# Rebuilding the Foundations of Math via (the “Theory”) ZFC;
# ZFC to Axiomatized Arithmetic (the “Theories” BA and PA)

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Troy, New York 12180 USA

IFLAI1
3/25/2021

# Questions?

AI in weapons …

The technical paper ...

# Reviewing the situation

…

# Types of Paradoxes

- Deductive Paradoxes
- Inductive Paradoxes — coming (e.g. The Lottery Paradox & The St Petersburg Paradox)

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https://www.megamillions.com

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1 in 302,575,350

Friday's Hill, Haslemere, 16 June 1902

Dear colleague,

For a year and a half I have been acquainted with your *Grundgesetze der Arithmetik*, but it is only now that I have been able to find the time for the thorough study I intended to make of your work. I find myself in complete agreement with you in all essentials, particularly when you reject any psychological element [[Moment]] in logic and when you place a high value upon an ideography [[Begriffsschrift]] for the foundations of mathematics and of formal logic, which, incidentally, can hardly be distinguished. With regard to many particular questions, I find in your work discussions, distinctions, and definitions that one seeks in vain in the works of other logicians. Especially so far as function is concerned (§ 9 of your *Begriffsschrift*), I have been led on my own to views that are the same even in the details. There is just one point where I have encountered a difficulty. You state (p. 17 [[p. 23 above]]) that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let $w$ be the predicate: to be a predicate that cannot be predicated of itself. Can $w$ be predicated of itself? From each answer its opposite follows. Therefore we must conclude that $w$ is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [[Menge]] does not form a totality.

I am on the point of finishing a book on the principles of mathematics and in it I should like to discuss your work very thoroughly.[1] I already have your books or shall buy them soon, but I would be very grateful to you if you could send me reprints of your articles in various periodicals. In case this should be impossible, however, I will obtain them from a library.

The exact treatment of logic in fundamental questions, where symbols fail, has remained very much behind; in your works I find the best I know of our time, and therefore I have permitted myself to express my deep respect to you. It is very regrettable that you have not come to publish the second volume of your *Grundgesetze*; I hope that this will still be done.

Very respectfully yours,

BERTRAND RUSSELL

The above contradiction, when expressed in Peano's ideography, reads as follows:

$$w = \mathrm{cls} \cap x \ni (x \sim\varepsilon\, x) . \supset : w \,\varepsilon\, w \,.=.\, w \sim\varepsilon\, w.$$

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# Russell’s Theorem

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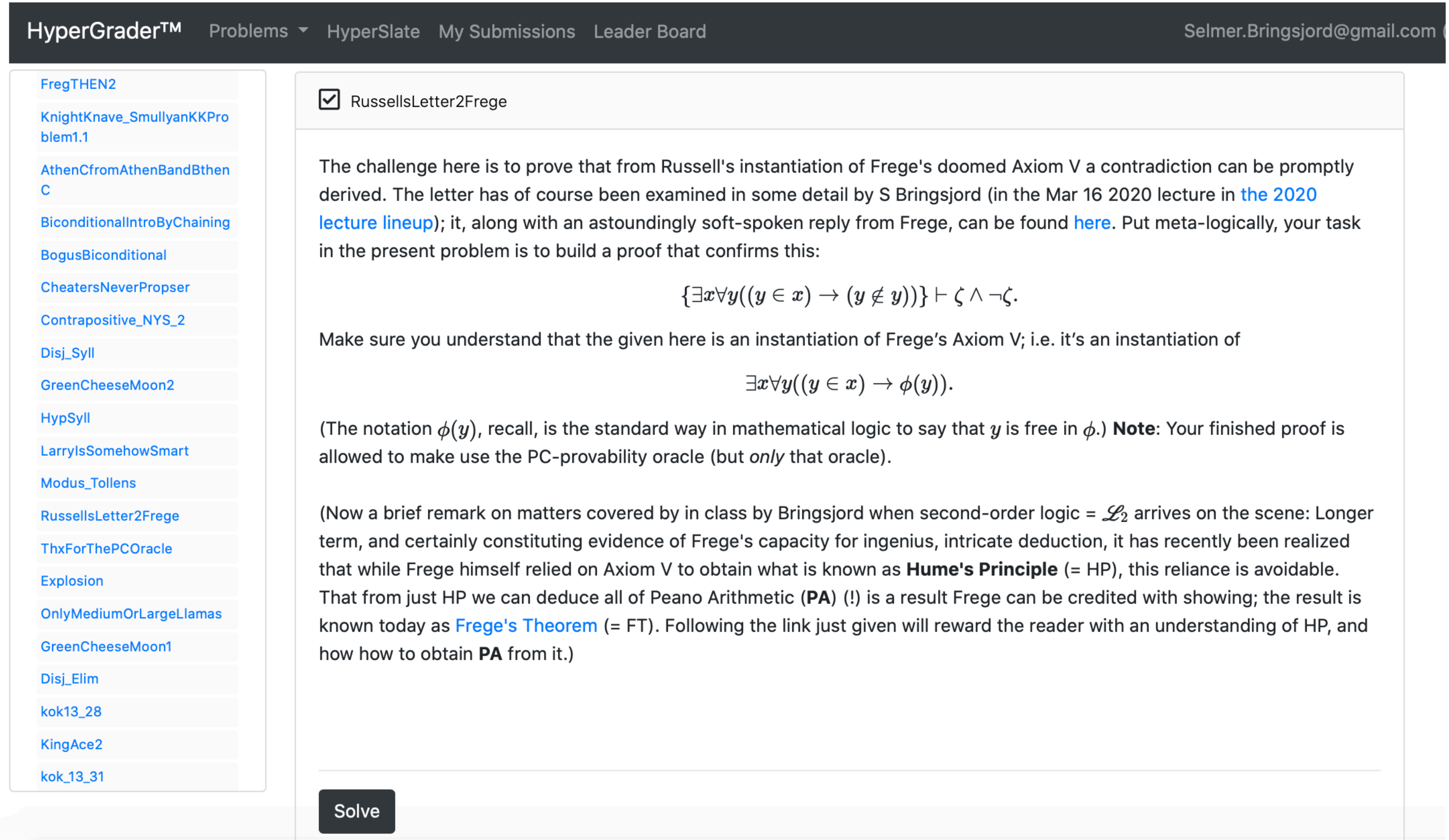

The challenge here is to prove that from Russell's instantiation of Frege's doomed Axiom V a contradiction can be promptly derived. The letter has of course been examined in some detail by S Bringsjord (in the Mar 16 2020 lecture in the 2020 lecture lineup); it, along with an astoundingly soft-spoken reply from Frege, can be found here. Put meta-logically, your task in the present problem is to build a proof that confirms this:

$$\{\exists x \forall y((y \in x) \rightarrow (y \notin y))\} \vdash \zeta \land \neg\zeta.$$

Make sure you understand that the given here is an instantiation of Frege's Axiom V; i.e. it's an instantiation of

$$\exists x \forall y((y \in x) \rightarrow \phi(y)).$$

(The notation $\phi(y)$, recall, is the standard way in mathematical logic to say that $y$ is free in $\phi$.) **Note**: Your finished proof is allowed to make use the PC-provability oracle (but *only* that oracle).

(Now a brief remark on matters covered by in class by Bringsjord when second-order logic = $\mathscr{L}_2$ arrives on the scene: Longer term, and certainly constituting evidence of Frege's capacity for ingenius, intricate deduction, it has recently been realized that while Frege himself relied on Axiom V to obtain what is known as **Hume's Principle** (= HP), this reliance is avoidable. That from just HP we can deduce all of Peano Arithmetic (**PA**) (!) is a result Frege can be credited with showing; the result is known today as Frege's Theorem (= FT). Following the link just given will reward the reader with an understanding of HP, and how how to obtain **PA** from it.)

# The Foundation Crumbles

The Rest of Math,
Engineering, etc.

Foundation

Axiom V etc.

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Foundation

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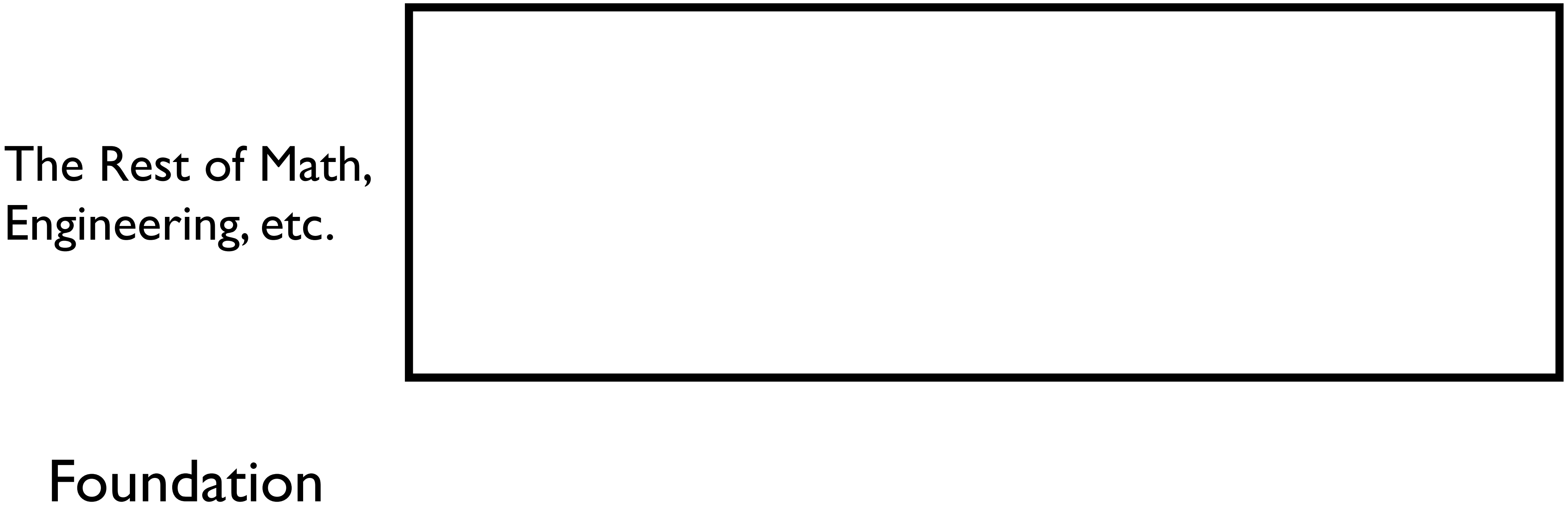

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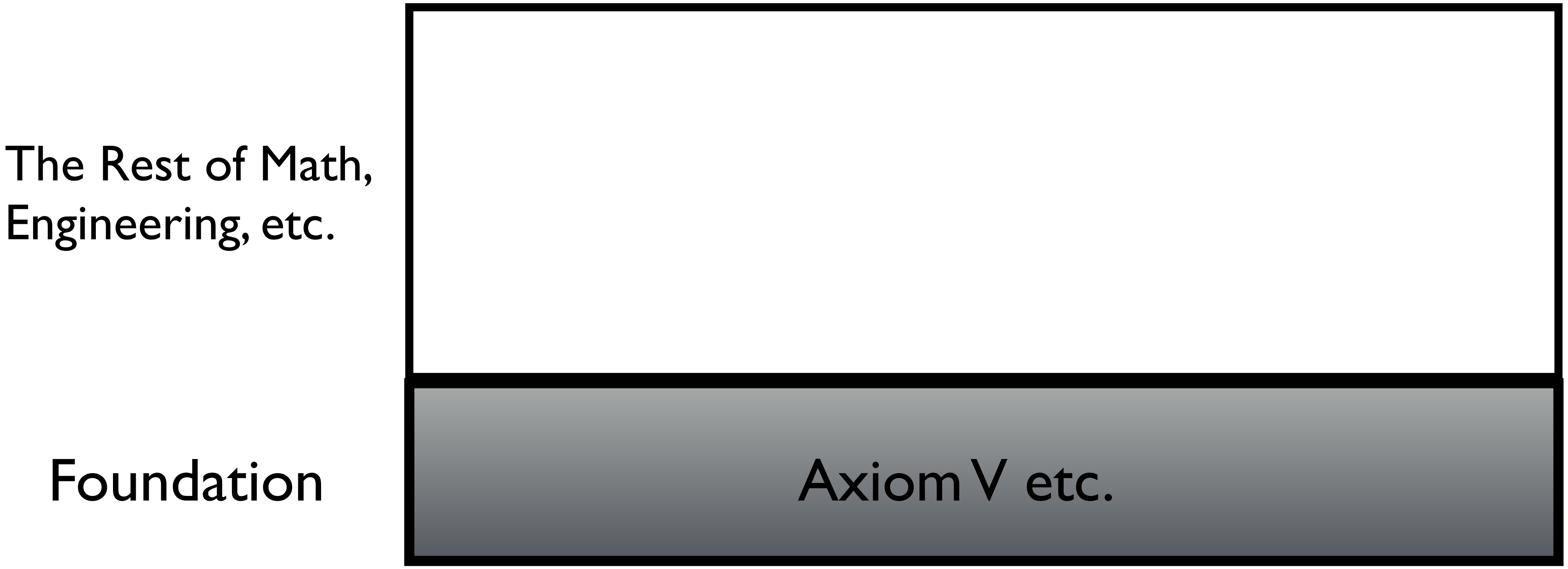

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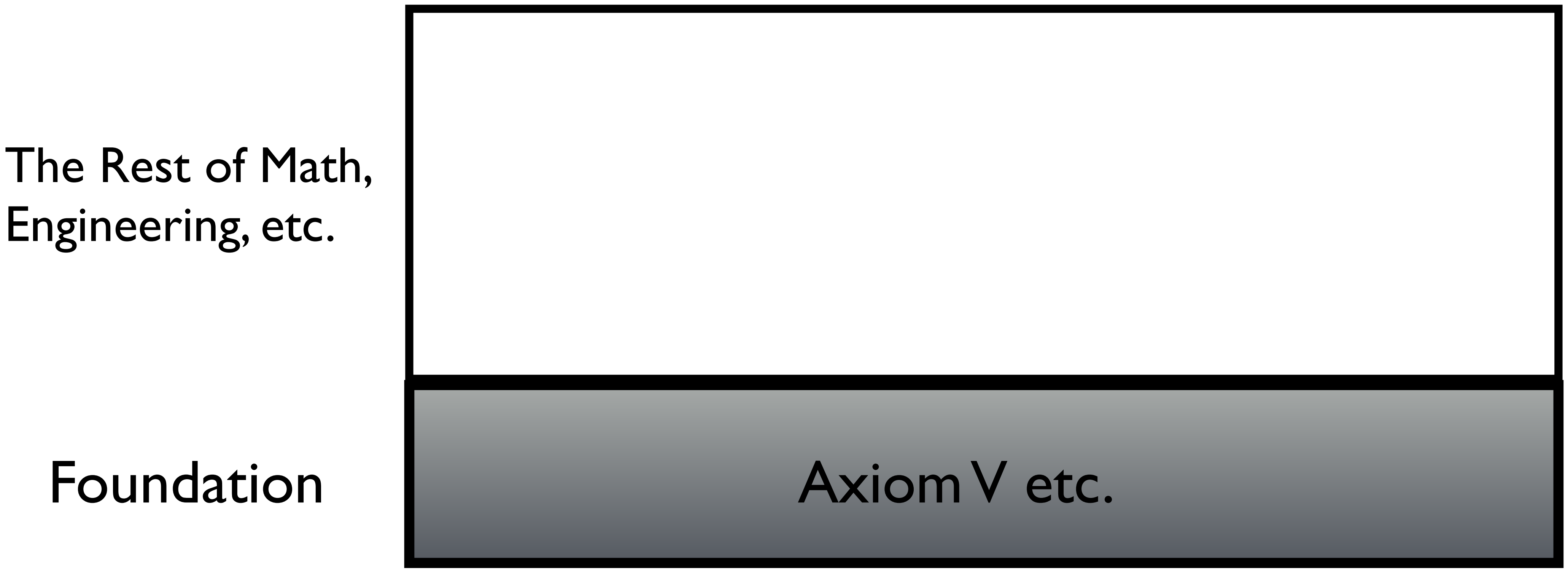

The Rest of Math,
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Axiom V etc.

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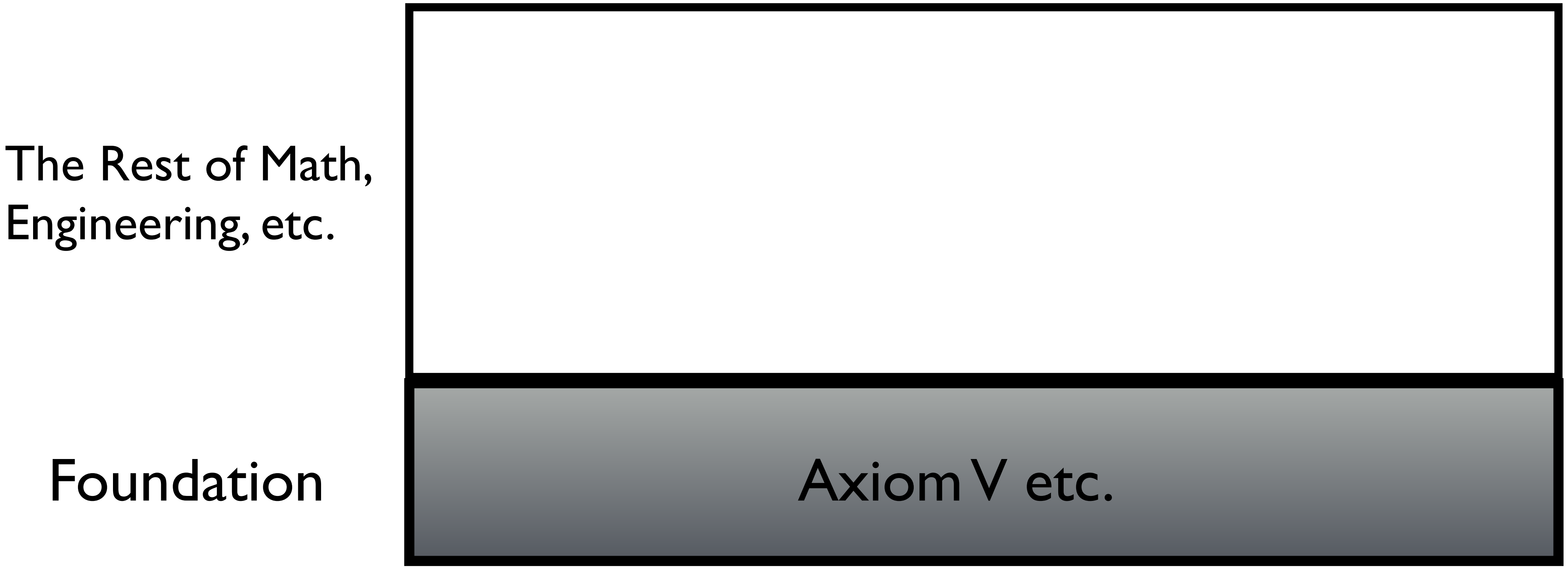

The Rest of Math, Engineering, etc.

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Axiom V etc.

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a formula of arbitrary size in which the variable *y* is free; this formula ascribes a property to *y*

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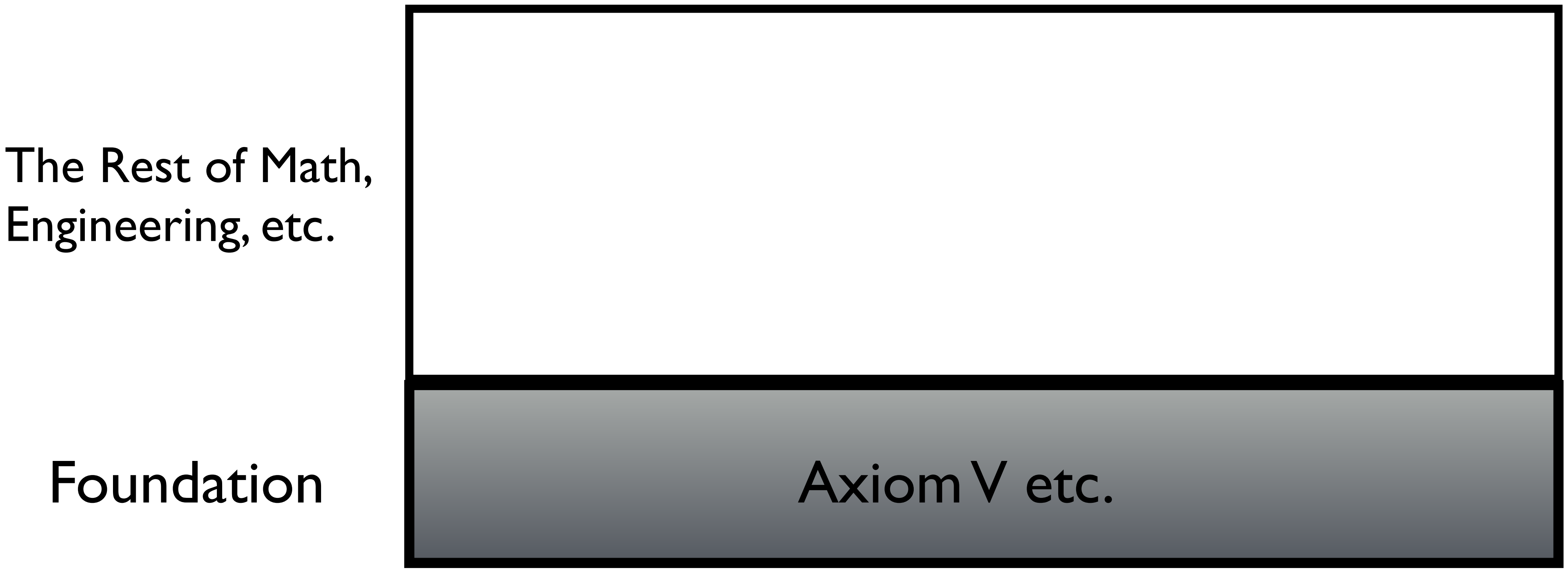

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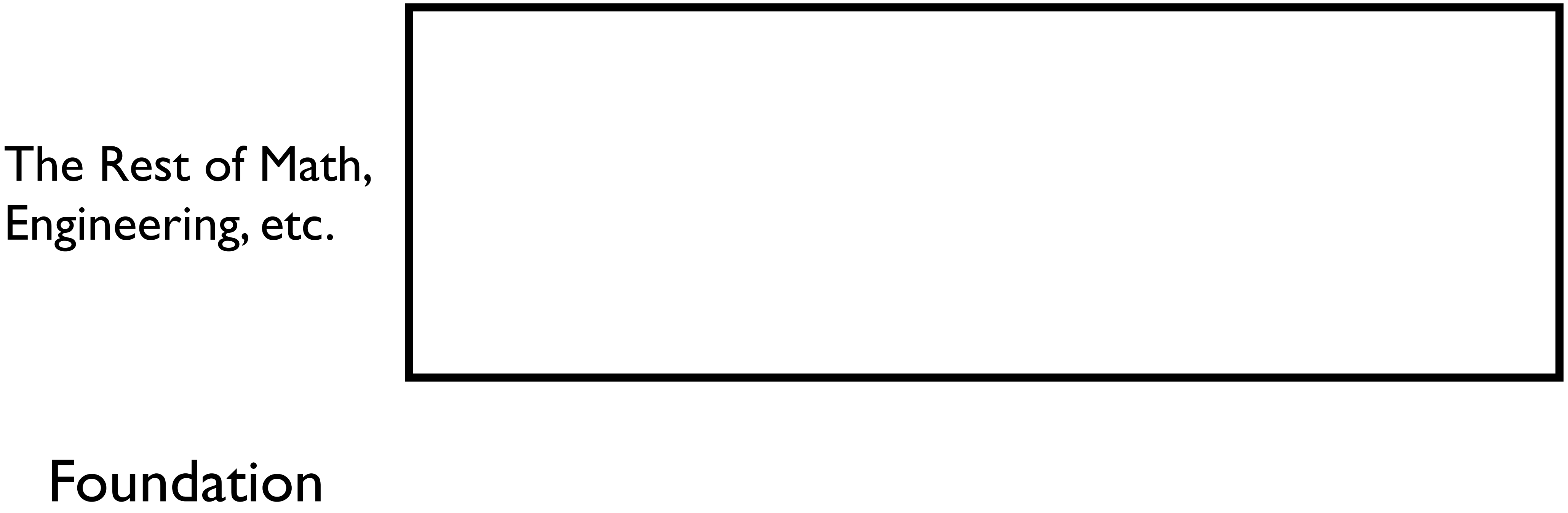

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# The Foundation Crumbles

The Rest of Math,
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Foundation

# It’s not just Russell’s Paradox that destroys naïve set theory:

# It’s not just Russell’s Paradox that destroys naïve set theory:

Richard’s Paradox …

a

a

b

a

b

⋮

a

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⋮

Doesn’t define a real number.

~~aaa~~

⋮

a

b

⋮

aa

ab

⋮

Doesn’t define a real number. ~~aaa~~

⋮

*E*

# Definition of Richard's *N*:

a
b
⋮
aa
ab
⋮
~~aaa~~
⋮
*E*

Doesn't define a real number.

## Definition of Richard's *N*:

a
b
⋮
aa
ab
⋮
~~aaa~~
⋮
*E*

Doesn't define a real number.

"The real number whose whole part is zero, and whose *n*-th decimal is *p* plus one if the *n*-th decimal of the real number defined by the *n*-th member of *E* is *p* and *p* is neither eight nor nine, and is simply one if this *n*-th decimal is eight or nine."

a
b
⋮
aa
ab
⋮
~~aaa~~ Doesn't define a real number.
⋮
*E*

# Definition of Richard's *N*:

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**Proof**: *N* is defined by a finite string taken from the English alphabet, so *N* is in the sequence *E*. But on the other hand, by definition of *N*, for every *m*, *N* differs from the *m*-th element of *E* in at least one decimal place; so *N* is not any element of *E*. Contradiction! **QED**

# Definition of Richard's *N*:

a
b
⋮
aa
ab
⋮
~~aaa~~
⋮

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~~aaa~~
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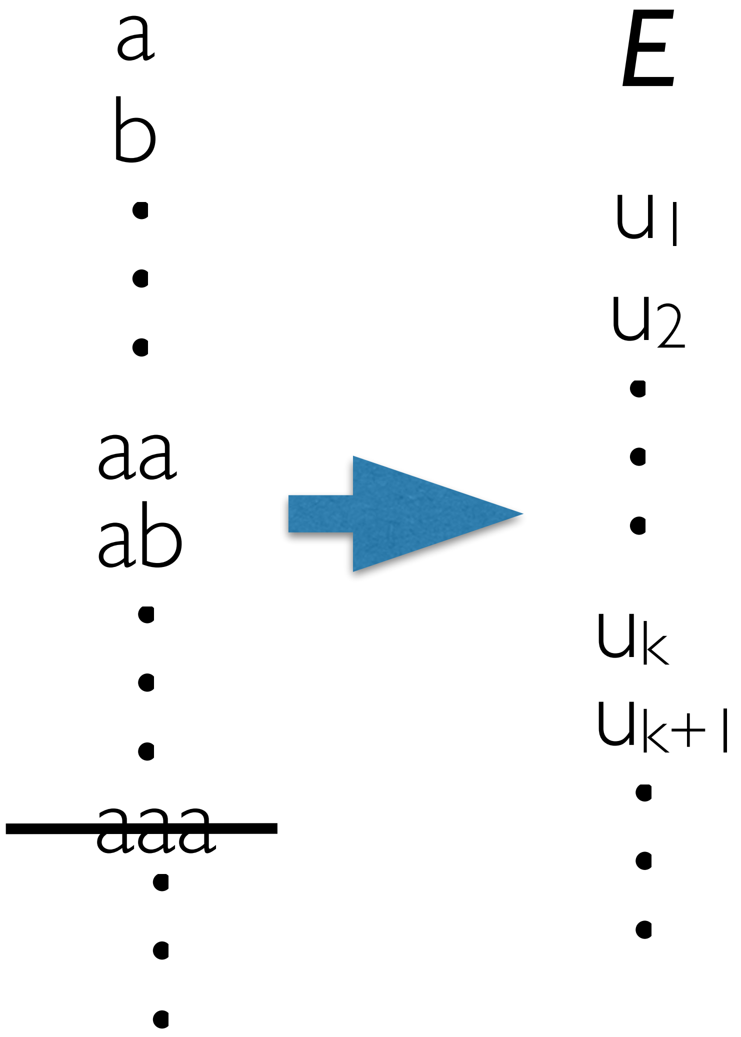

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~~aaa~~
⋮

*E*
$u_1$
$u_2$
⋮
$u_k$
$u_{k+1}$
⋮

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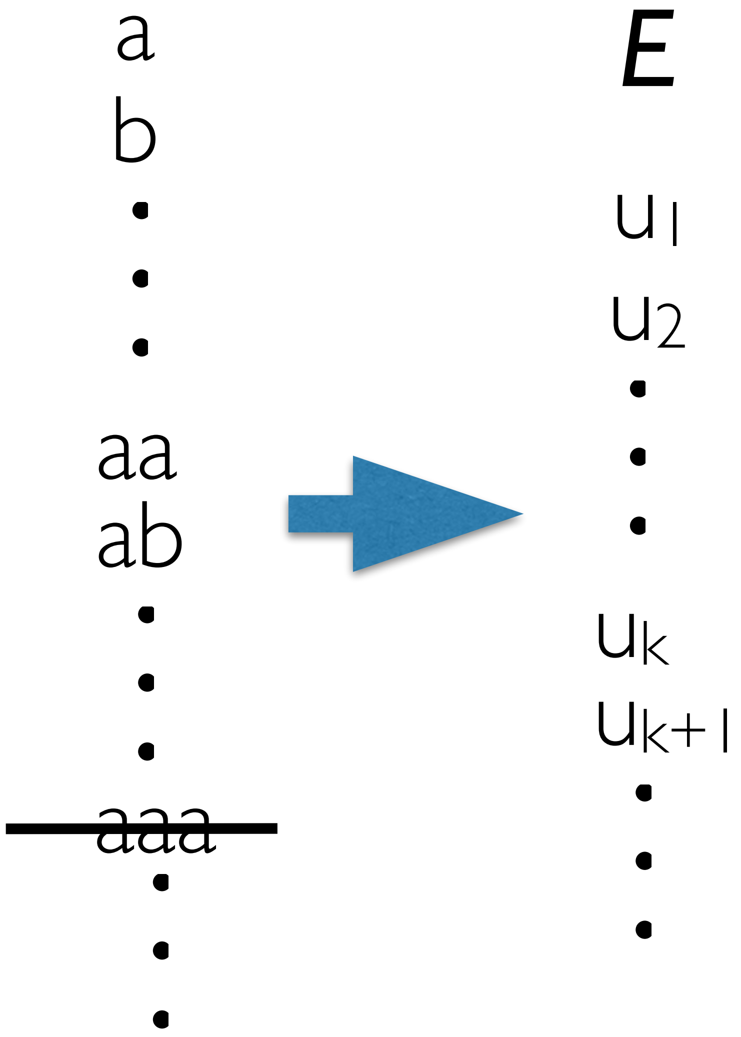

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*E*

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$u_2$

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$u_{k+1}$

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**Suppose *N* is $u_m$.**

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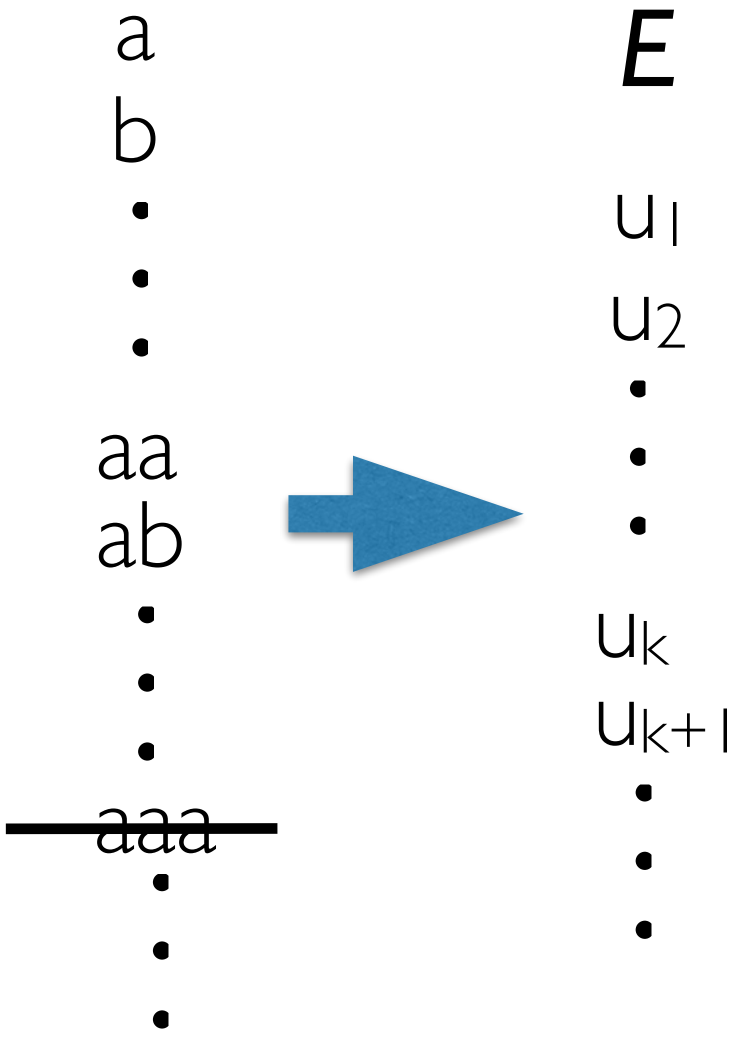

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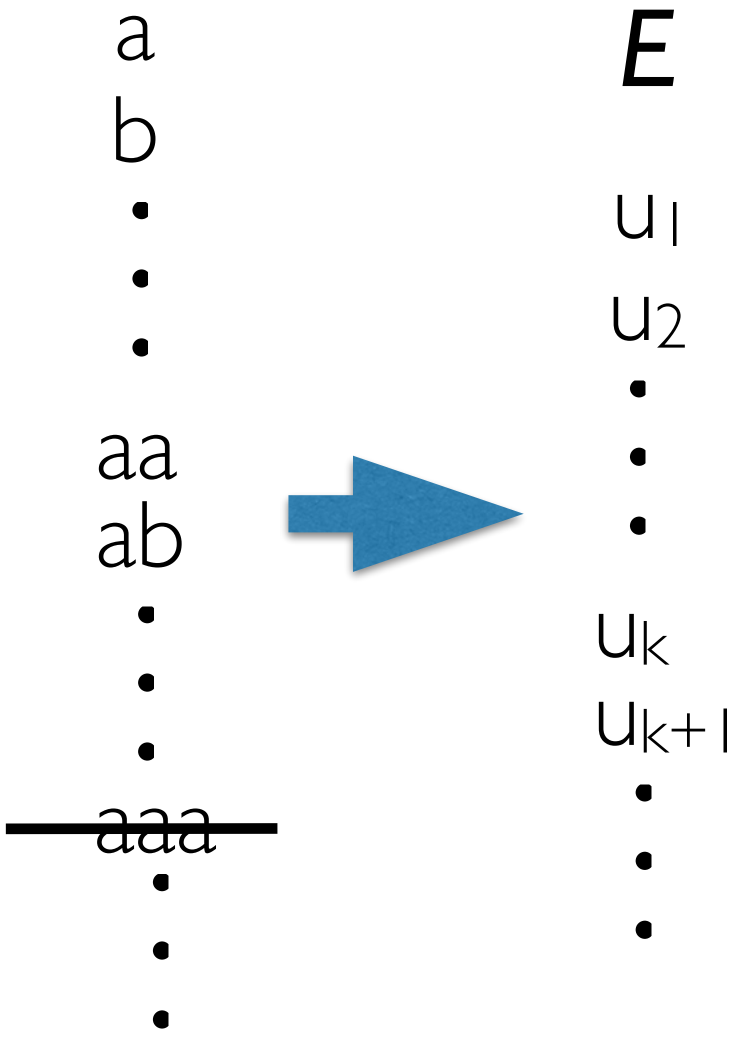
**Suppose $u_m$ is 0.xxxx…xxx…**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**Suppose $u_m$ is 0.xxxx…$\underset{\substack{\uparrow \\ \text{mth}}}{\text{x}}$xx…**

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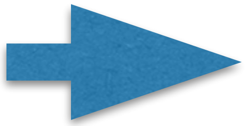

$E$

$u_1$
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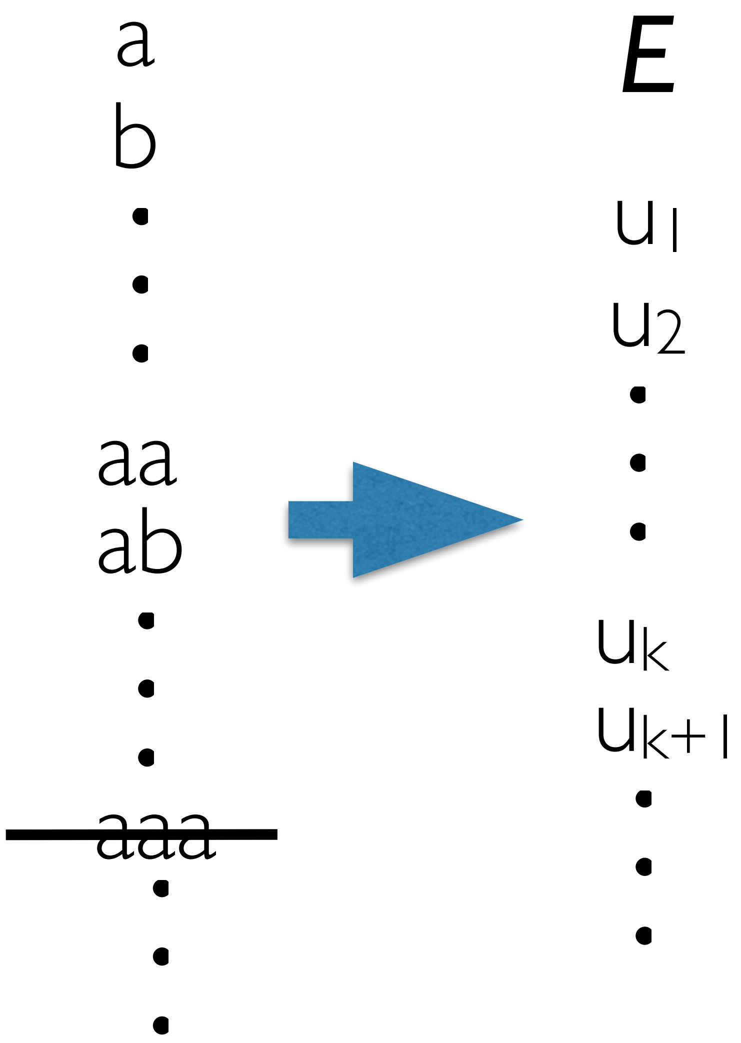
**Suppose $u_m$ is 0.xxxx…xxx…**

$\uparrow$
mth

**Suppose $u_m$ is 0.xxxx…8xx…**

$\uparrow$
mth



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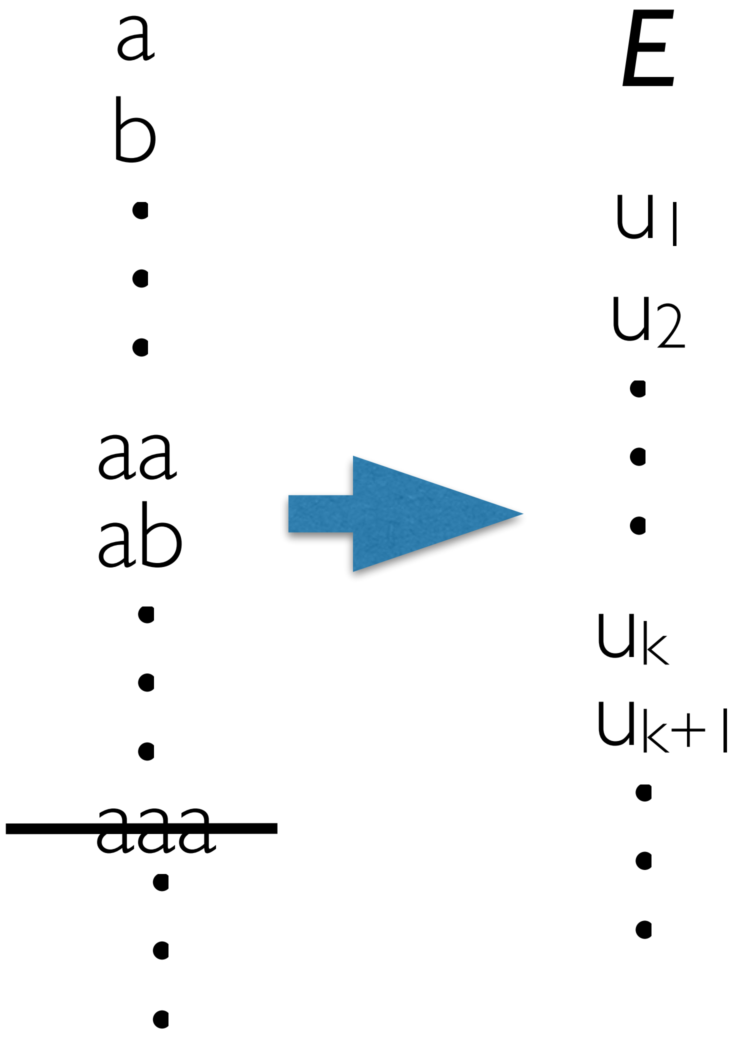

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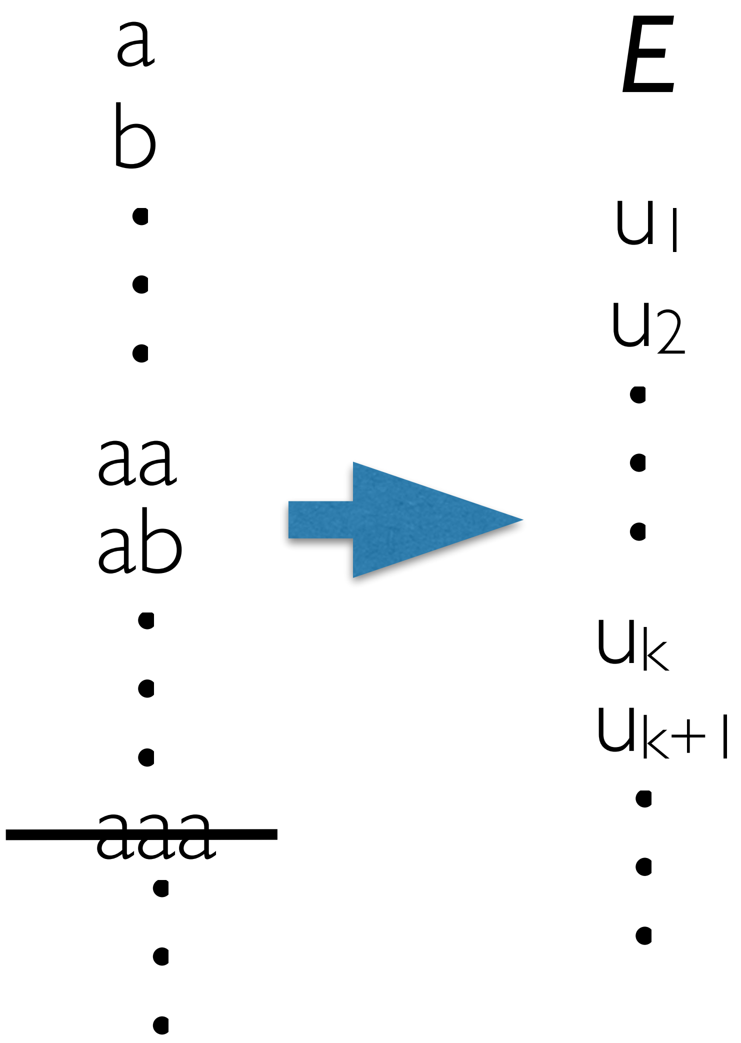
mth

**Then *N* is 0.xxxx…1xx…**

↑
mth



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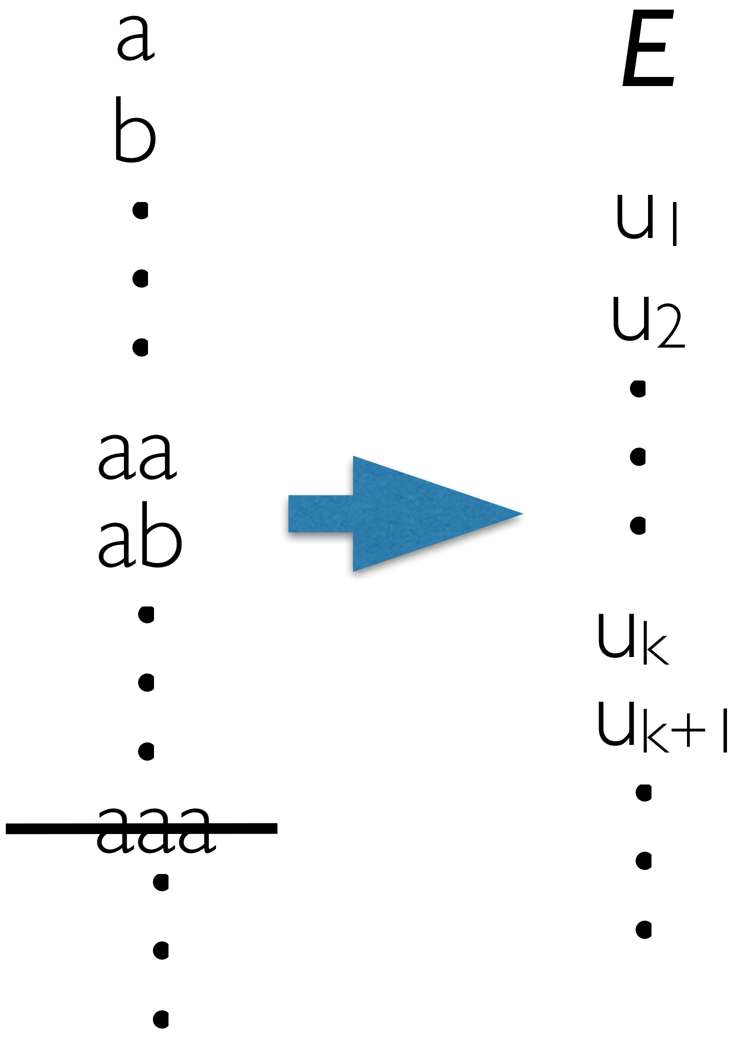
**Since 8 ≠ 1, *N* can't be $u_m$!**



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*E*

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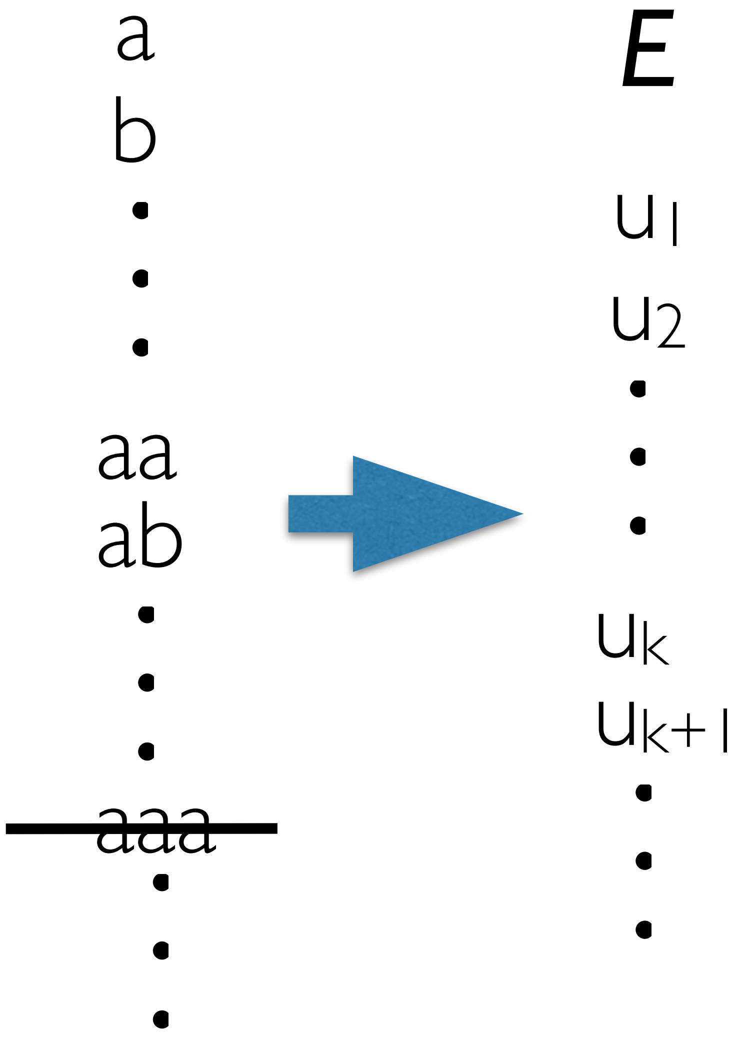

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↑
mth

**Suppose $u_m$ is 0.xxxx…8xx…**

↑
mth

**Suppose $u_m$ is 0.xxxx…5xx…**

↑
mth

**Then *N* is 0.xxxx…1xx…**

↑
mth

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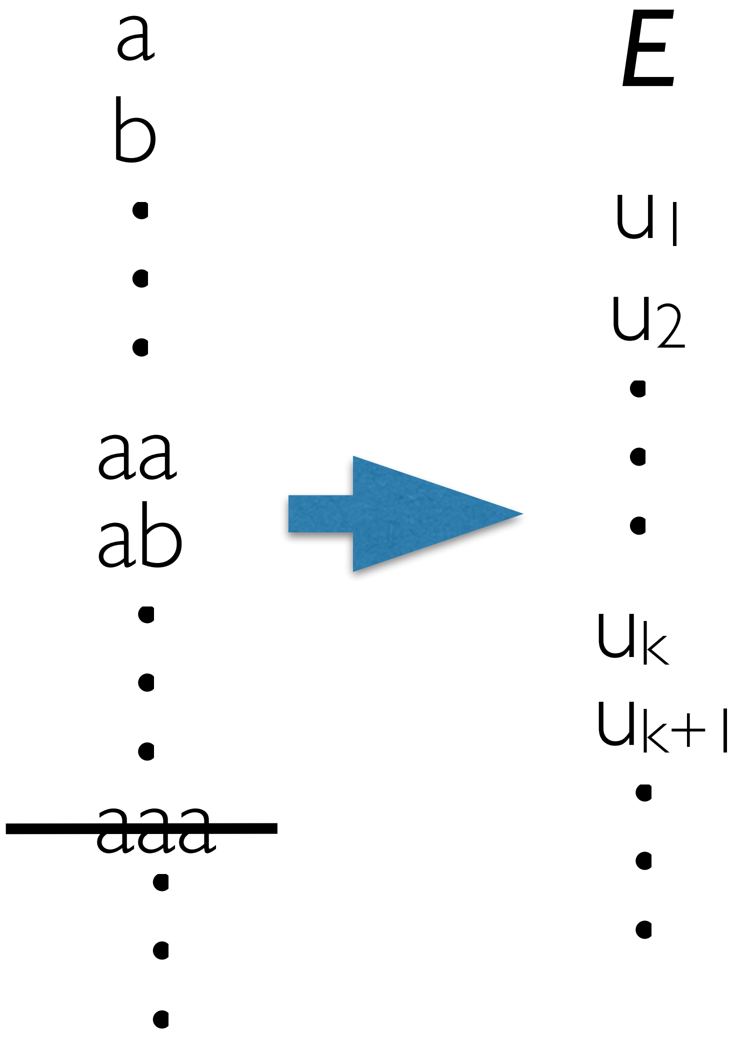
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mth

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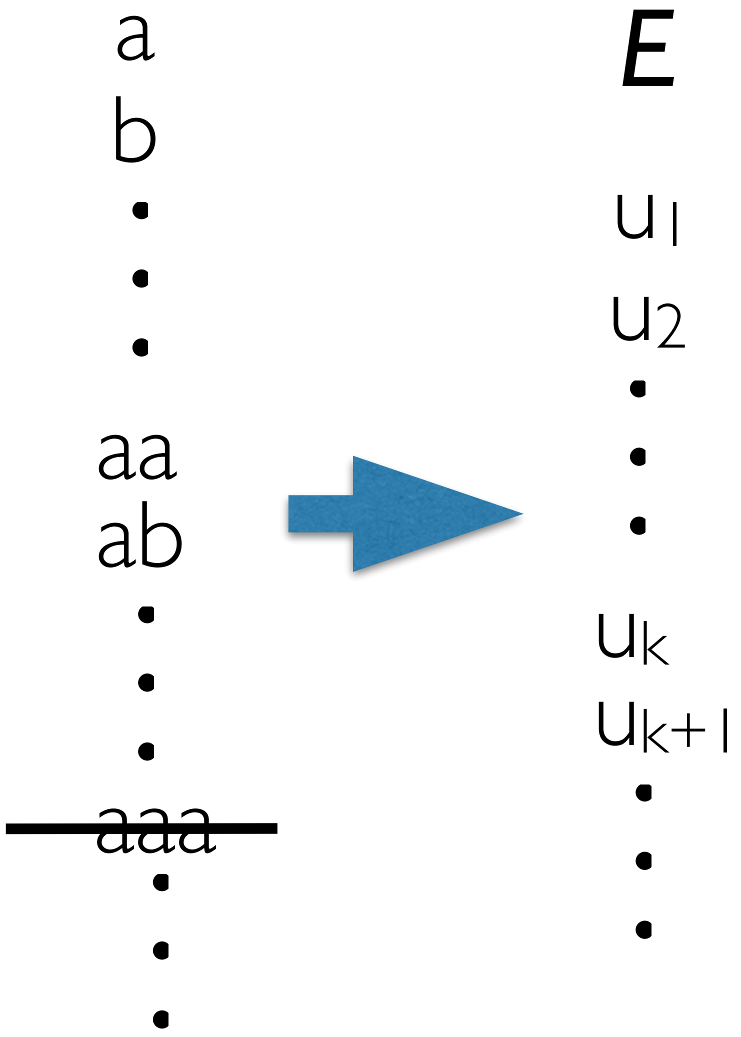

a
b
.
.
.
aa
ab
.
.
.
~~aaa~~
.
.
.

*E*

$u_1$
$u_2$
.
.
.
$u_k$
$u_{k+1}$
.
.
.

## Definition of Richard's *N*:

"The real number whose whole part is zero, and whose *n*-th decimal is *p* plus one if the *n*-th decimal of the real number defined by the *n*-th member of *E* is *p* and *p* is neither eight nor nine, and is simply one if this *n*-th decimal is eight or nine."

**Suppose *N* is $u_m$.**

**Suppose $u_m$ is 0.xxxx...xxx...**
↑
mth

**Suppose $u_m$ is 0.xxxx...8xx...**
↑
mth

**Then *N* is 0.xxxx...1xx...**
↑
mth

**Suppose $u_m$ is 0.xxxx...5xx...**
↑
mth

**Then *N* is 0.xxxx...6xx...**
↑
mth

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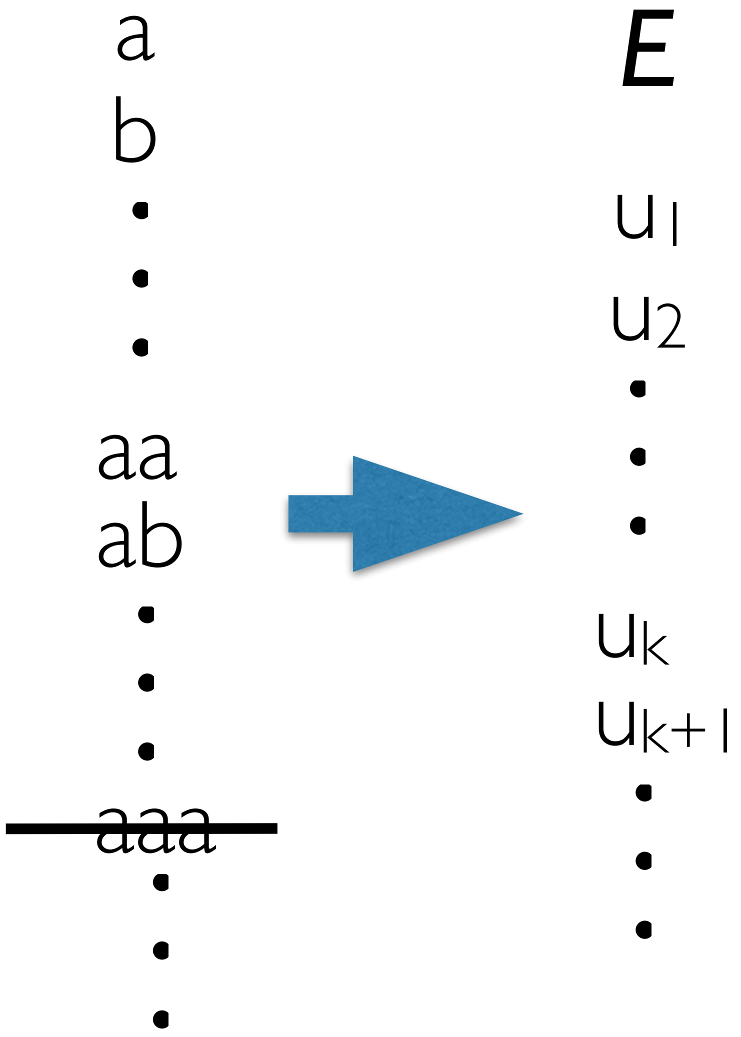

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**Then *N* is $0.xxxx\ldots\underset{\overset{\uparrow}{\text{mth}}}{6}xx\ldots$**

**Since 5 ≠ 6, *N* can't be $u_m$!**

# The Foundation Rebuilt

The Rest of Math,
Engineering, etc.

New Foundation

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New Foundation

ZFC

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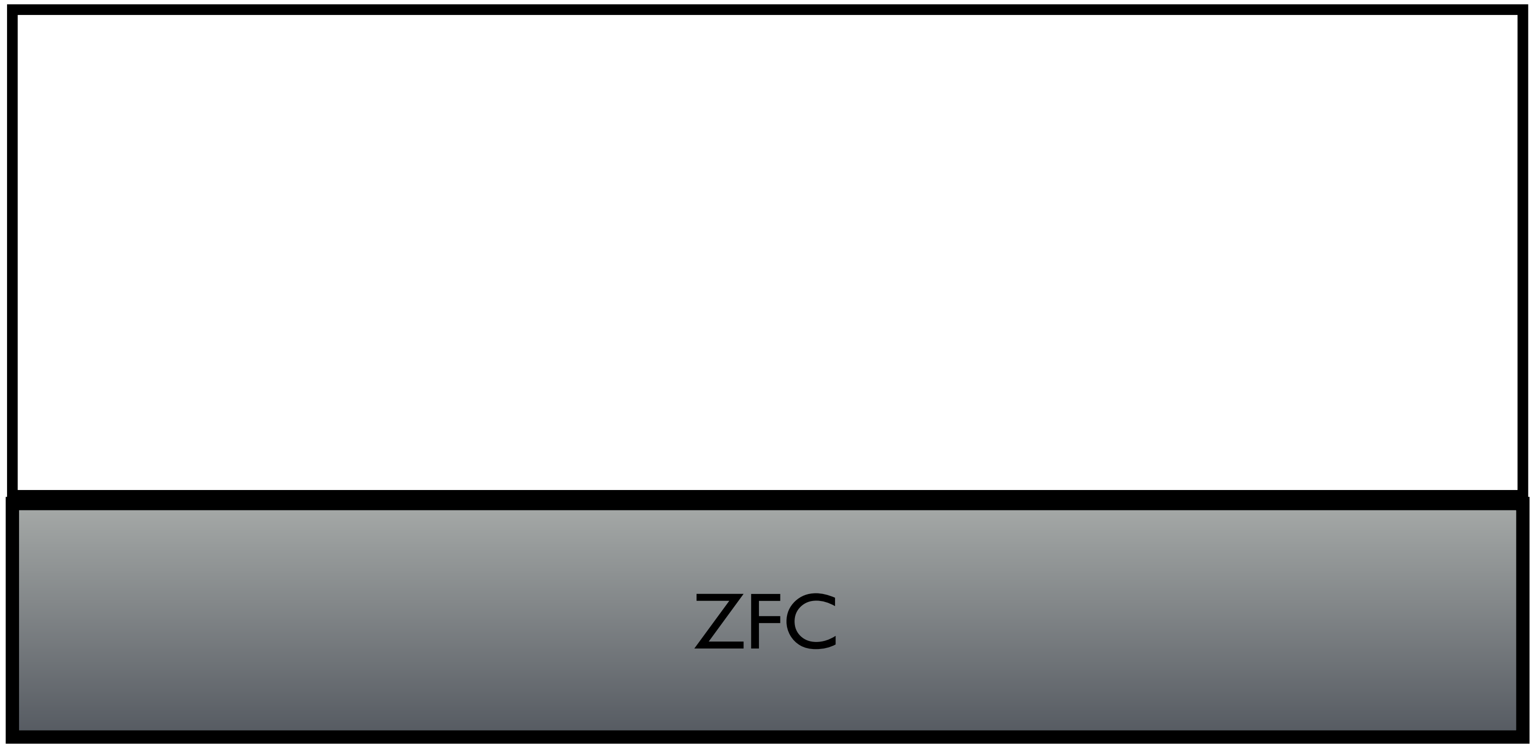

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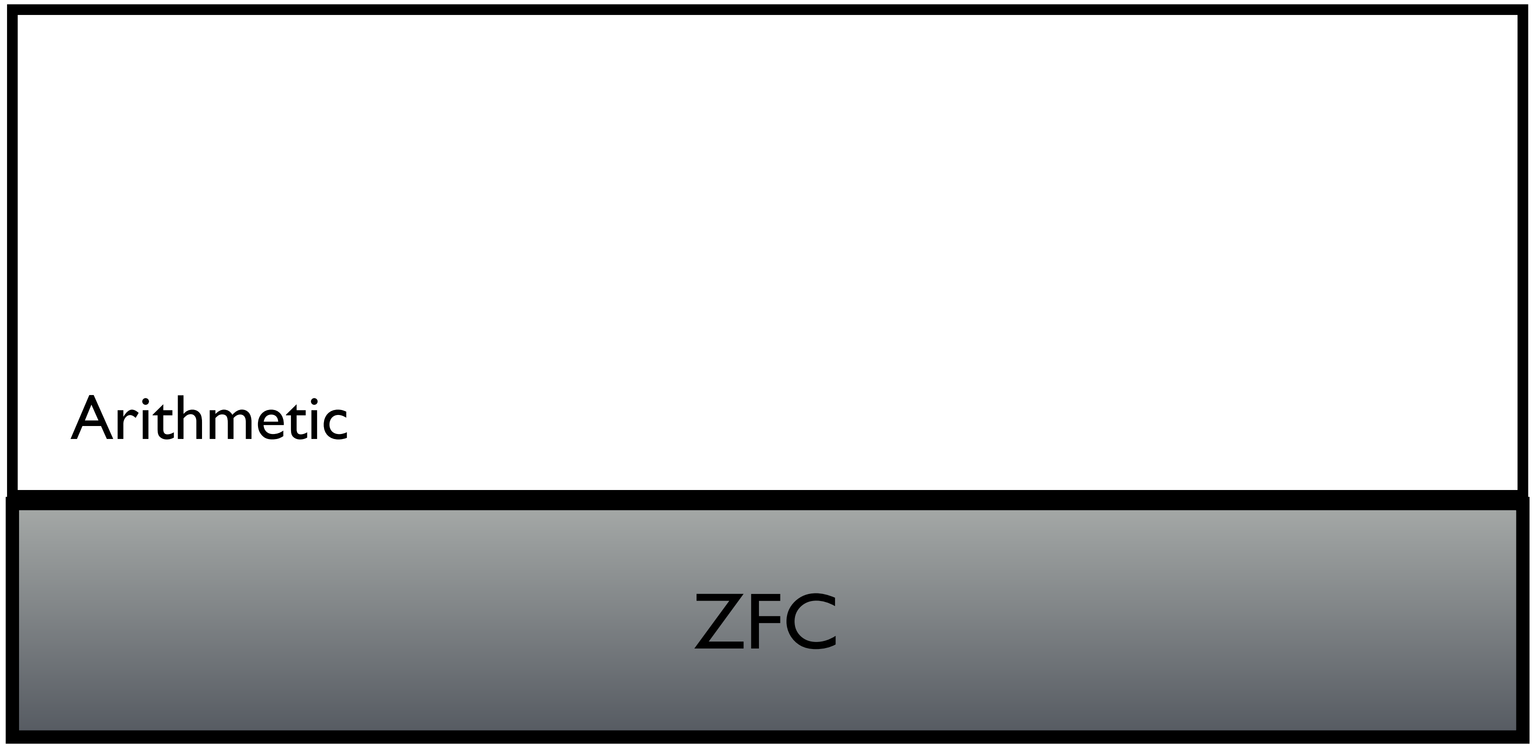

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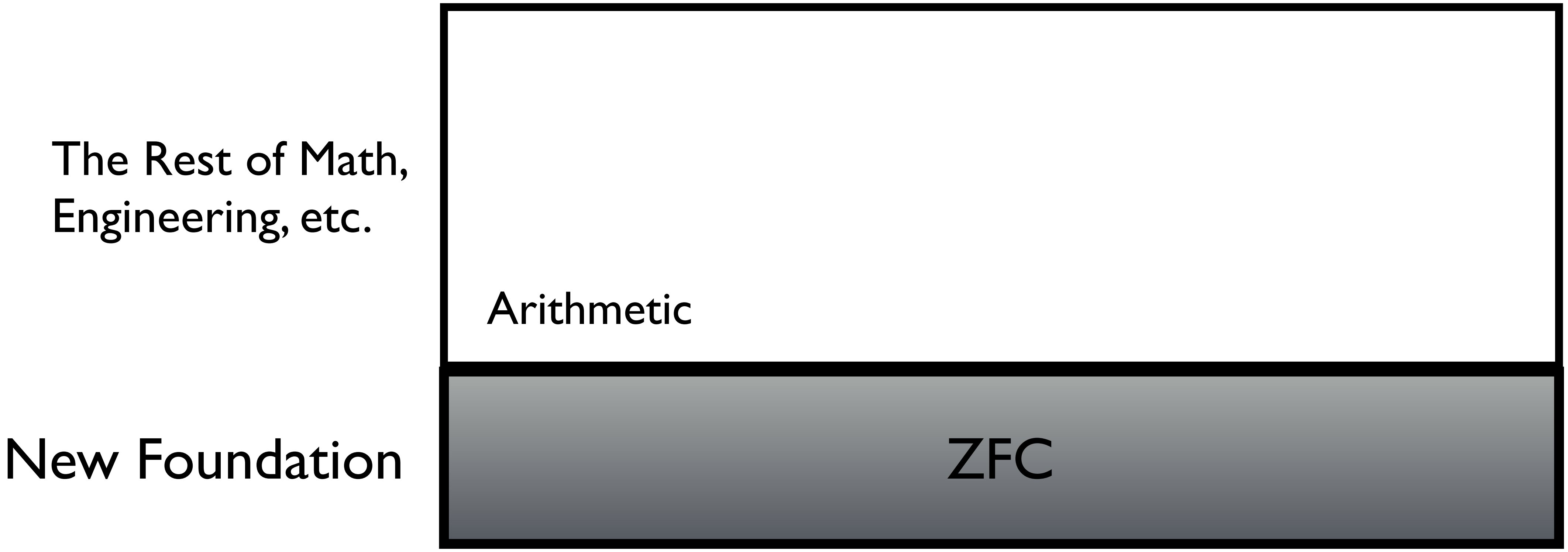

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So what are the axioms in ZFC?

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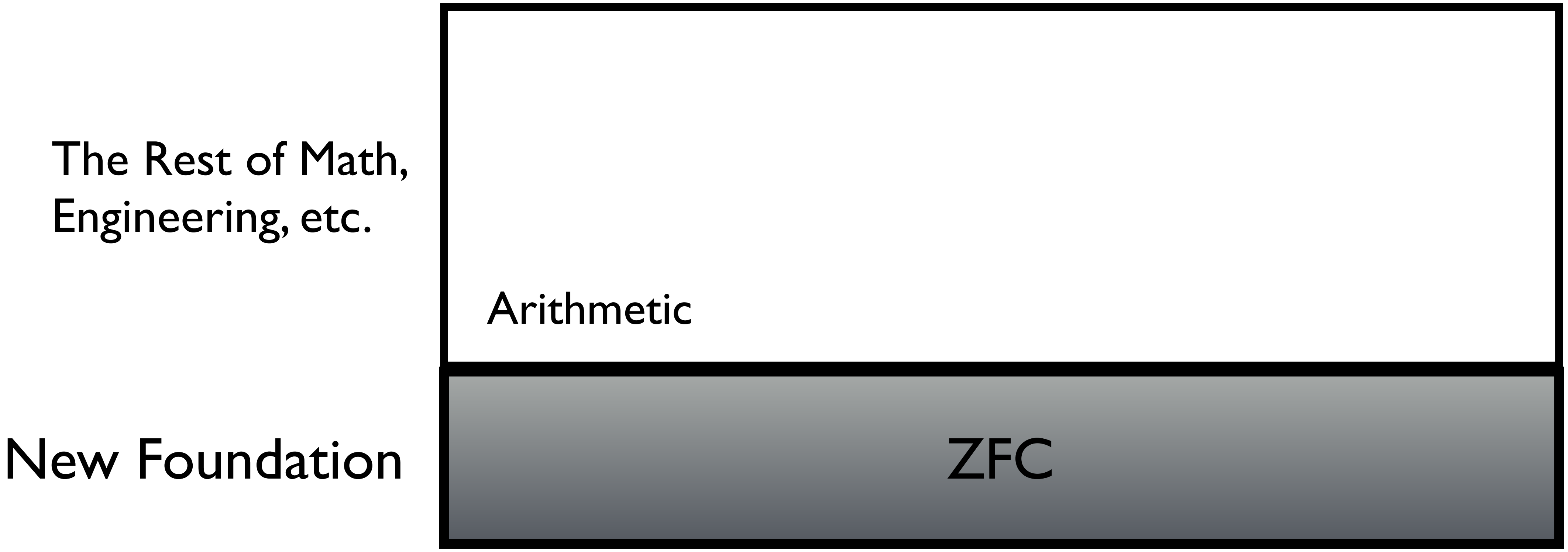

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# Axiom *Schema* of Separation (SEP)

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$$\forall x_1 \ldots \forall x_k \forall x \exists y \forall z [z \in y \leftrightarrow (z \in x \wedge \phi(z, x_1, \ldots, x_k))]$$

where $x$ and $y$ are distinct, and are both distinct from $z$ and the $x_i$;
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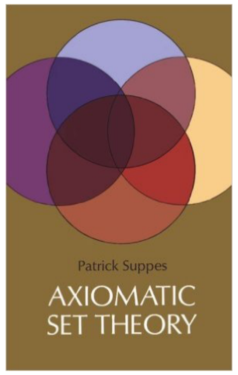
How does this neutralize Russell's letter to Frege?

# Russell's Paradox … to ZFC

$$\vdash \neg \exists x \forall y (y \in x \leftrightarrow y \notin y)$$

(Russell's Theorem; poor Frege!)

http://plato.stanford.edu/entries/russell-paradox/#HOTP



Supplant Cantor's/Frege's Axiom V with the Axiom Schema of Separation (& put on our thinking caps …) and try to show Theorem 1 from Suppes:

$$\vdash \forall x (x \notin \emptyset)$$

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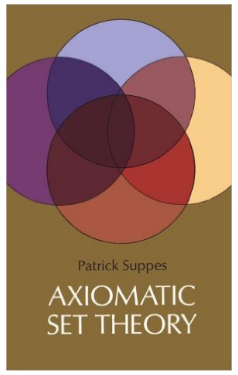

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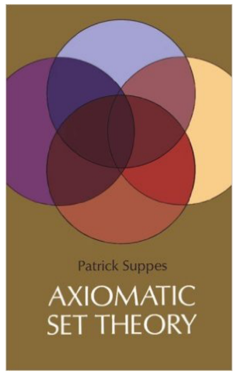

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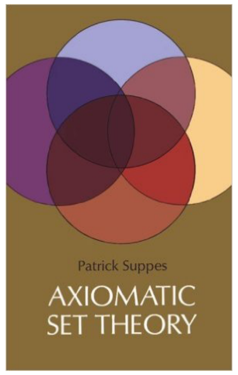

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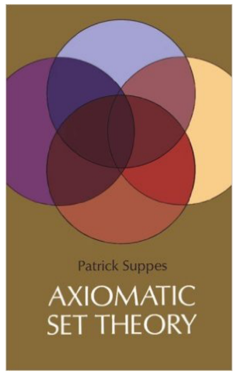

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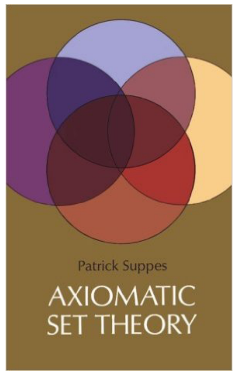

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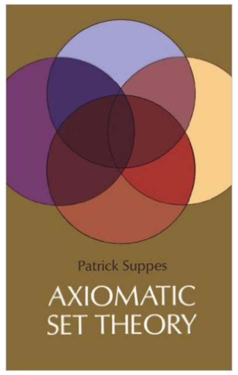

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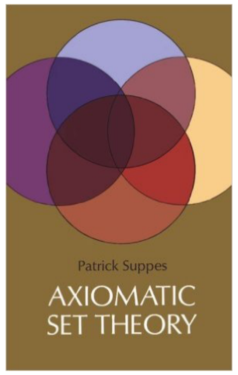

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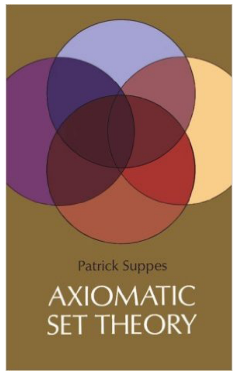

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With this definition, can you prove (Theorem 3) that every set is a subset of itself?

# Formal Natural-Number Arithmetic …

# **PA** (Peano Arithmetic)

A1 $\forall x(0 \neq s(x))$

A2 $\forall x \forall y(s(x) = s(y) \rightarrow x = y)$

A3 $\forall x(x \neq 0 \rightarrow \exists y(x = s(y)))$

A4 $\forall x(x + 0 = x)$

A5 $\forall x \forall y(x + s(y) = s(x + y))$

A6 $\forall x(x \times 0 = 0)$

A7 $\forall x \forall y(x \times s(y) = (x \times y) + x)$

And, every sentence that is the universal closure of an instance of

$$([\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x))] \rightarrow \forall x \phi(x))$$

where $\phi(x)$ is open wff with variable $x$, and perhaps others, free.

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This open wff $\phi(x)$ expresses the arithmetic property 'even.'

# Test 2 Orientation …

# *Slutten*