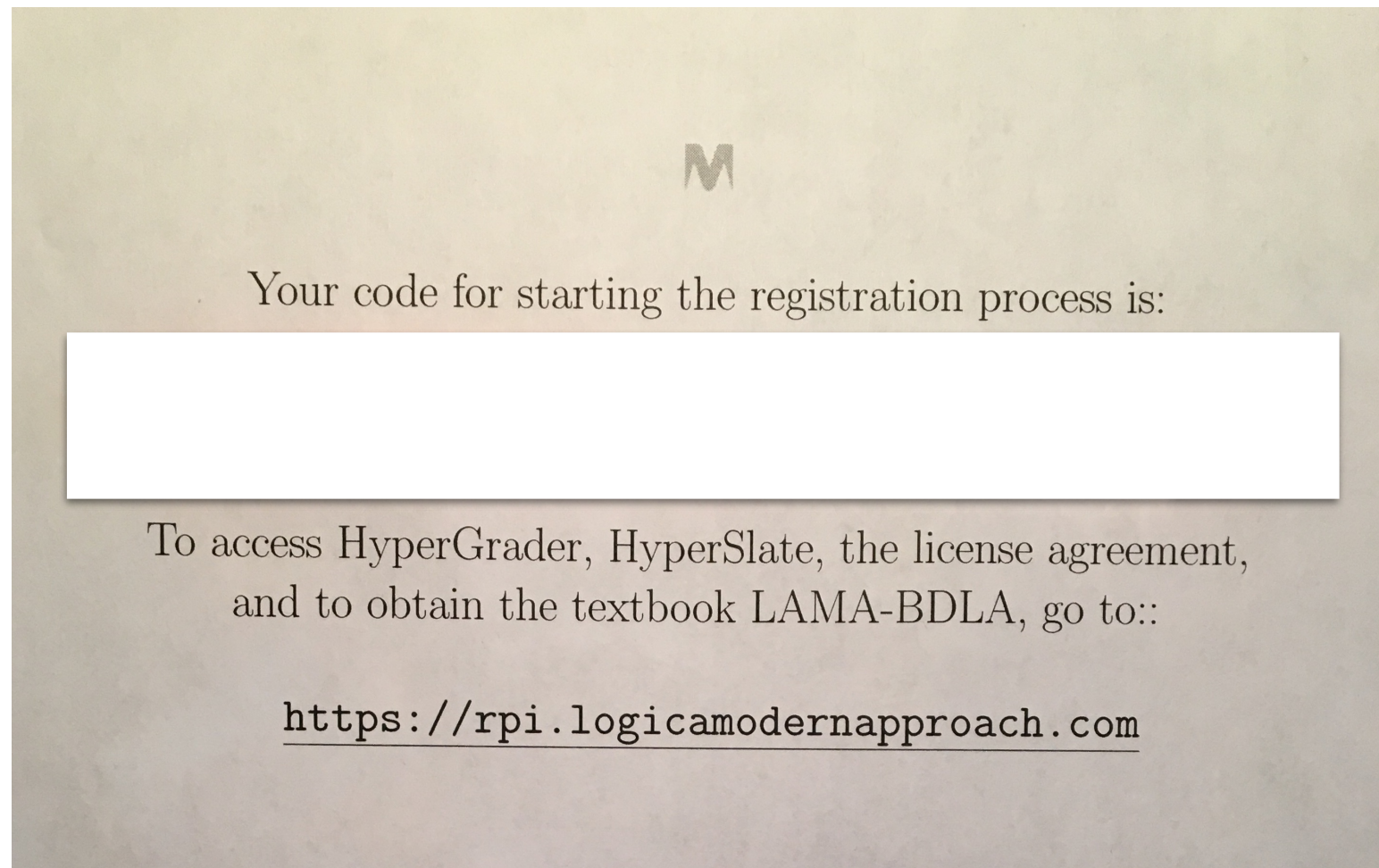
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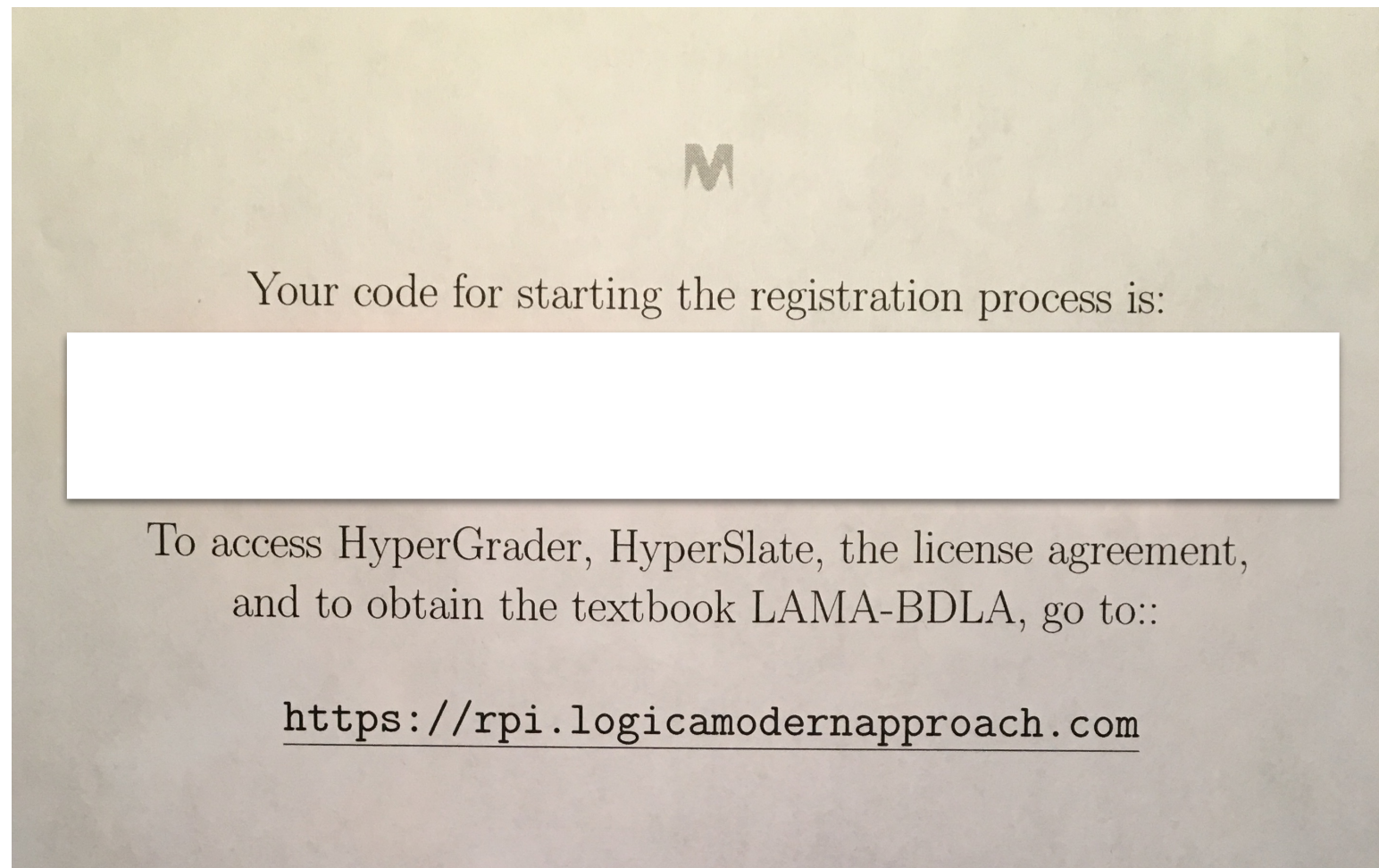
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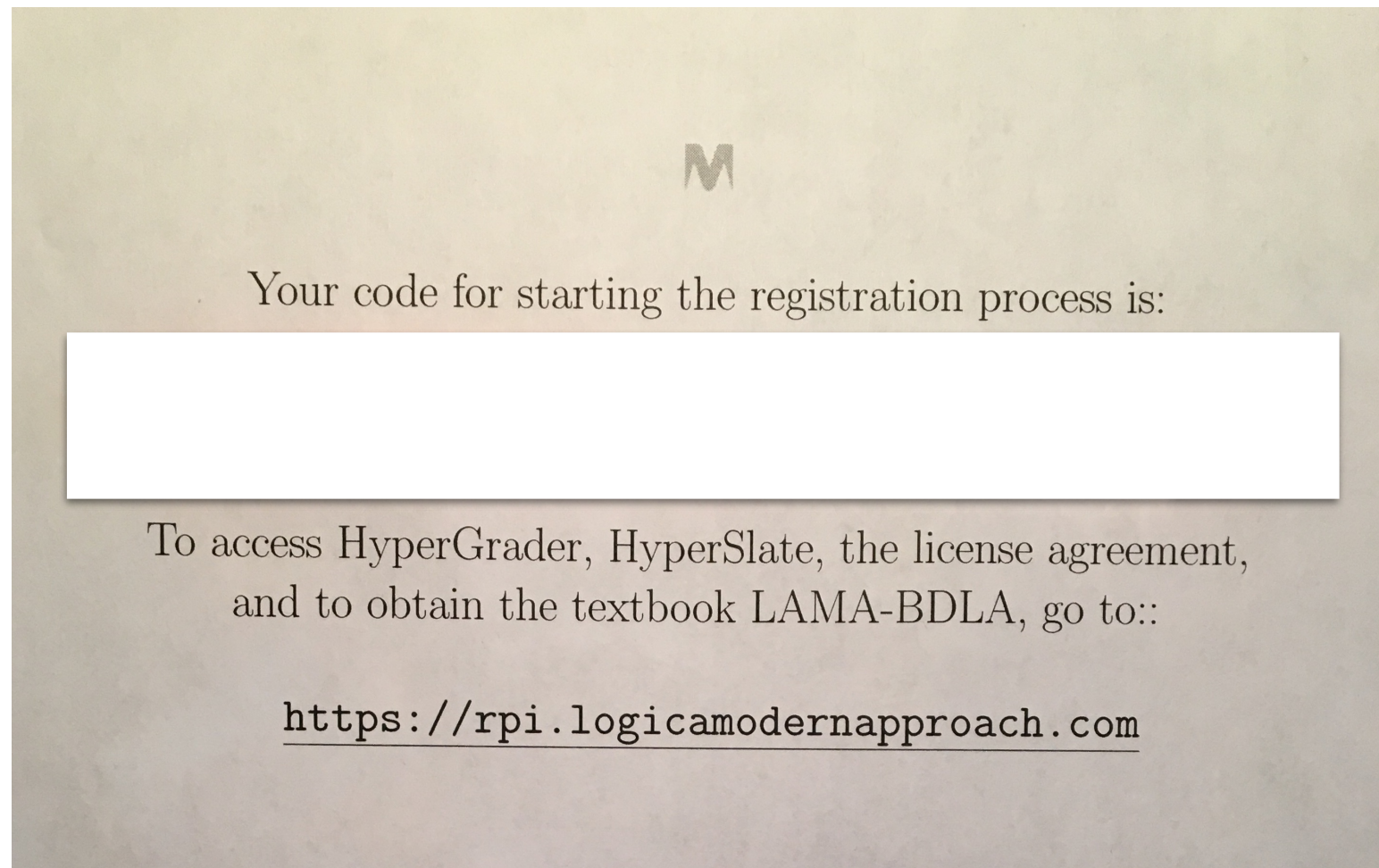
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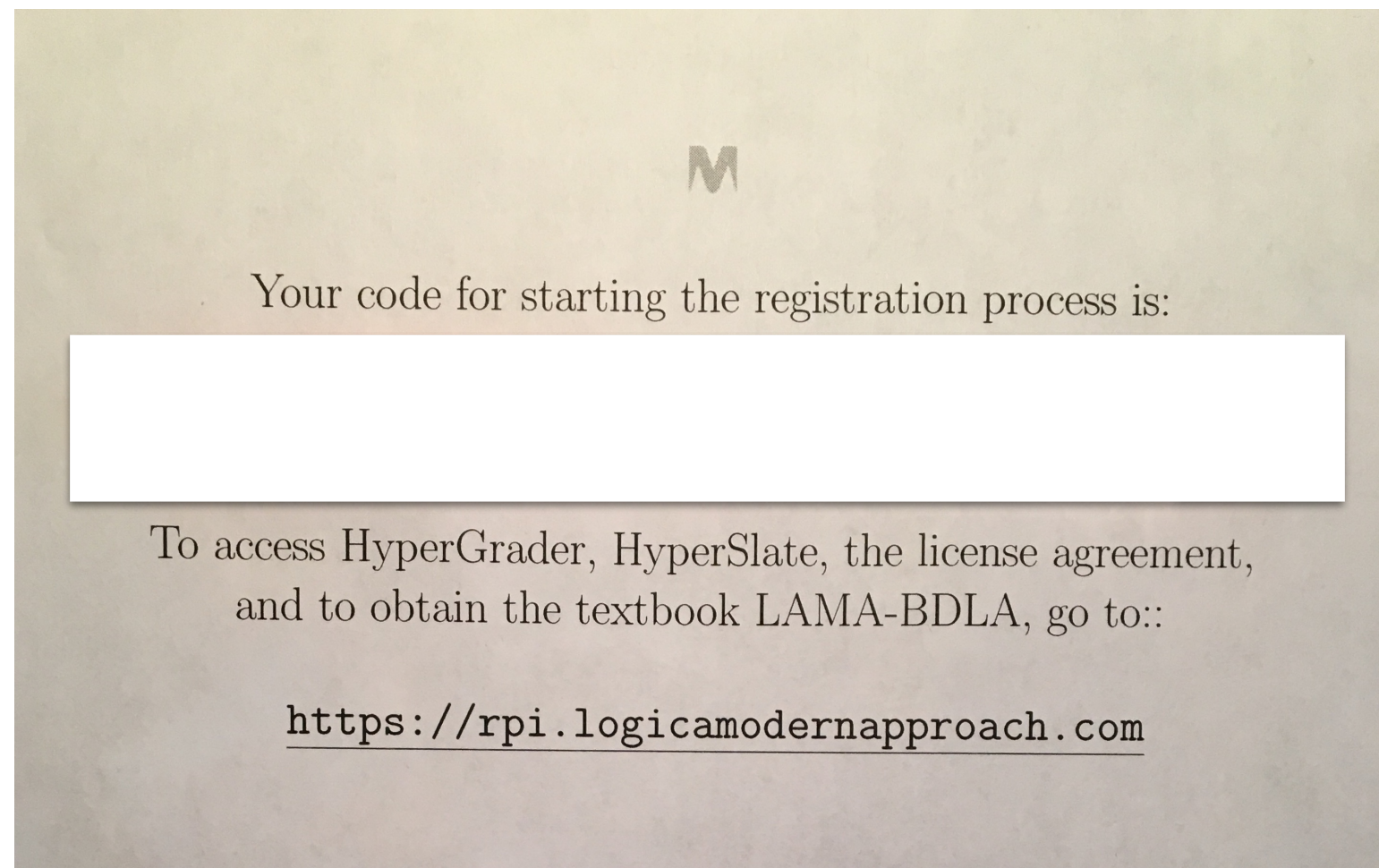
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Watch that the link doesn't end up being classified as spam.



The Starting Code Purchased in Bookstore Should  
By Now've Been Used to Register & Subsequently Sign In

Batch of prop. calc. (Homework) Problems in;  
easiest starting place: `switching_conjuncts_fine`;  
but should be done with some others by now.



# E-Housekeeping Pts, Redux

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- Make sure browser fully up-to-date.
- Chrome best (but I use Safari).
- Always work in the same browser window with multiple tabs; must do this with email and HyperGrader™ & HyperSlate™.

# Some HyperSlate Pts



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- Don't fall behind on (Homework) Problems!!  
Many more are coming for this self-paced 40%-of-grade part of your learning ...

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- We are not covering all inference rules/schemata in class.

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- “Where are my files?”
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- conditional intro, briefly ...

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*Reductio ...*



“Reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.”

—G. H. Hardy

**A Greek-shocking Example ...**

$$\frac{p}{q}$$

# What are rational numbers?

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- Any number that can be expressed in the form of  $\frac{p}{q}$  such that we have a numerator  $p$  and a non-zero denominator.
- Rational numbers are a subset of real numbers!
- Examples of *irrational* numbers?

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$$\sqrt{2}$$





Prove that:

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$$\sqrt{2}$$

Prove that:

$\sqrt{2}$  is irrational

Suppose  $\sqrt{2}$  is **rational**. That means it can be written as the ratio of two integers  $p$  and  $q$

$$\sqrt{2} = \frac{p}{q} \quad (1)$$

where we may assume that  $p$  and  $q$  **have no common factors**. (If there are any common factors we cancel them in the numerator and denominator.) Squaring in (1) on both sides gives

$$2 = \frac{p^2}{q^2} \quad (2)$$

which implies

$$p^2 = 2q^2 \quad (3)$$

Thus  $p^2$  is even. The only way this can be true is that  $p$  itself is even. But then  $p^2$  is actually divisible by 4. Hence  $q^2$  and therefore  $q$  must be even. So  $p$  and  $q$  are both even which is a contradiction to our assumption that they have no common factors. The square root of 2 cannot be rational!

# Sizing Some Sets

$\mathbb{Q}$

?

$\mathbb{Z}^+$

$\mathbb{N}$

$\mathbb{R}$

$\mathbb{I}$



And now, what are *prime* numbers?



# And now, what are *prime* numbers?

- A number that can be divided by only two numbers, one and itself.
- Must be a whole number.
- Example: 2,3,5,7.....

# And recall: Euclidean “Magic”

**Theorem:** There are infinitely many primes.

**Proof:** We take an indirect route. Let  $\Pi = p_1 = 2, p_2 = 3, p_3 = 5, \dots, p_k$  be a finite, exhaustive consecutive sequence of prime numbers. Next, let  $M_\Pi$  be  $p_1 \times p_2 \times \dots \times p_k$ , and set  $M'_\Pi$  to  $M_\Pi + 1$ . Either  $M'_\Pi$  is prime, or not; we thus have two (exhaustive) cases to consider.

- C1 Suppose  $M'_\Pi$  is prime. In this case we immediately have a prime number beyond any in  $\Pi$  — contradiction!
- C2 Suppose on the other hand that  $M'_\Pi$  is *not* prime. Then some prime  $p$  divides  $M'_\Pi$ . (Why?) Now,  $p$  itself is either in  $\Pi$ , or not; we hence have two sub-cases. Supposing that  $p$  is in  $\Pi$  entails that  $p$  divides  $M_\Pi$ . But we are operating under the supposition that  $p$  divides  $M'_\Pi$  as well. This implies that  $p$  divides 1, which is absurd (a contradiction). Hence the prime  $p$  is outside  $\Pi$ .

Hence for *any* such list  $\Pi$ , there is a prime outside the list. That is, there are infinitely many primes. **QED**

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Notice that this proof uses two applications of *indirect proof*, and one application of *proof by cases*.

Study it word by word until you endorse it with your very soul!



# From Algebra 2 in High School ...

(Pearson Common-Core Compliant Textbook)

A proof involving indirect reasoning is an **indirect proof**. Often in an indirect proof, a statement and its negation are the only possibilities. When you see that one of these possibilities leads to a conclusion that contradicts a fact you know to be true, you can eliminate that possibility. For this reason, indirect proof is sometimes called *proof by contradiction*.

### TAKE NOTE Key Concept

#### Writing an Indirect Proof

- Step 1** State as a temporary assumption the opposite (negation) of what you want to prove.
- Step 2** Show that this temporary assumption leads to a contradiction.
- Step 3** Conclude that the temporary assumption must be false and that what you want to prove must be true.

### Problem 3 Writing an Indirect Proof

#### Proof

**Given:**  $\triangle ABC$  is scalene.

**Prove:**  $\angle A$ ,  $\angle B$ , and  $\angle C$  all have different measures.

#### THINK

Assume temporarily the opposite of what you want to prove.

Show that this assumption leads to a contradiction.

Conclude that the temporary assumption must be false and that what you want to prove must be true.

#### WRITE

Assume temporarily that two angles of  $\triangle ABC$  have the same measure. Assume that  $m\angle A = m\angle B$ .

By the Converse of the Isosceles Triangle Theorem, the sides opposite  $\angle A$  and  $\angle B$  are congruent. This contradicts the given information that  $\triangle ABC$  is scalene.

The assumption that two angles of  $\triangle ABC$  have the same measure must be false. Therefore,  $\angle A$ ,  $\angle B$ , and  $\angle C$  all have different measures.

 SHOW SOLUTION

 GOT IT?

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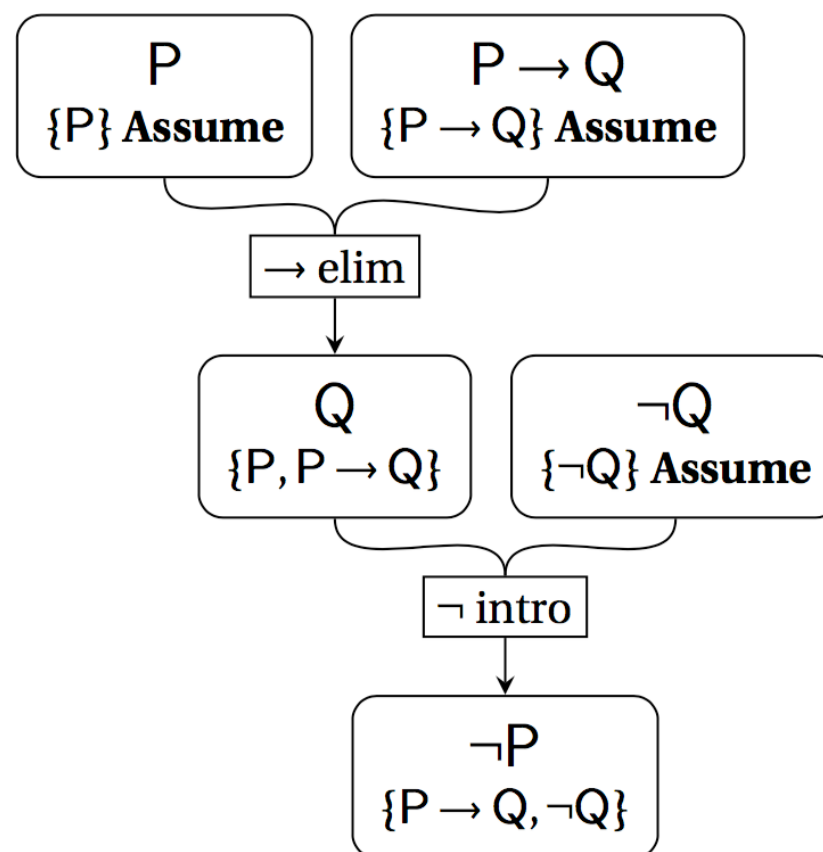
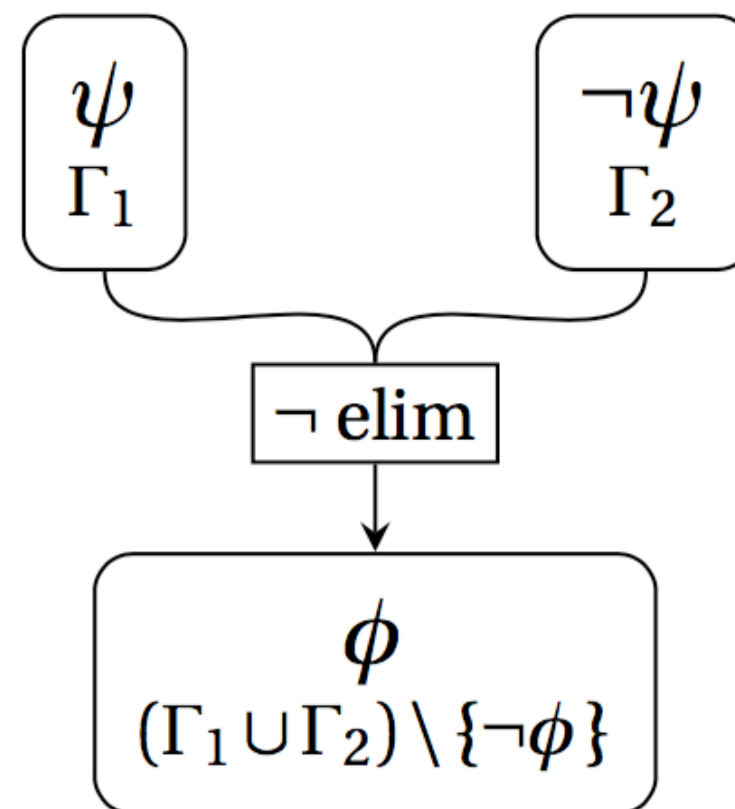
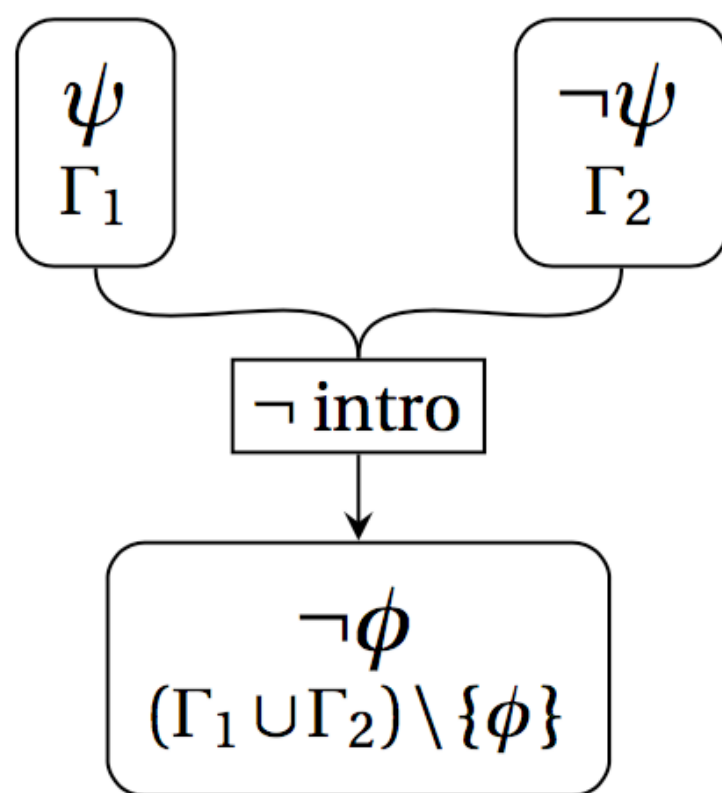
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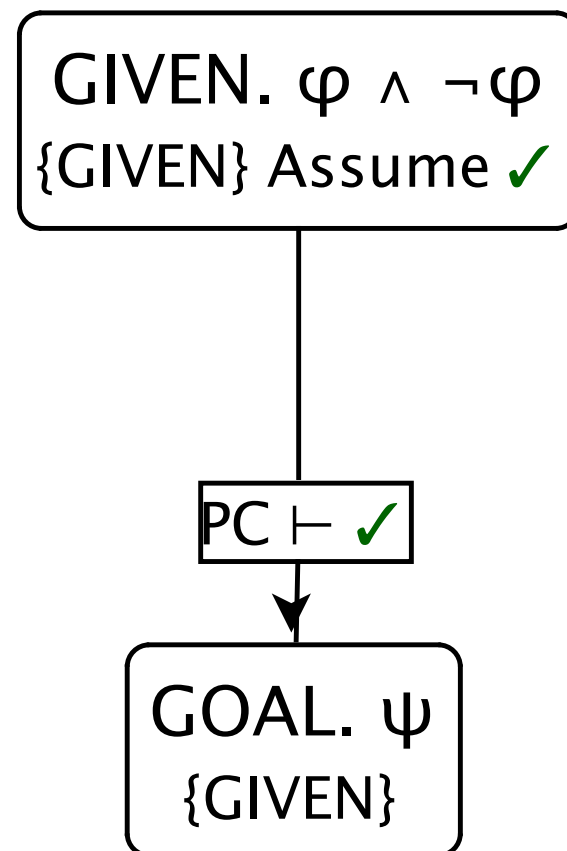
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  Year = 1994}
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▶ SHOW SOLUTION

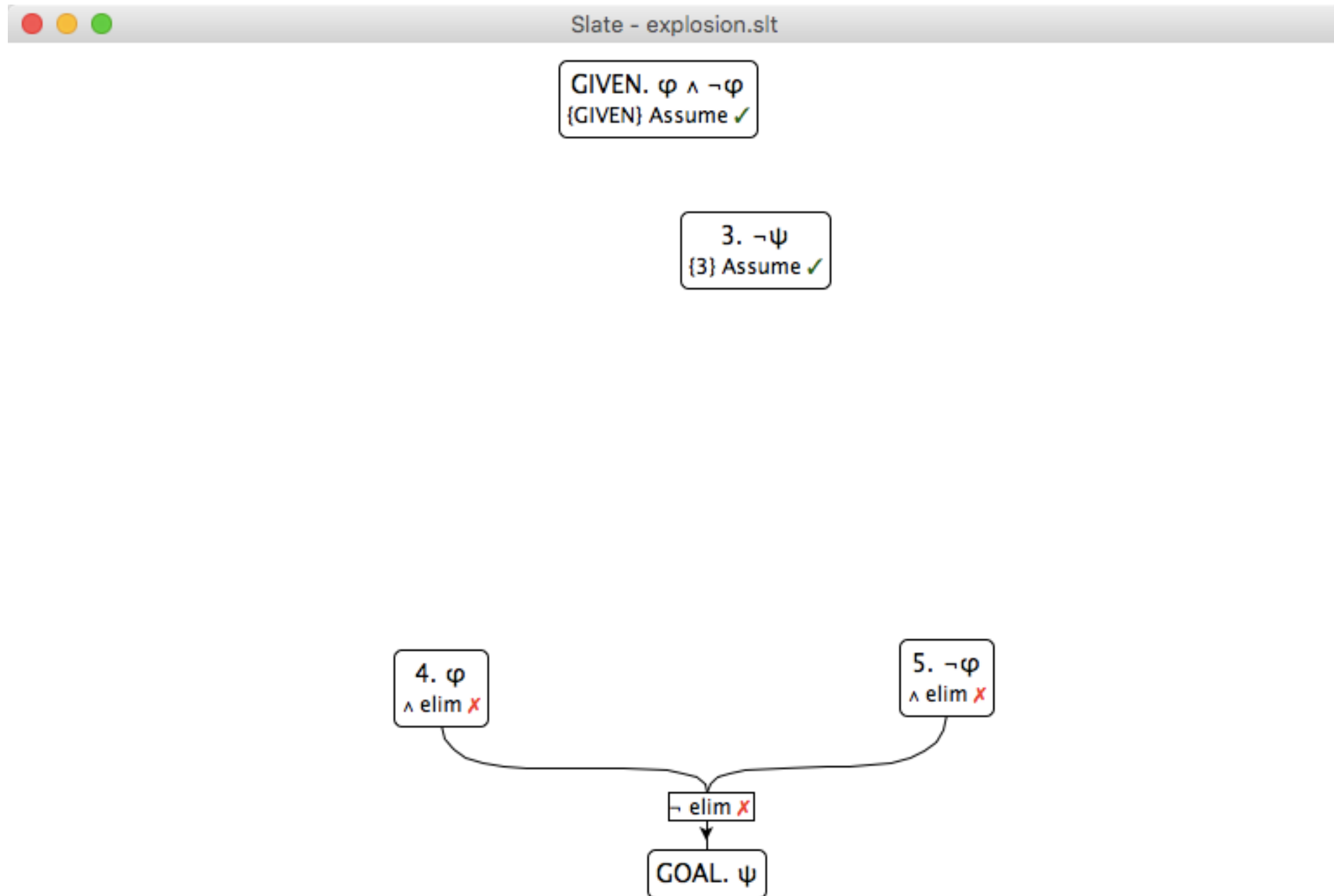
▶ GOT IT?



# Explosion

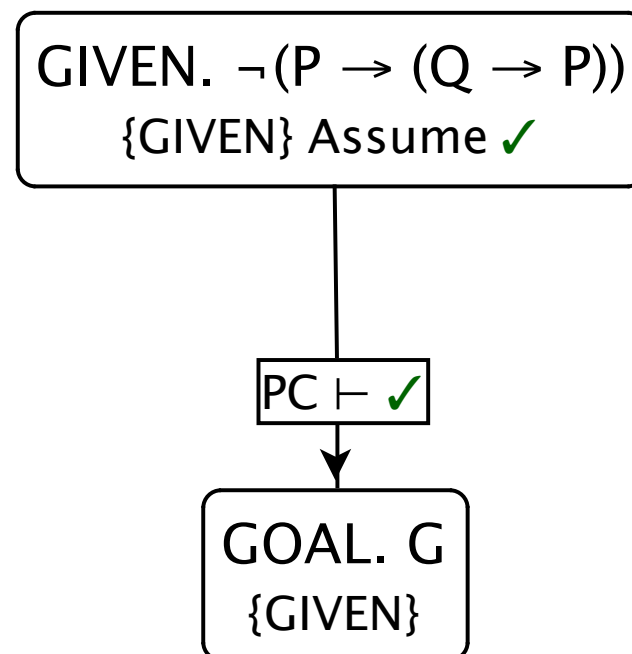


# Explosion: Partial Proof Plan





# GreenCheeseMoon I





# GreenCheeseMoon I: Partial Proof Plan

Slate - GreenCheeseMoon1.slt

GIVEN.  $\neg(P \rightarrow (Q \rightarrow P))$   
{GIVEN} Assume ✓

Sub-Proof Here

5. P  
{5} Assume ✓

3.  $P \rightarrow (Q \rightarrow P)$   
PC ✓

4.  $\neg G$   
{4} Assume ✓

GOAL. G  
 $\neg$  elim ✗

# Pop Problem

