Gödel's …
First Incompleteness Theorem
(G1)

Selmer Bringsjord*

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Department of Computer Science
Lally School of Management & Technology
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Troy, New York 12180 USA

4/11/2022 (improvements for 4/18/22)

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*With Naveen Sundar G's introduction to Gödel numbering, and his depiction of GT growth, etc.

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Note: This is a version designed for those who have had at least one robust, proof-intensive university-level course in formal logic to the level of $\mathscr{L}_2$.

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Background Context …

Gödel's Great Theorems (OUP)

by Selmer Bringsjord

- Introduction ("The Wager")
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
- The Continuum-Hypothesis Theorem
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- Gödel's "God Theorem"
- Could a Finite Machine Match Gödel's Greatness?

STOP & REVIEW IF NEEDED!

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A corollary of the First Incompleteness Theorem: *We cannot prove (in classical mathematics) that mathematics is consistent.*

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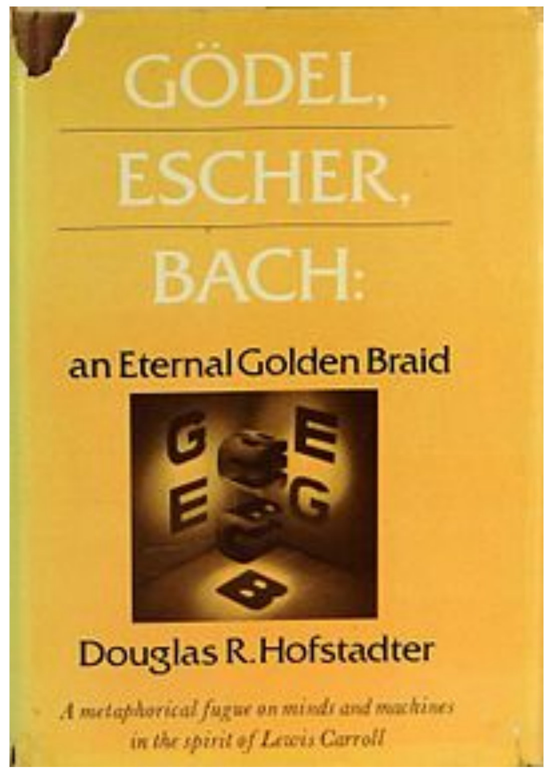
By far the greatest of GGT; Selm's analysis based Sherlock Holmes' mystery "Silver Blaze."

Deficient; Beware

Deficient; Beware



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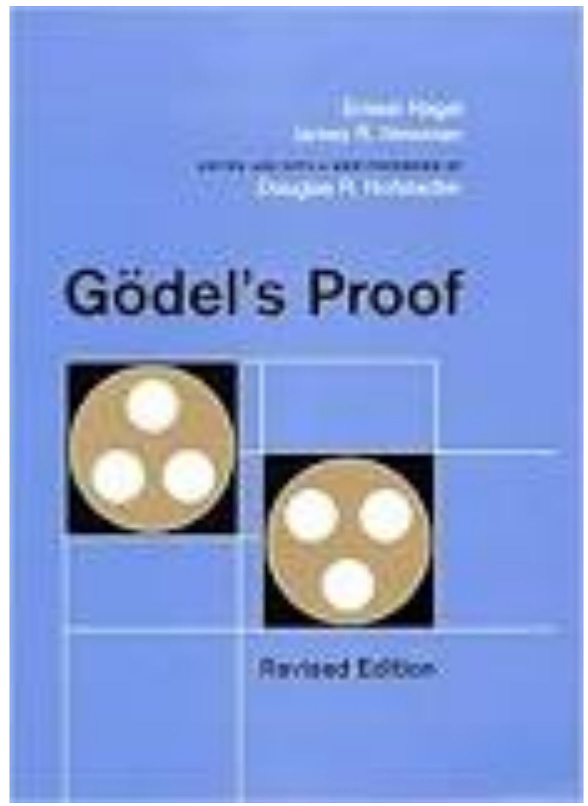

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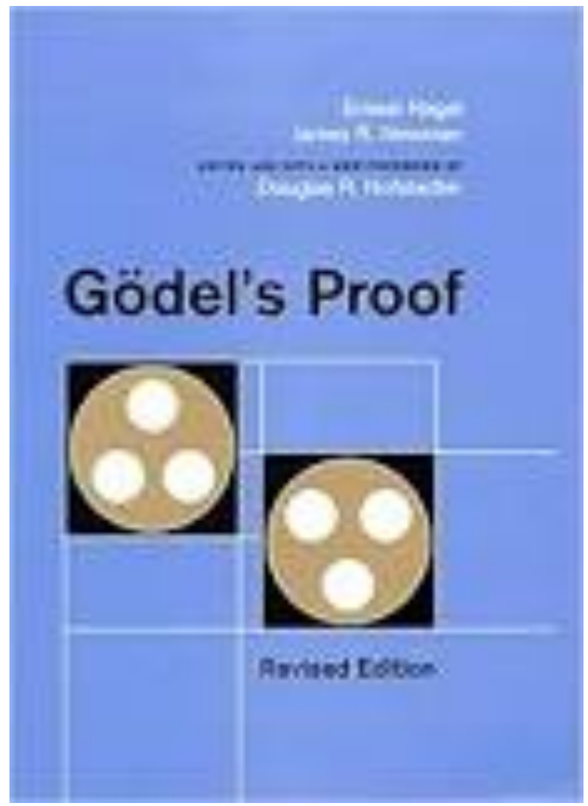

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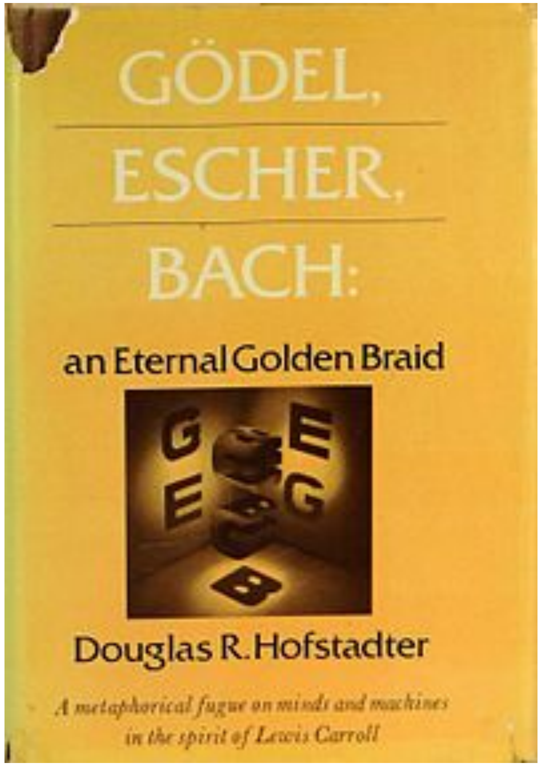

Some Timeline Points

1978 Princeton NJ USA.

1940 Back to USA, for good.

1936 Schlick murdered; Austria annexed

1933 Hitler comes to power.

1930 Announces (First) *In*completeness Theorem

1929 Doctoral Dissertation: Proof of Completeness Theorem

Undergrad in seminar by Schlick

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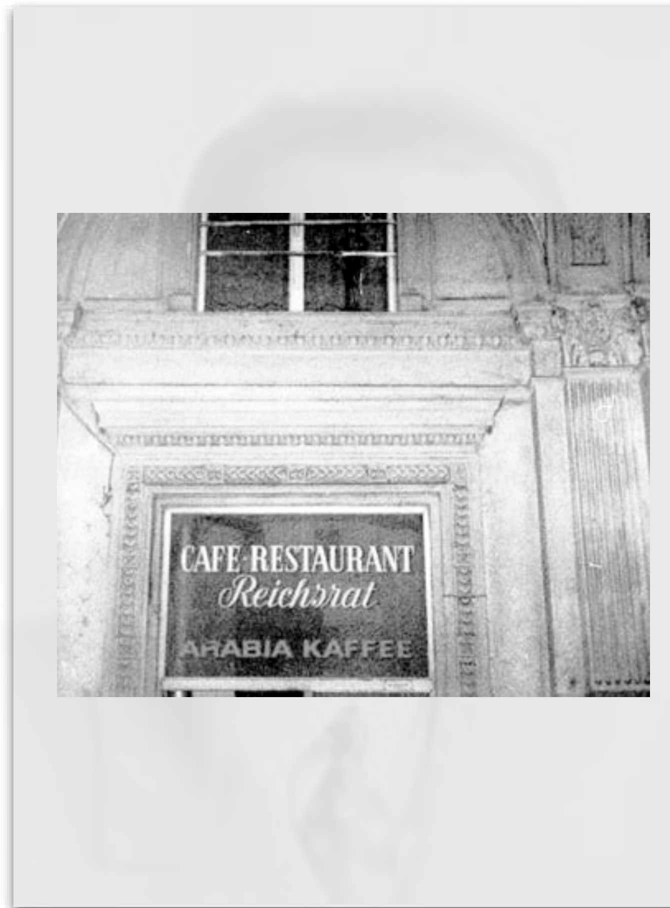

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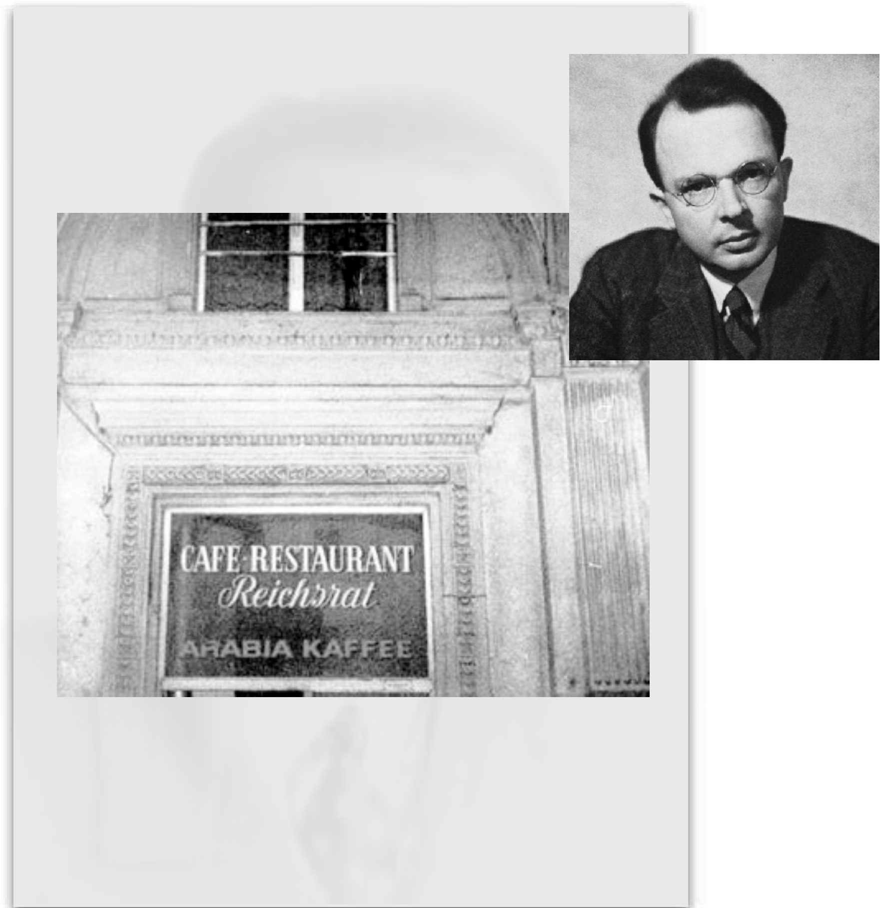

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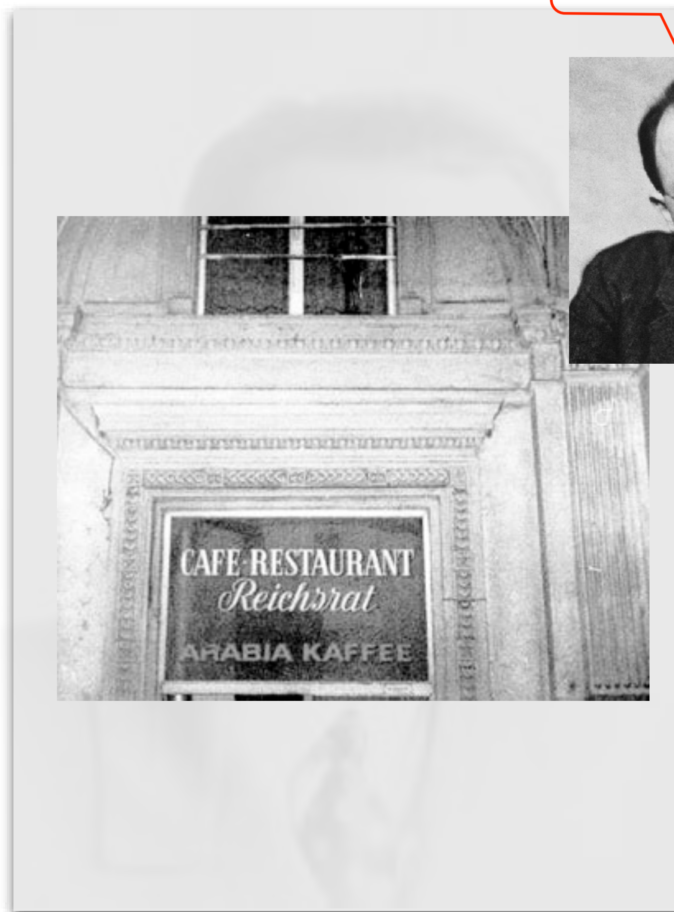

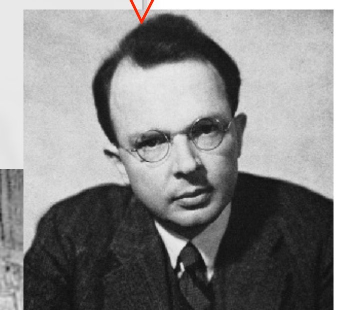

The “Liar Tree”

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The Liar Paradox

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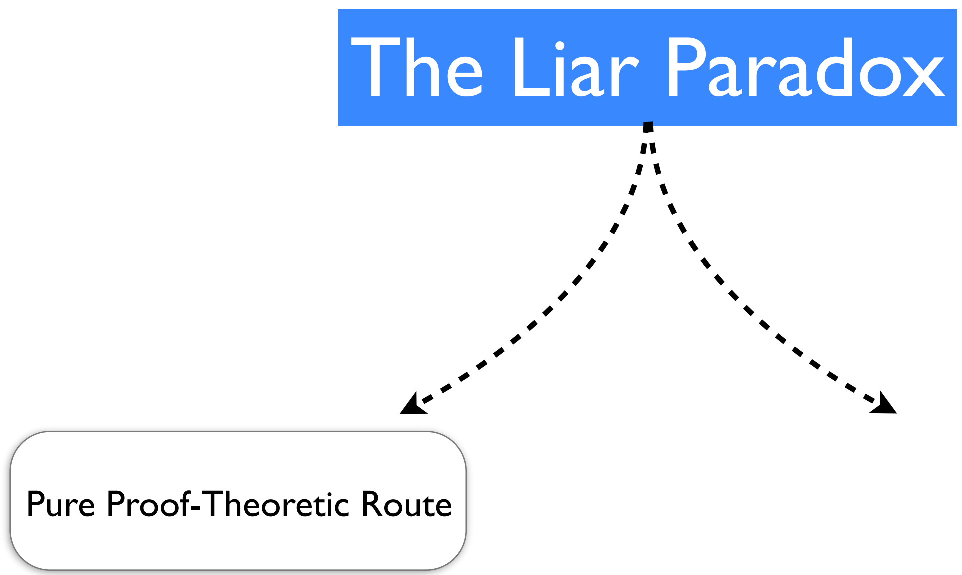
The Liar Paradox

Pure Proof-Theoretic Route

The "Liar Tree"



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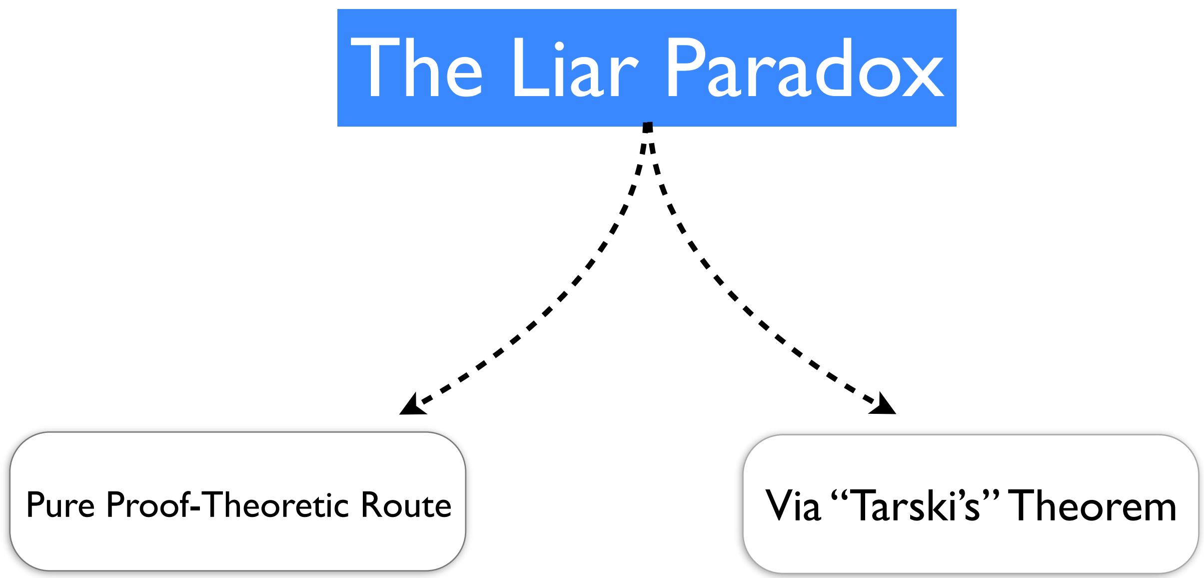

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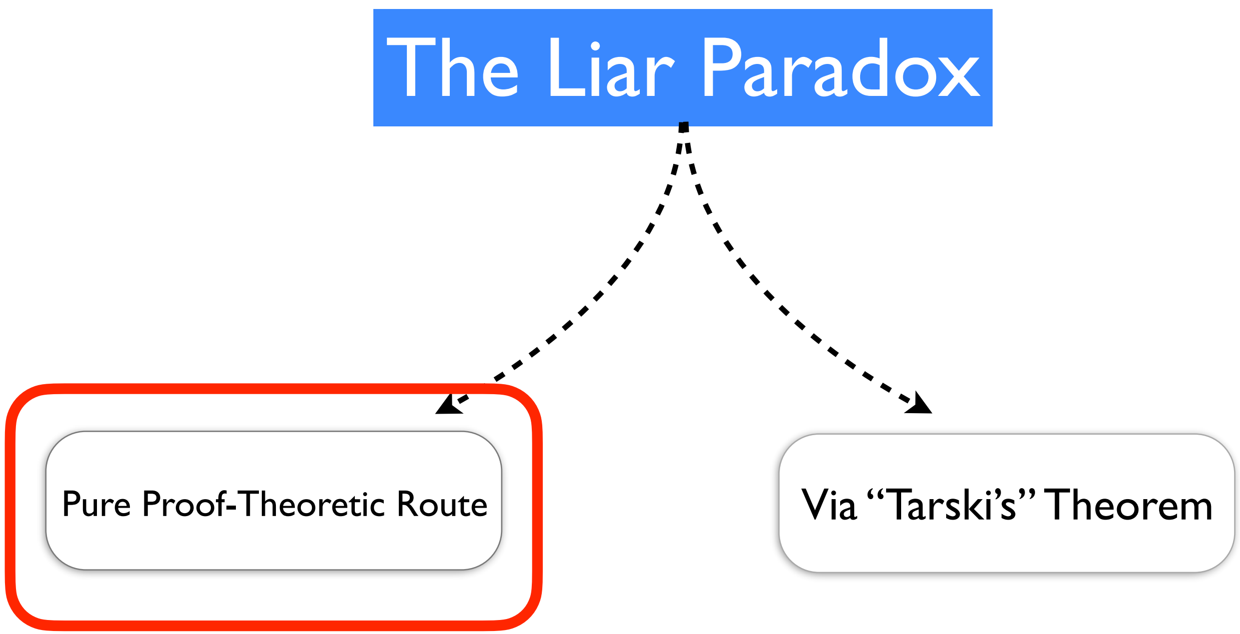

The “Liar Tree”

The Liar Paradox

Pure Proof-Theoretic Route

Via “Tarski’s” Theorem

Paul Erdős

“The Book”

The "Liar Tree"

The Liar Paradox

Pure Proof-Theoretic Route

Via "Tarski's" Theorem

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"The Book"

Ergo, step one: What *is* LP?

“The (Economical) Liar”

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L: This sentence is false.

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Suppose that T(**L**); then ¬T(**L**).

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Contradiction!

The “Gödelian” Liar (from me)

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$\bar{P}$: This sentence is unprovable.

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Suppose that $\bar{P}$ is true. Then we can immediately deduce that $\bar{P}$ is provable, because here is a proof: $\bar{P} \rightarrow \bar{P}$ is an easy theorem, and from it and our supposition we deduce $\bar{P}$ by *modus ponens*. But since what $\bar{P}$ says is that it's unprovable, we have deduced that $\bar{P}$ is false under our initial supposition.

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Suppose on the other hand that $\bar{P}$ is false. Then we can immediately deduce that $\bar{P}$ is unprovable: Suppose for *reductio* that $\bar{P}$ *is* provable; then $\bar{P}$ holds as a result of some proof, but what $\bar{P}$ says is that it's unprovable; and so we have contradiction. But since what $\bar{P}$ says is that it's unprovable, and we have just proved that under our supposition, we arrive at the conclusion that $\bar{P}$ is true.

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Contradiction!

All of this is fishy; but Gödel transformed it into utterly precise, impactful, indisputable reasoning …

PA (Peano Arithmetic):

$$
\begin{array}{ll}
\text{A1} & \forall x(0 \neq s(x)) \\
\text{A2} & \forall x \forall y(s(x) = s(y) \rightarrow x = y) \\
\text{A3} & \forall x(x \neq 0 \rightarrow \exists y(x = s(y))) \\
\text{A4} & \forall x(x + 0 = x) \\
\text{A5} & \forall x \forall y(x + s(y) = s(x + y)) \\
\text{A6} & \forall x(x \times 0 = 0) \\
\text{A7} & \forall x \forall y(x \times s(y) = (x \times y) + x)
\end{array}
$$

And, every sentence that is the universal closure of an instance of

$$([\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x))] \rightarrow \forall x \phi(x))$$

where $\phi(x)$ is open wff with variable x, and perhaps others, free.

Arithmetic Crucial Part of All Things Sci/Eng/Tech!

but alas, courtesy of Gödel: An infinite number of arithmetic propositions impossible to settle/decide.

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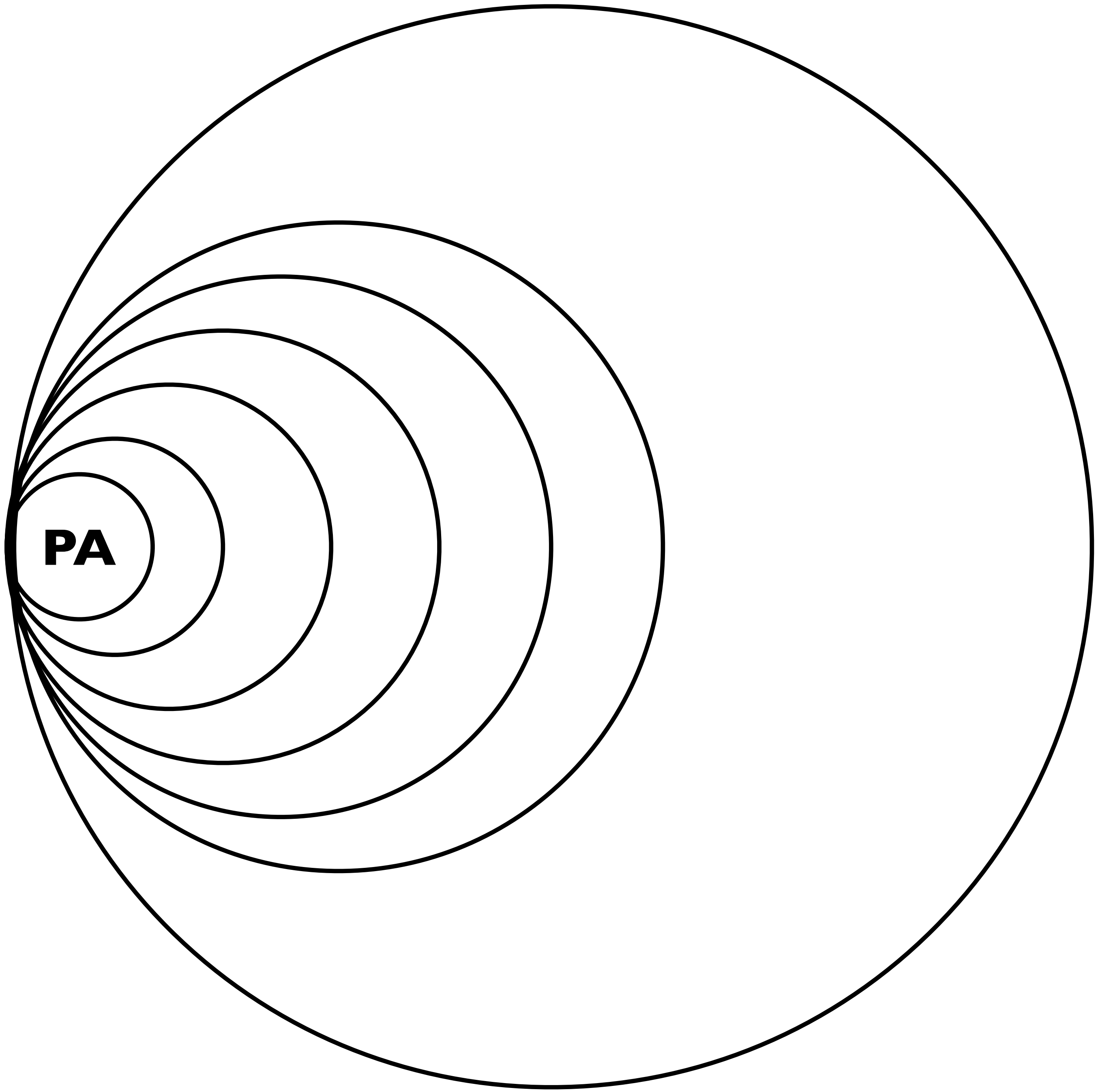

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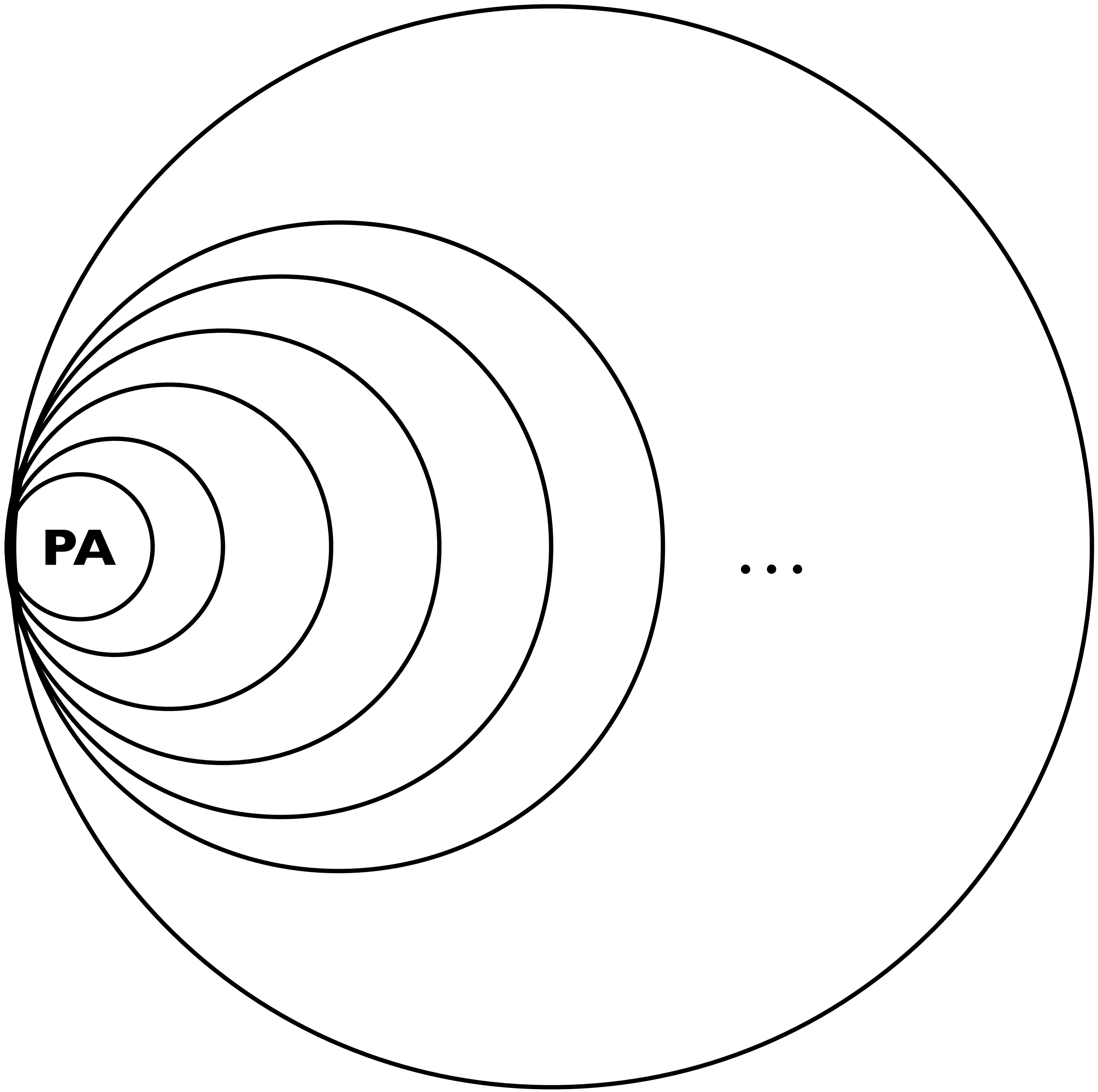

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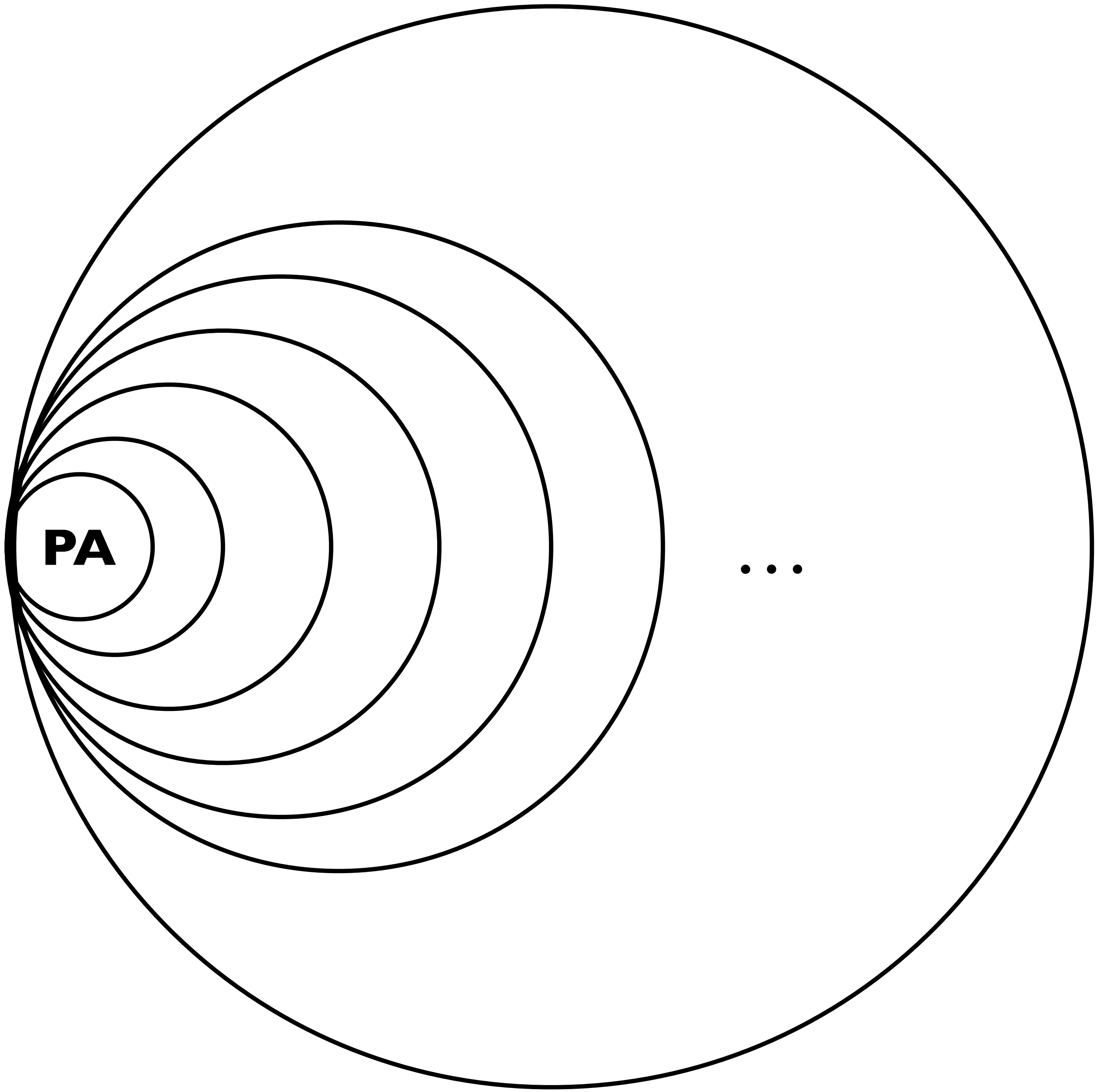

Each circle is a larger part of the formal sciences.

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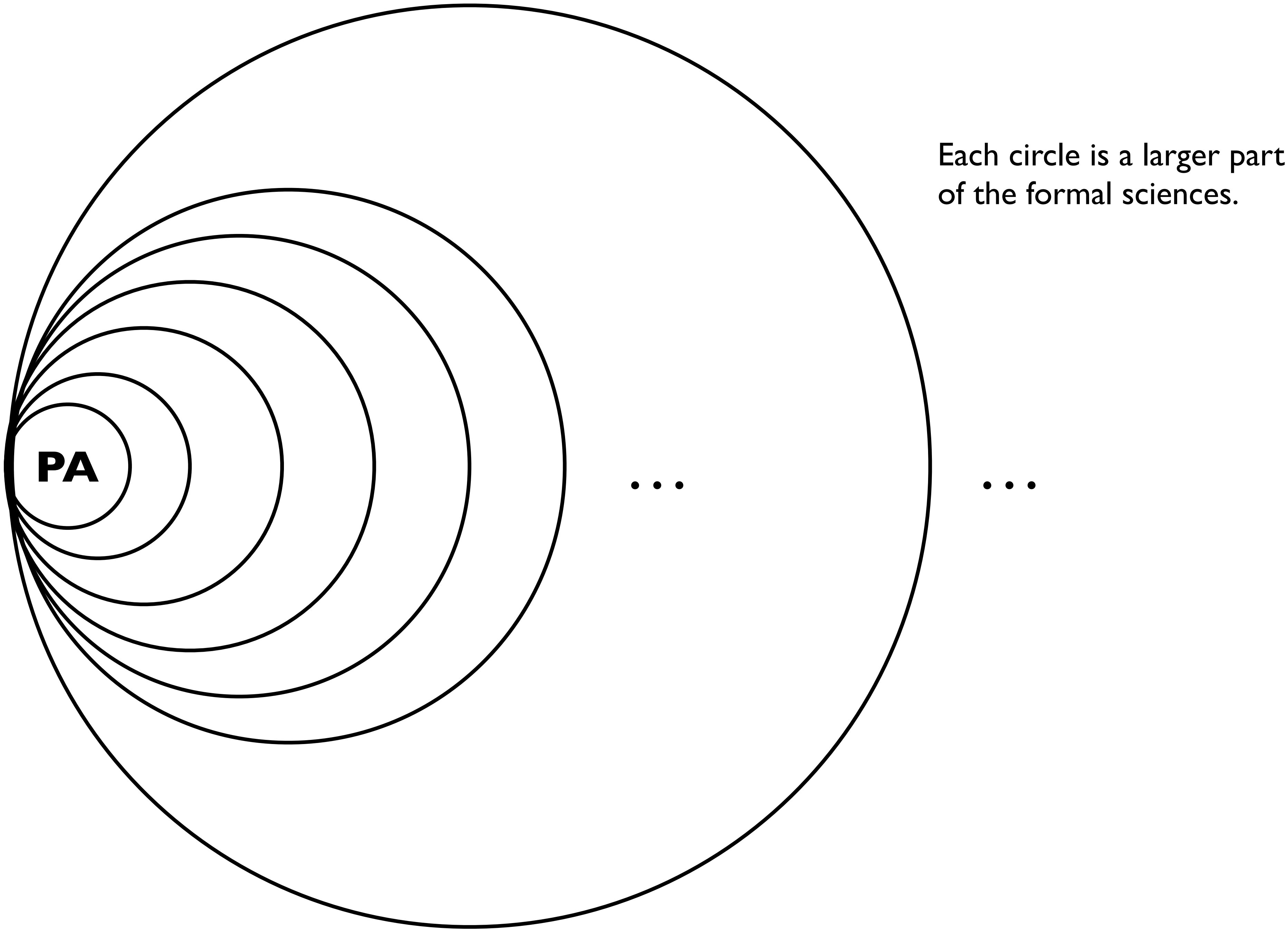

Gödel Numbering

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Problem: How do we enable a formula to refer to other formulae and itself (and also other objects like proofs, terms etc.), in a perfectly consistent way?

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$$\boxed{\begin{array}{l} \phi \\ \phi \rightarrow \psi \\ f(x, a) \end{array}}$$

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Object-level objects in the language of $\mathscr{L}_1$

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ϕ

$\phi \rightarrow \psi$

$f(x, a)$

323

23432

142323

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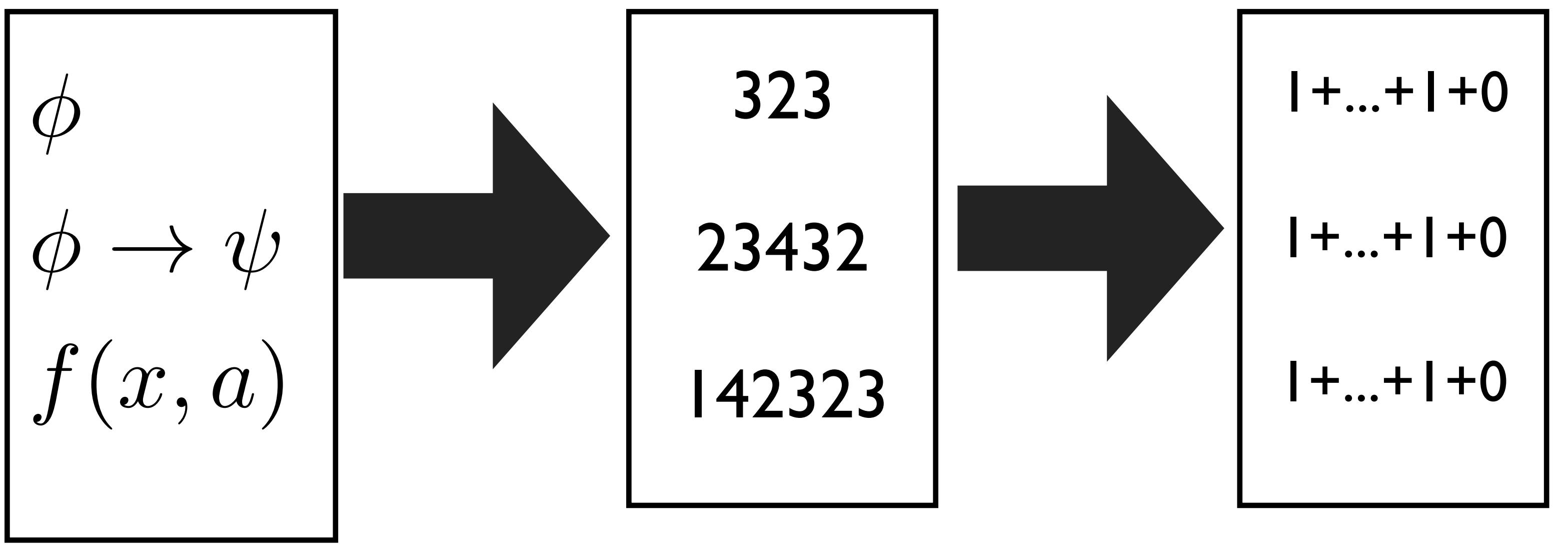

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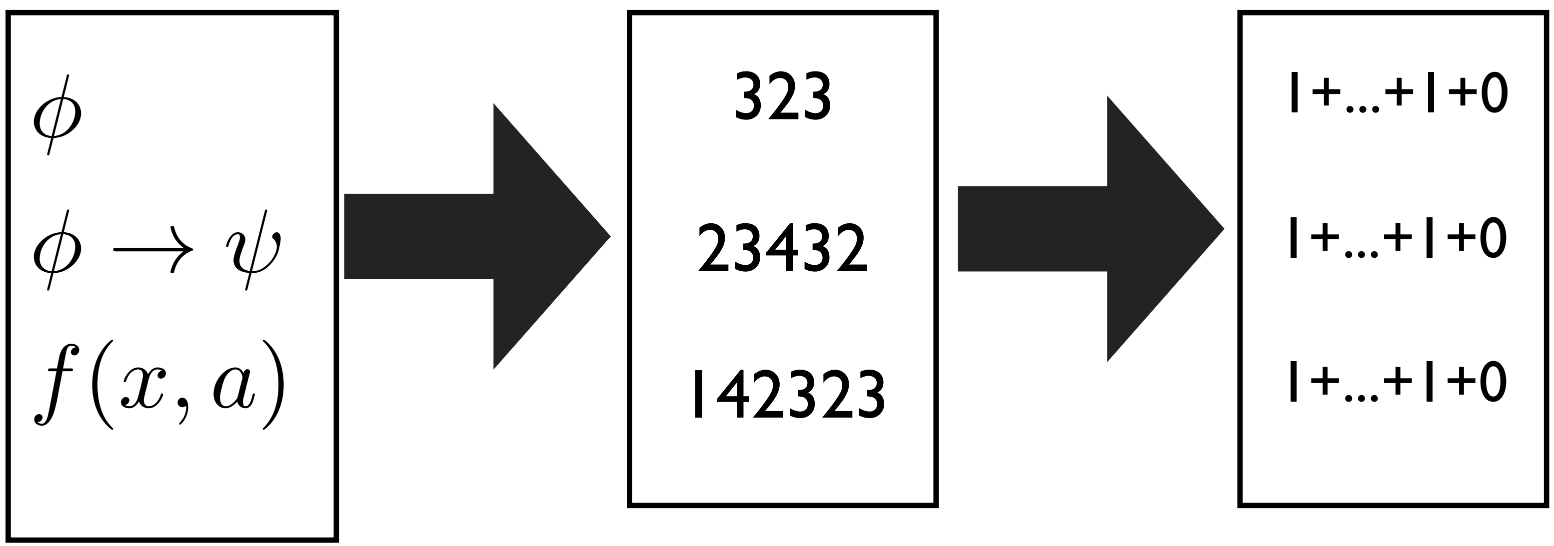

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S will often conflate.

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Just realize that every entry in a dictionary is named by a number *n*, and by the same basic lexicographic ordering, every computer program, formula, etc. is named by a number *m* in a lexicographic ordering going from 1, to 2, to …

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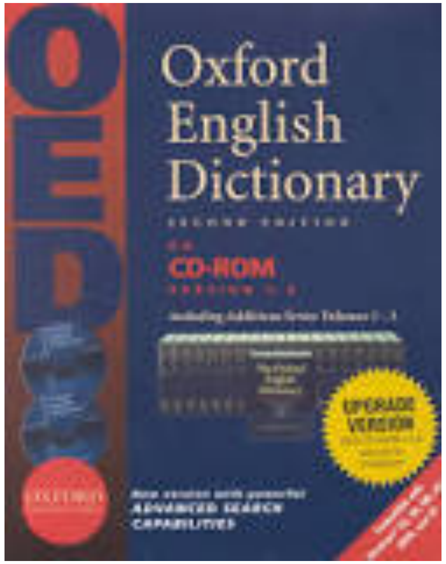

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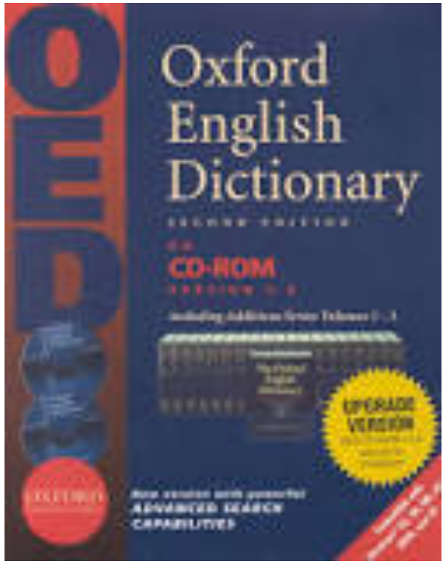

So, `gimcrack` is named by some positive integer *k*. Hence, I can just refer to this word as "*k*" Or in the notation I prefer: k^{gimcrack}.

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Or, every syntactically valid computer program in Clojure that you will ever write can be uniquely denoted by some number *m* in the lexicographic ordering of all syntactically valid such programs. So your program π can just be coded as a numeral m^{π} in a formal language that captures arithmetic (i.e., an *arithmetic language*).

Gödel's First Incompleteness Theorem

Let Φ be a set of arithmetic sentences that is

(i) consistent (i.e. no contradiction $\phi \wedge \neg\phi$ can be deduced from Φ);
(ii) s.t. an algorithm is available to decide whether or not a given string u is a member of Φ; and
(iii) sufficiently expressive to capture all of the operations of a standard computing machine (e.g. a Turing machine, register machine, KU machine, etc.).

Then there is an "undecidable" arithmetic sentence $\mathscr{G}$ from Gödel that can't be proved from Φ, nor can the negation of this sentence (i.e. $\neg\mathscr{G}$) be proved from Φ!

Alas, that’s painfully verbose.

Gödel's First Incompleteness Theorem

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Suppose $\Phi \supset \mathbf{PA}$ (= Φ contains **PA**) that is

(i) Con Φ;
(ii) Turing-decidable, and
(iii) sufficiently expressive to capture all of the operations of a Turing machine (i.e. Repr Φ).

Then there is an arithmetic sentence $\mathcal{G}$ s.t.
$\Phi \nvdash \mathcal{G}$ and $\Phi \nvdash \neg\mathcal{G}$.

Gödel's First Incompleteness Theorem

Remember Church's Theorem!

Suppose $\mathbf{\Phi} \supset \mathbf{PA}$ (= Φ contains **PA**) that is

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Gödel's First Incompleteness Theorem

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Suppose $\mathbf{\Phi} \supset \mathbf{PA}$ (= $\mathbf{\Phi}$ contains **PA**) that is

(i) Con $\mathbf{\Phi}$;
(ii) Membership in $\mathbf{\Phi}$ is Turing-decidable, and
(iii) Alll recursive = Turing-decidable arithmetic functions and relations is representable in $\mathbf{\Phi}$ (i.e. Repr $\mathbf{\Phi}$). In other words, we can logicize any meta-logical statement that says some Turing-decidable relation holds of some natural numbers.

Then there is an arithmetic sentence $\mathscr{G}$ s.t.
$\mathbf{\Phi} \nvdash \mathscr{G}$ and $\mathbf{\Phi} \nvdash \neg\mathscr{G}$.

To prove G1, we shall allow ourselves …

The Fixed Point Theorem (FPT)

Assume that Φ is a set of arithmetic sentences such that Repr Φ. Then for every arithmetic formula $\psi(x)$ with one free variable x, there is an arithmetic sentence ϕ s.t.

$\Phi \vdash \phi \leftrightarrow \psi(\hat{n}^{\phi})$.

We can intuitively understand ϕ to be saying:
"I have the property ascribed to me by the formula ψ."

“I thought there was no free lunch!”

[W]e “would hope that such a deep theorem would have an insightful proof. No such luck. I am going to write down a sentence … and verify that it works. What I won’t do is give you a satisfactory explanation for why I write down the particular formula that I do. I write down the formula because Gödel wrote down the formula, and Gödel wrote down the formula because, when he played the logic game he was able to see seven or eight moves ahead, whereas you and I are only able to see one or two moves ahead. I don’t know anyone who thinks he has a fully satisfying understanding of why the Self-referential Lemma [= FPT] works. It has a rabbit-out-of-a-hat quality for everyone.”

—V. McGee, 2002; as quoted in (Salehi 2020)

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Ok; so let’s do it …

Proof: Let Φ be a set of arithmetic sentences, and suppose (for conditional intro) the antecedent of **G1** holds, i.e. (i)–(iii) hold. We must show that neither $\mathscr{G}$, nor the negation of this (Liar-Paradox-inspired) arithmetic sentence, can be proved from Φ. We know, respectively, that for any theorem ϕ of Φ, and from an instantiation of FPT, that:

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(FPT*) = (2) $\mathbf{\Phi} \vdash \mathscr{G}$ if and only if $\neg \mathscr{T}(\hat{n}^{\mathscr{G}})$.

Proof: Let $\mathbf{\Phi}$ be a set of arithmetic sentences, and suppose (for conditional intro) the antecedent of **G I** holds, i.e. (i)–(iii) hold. We must show that neither $\mathscr{G}$, nor the negation of this (Liar-Paradox-inspired) arithmetic sentence, can be proved from $\mathbf{\Phi}$. We know, respectively, that for any theorem ϕ of $\mathbf{\Phi}$, and from an instantiation of FPT, that:

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Here, ϕ is of course a variable in (1) for any formula; and $\mathscr{T}$ is a logicization of Thm. Now suppose $\mathbf{\Phi} \vdash \mathscr{G}$. By right-to-left on (1) we deduce Thm$(n^{\mathscr{G}})$. We can logicicize this as $\neg\neg\mathscr{T}(\hat{n}^{\mathscr{G}})$. Then by *modus tollens* $\mathbf{\Phi} \nvdash \mathscr{G}$, by right-to-left on (2). Contradiction!

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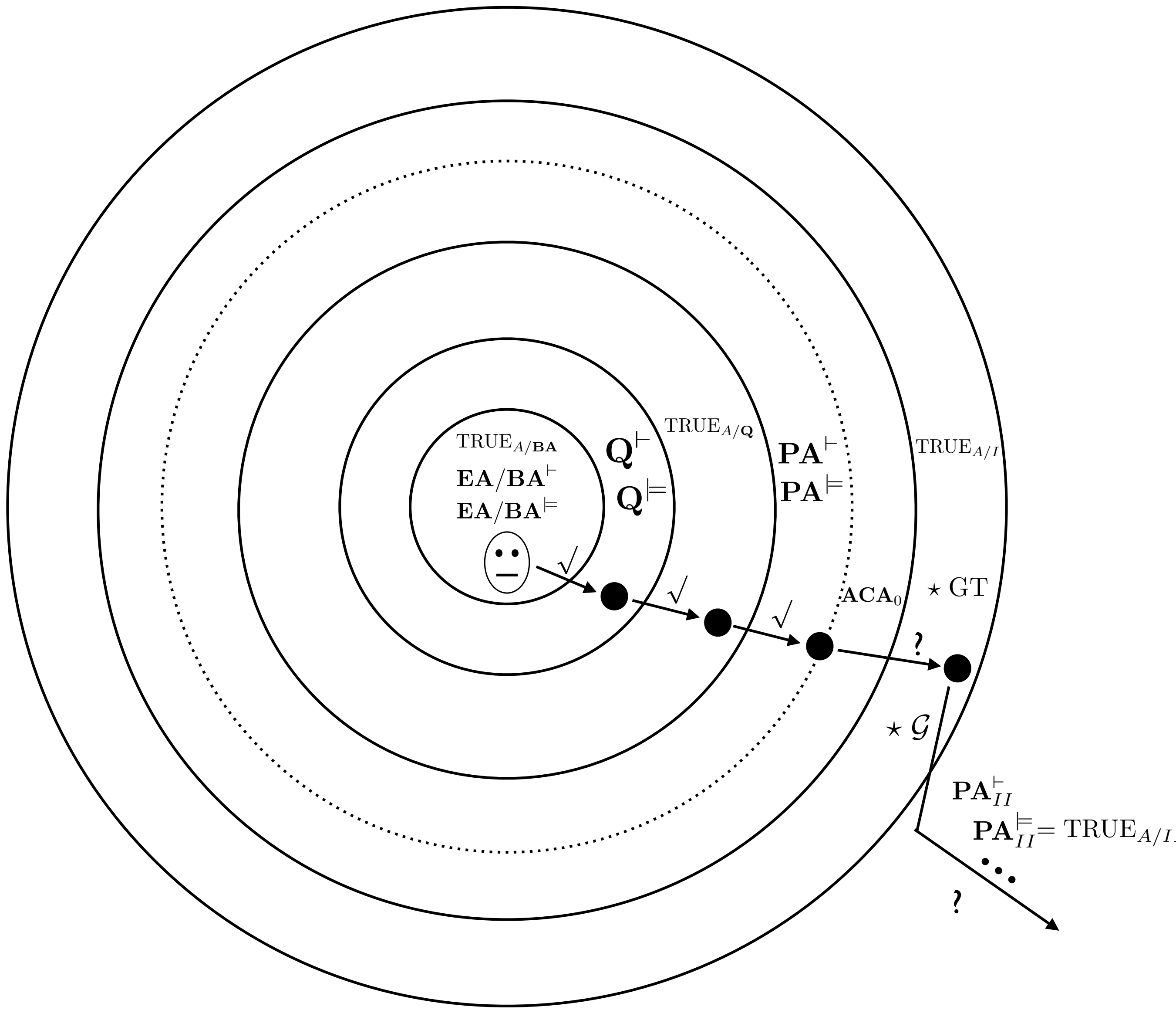
Suppose on the other hand $\Phi \vdash \neg\mathscr{G}$. And, suppose for *reductio* that $\neg\text{Thm}(n^{\mathscr{G}})$. We can logicize this as $\neg\mathscr{T}(\hat{n}^{\mathscr{G}})$, and then we can use this right-to-left on (2) to deduce $\Phi \vdash \mathscr{G}$. But this entails Inc Φ = Con Φ. Yet our original assumptions (it's (i), specifically) include Con Φ, so: contradiction. Therefore (by negation elim) we have $\text{Thm}(n^{\mathscr{G}})$. But from this, left-to-right on (1), we have $\Phi \vdash \mathscr{G}$. But then we have that $\mathscr{G}$ is both provable and not provable from Φ, which is a contradiction with (i) = Con Φ! **QED**

“Silly abstract nonsense! There aren’t any concrete examples of $\mathscr{G}$!”

Astrologic:

Rational Aliens Will be on the Same "Race Track"!



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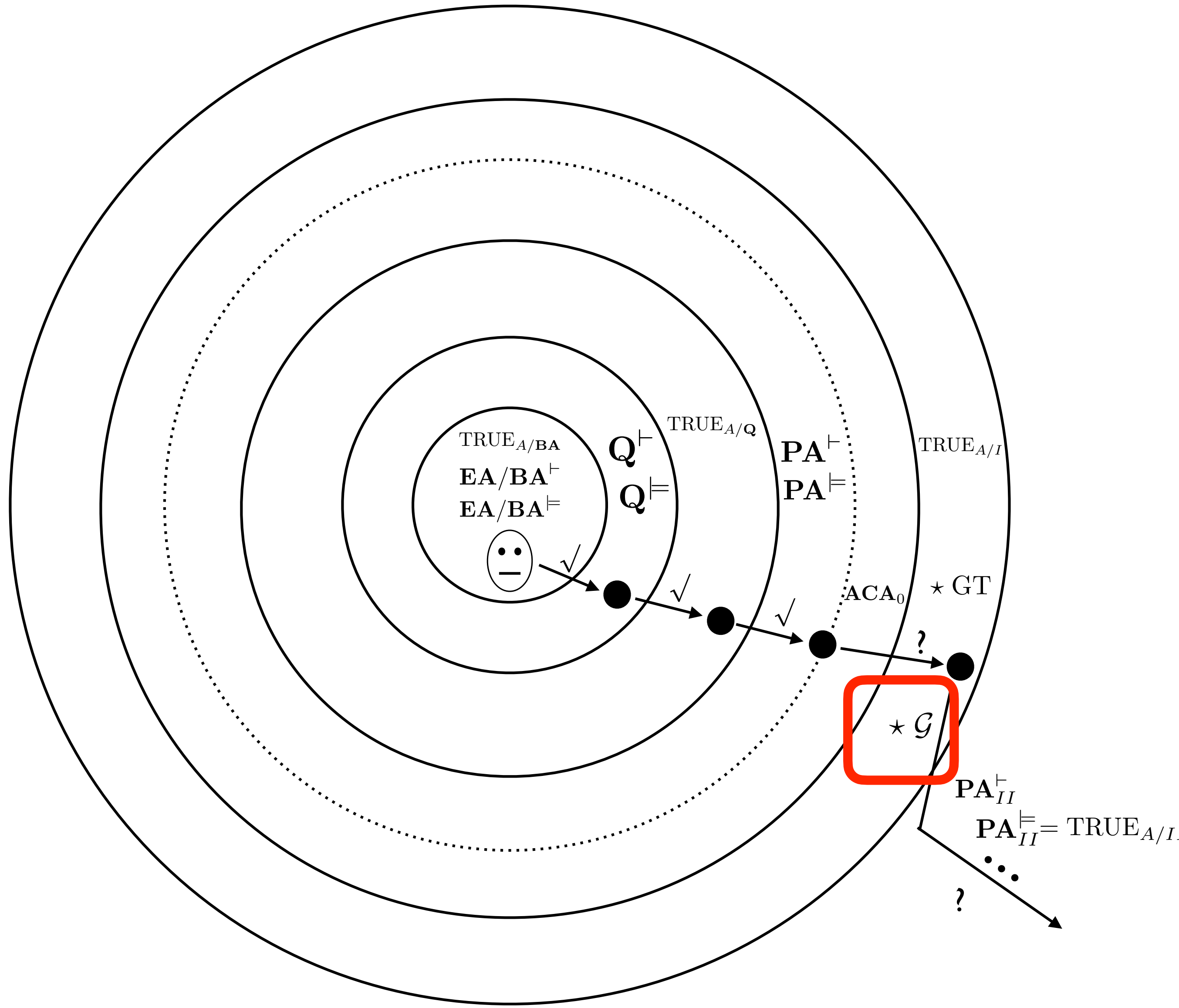

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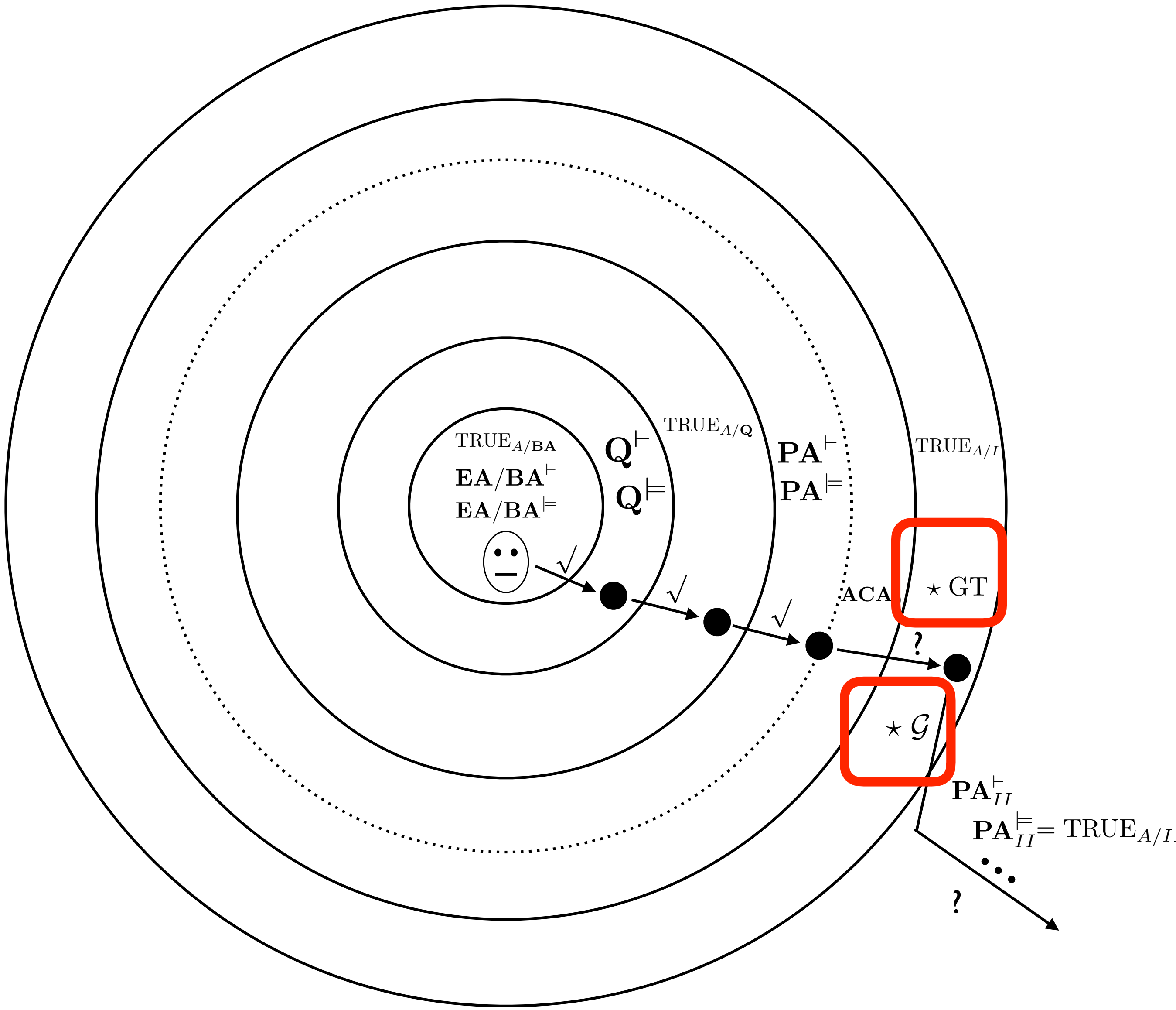

Ah, but e.g.: Goodstein's Theorem!

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The Goodstein Sequence goes to zero!

Pure base *n* representation of a number *r*

- Represent *r* as only sum of powers of *n* in which the exponents are also powers of *n*, etc.

$$266 = 2^{2^{(2^{2^0}+2^0)}} + 2^{(2^{2^0}+2^0)} + 2^{2^0}$$

Grow Function

$Grow_k(n)$:

1. Take the pure base k representation of n
2. Replace all k by $k+1$. Compute the number obtained.
3. Subtract one from the number

Example of **Grow**

$$Grow_2(19)$$

$$19 = 2^{2^{2^{2^0}}} + 2^{2^0} + 2^0$$

$$3^{3^{3^{3^0}}} + 3^{3^0} + 3^0$$

$$3^{3^{3^{3^0}}} + 3^{3^0} + 3^0 - 1$$

$$7625597484990$$

Goodstein Sequence

- For any natural number *m*

$$
\begin{aligned}
&m \\
&Grow_2(m) \\
&Grow_3(Grow_2(m)) \\
&Grow_4(Grow_3(Grow_2(m))), \\
&\ldots
\end{aligned}
$$

Sample Values

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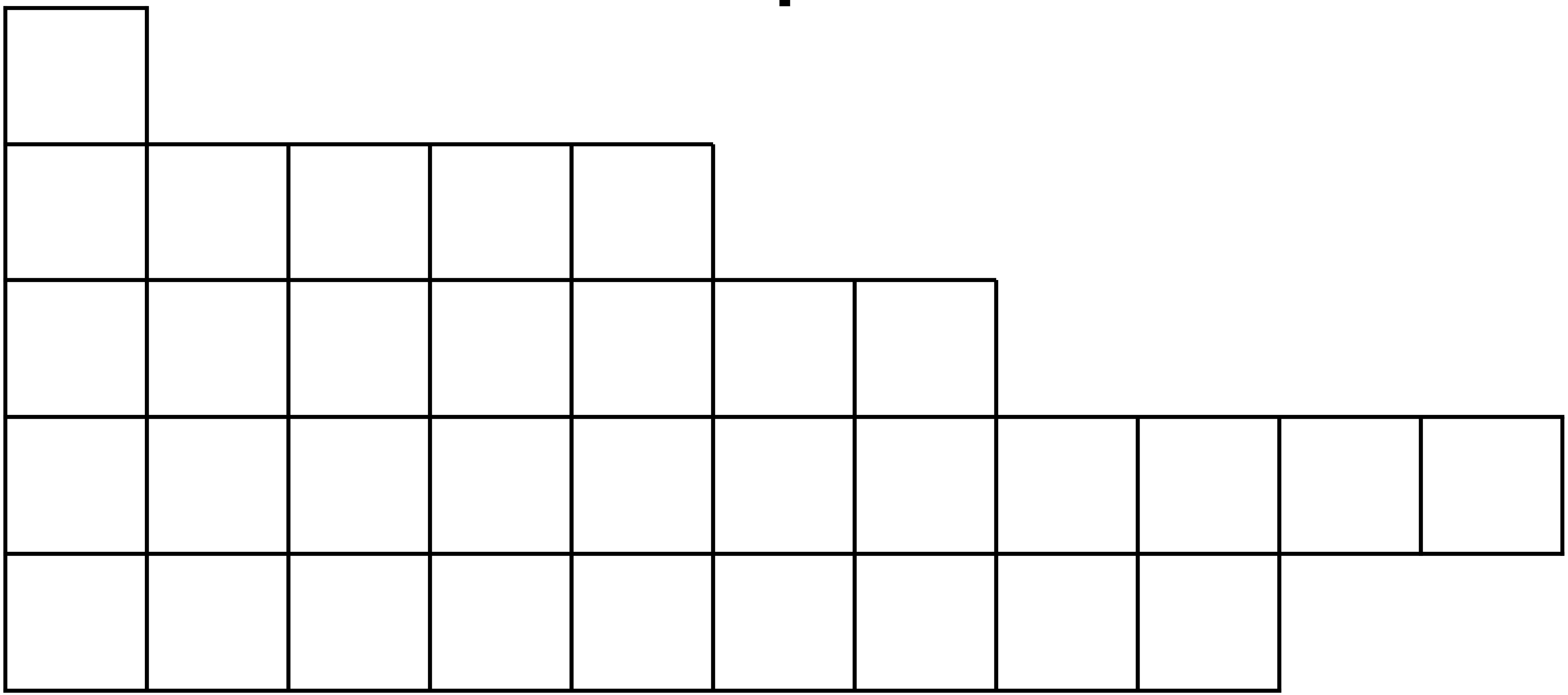

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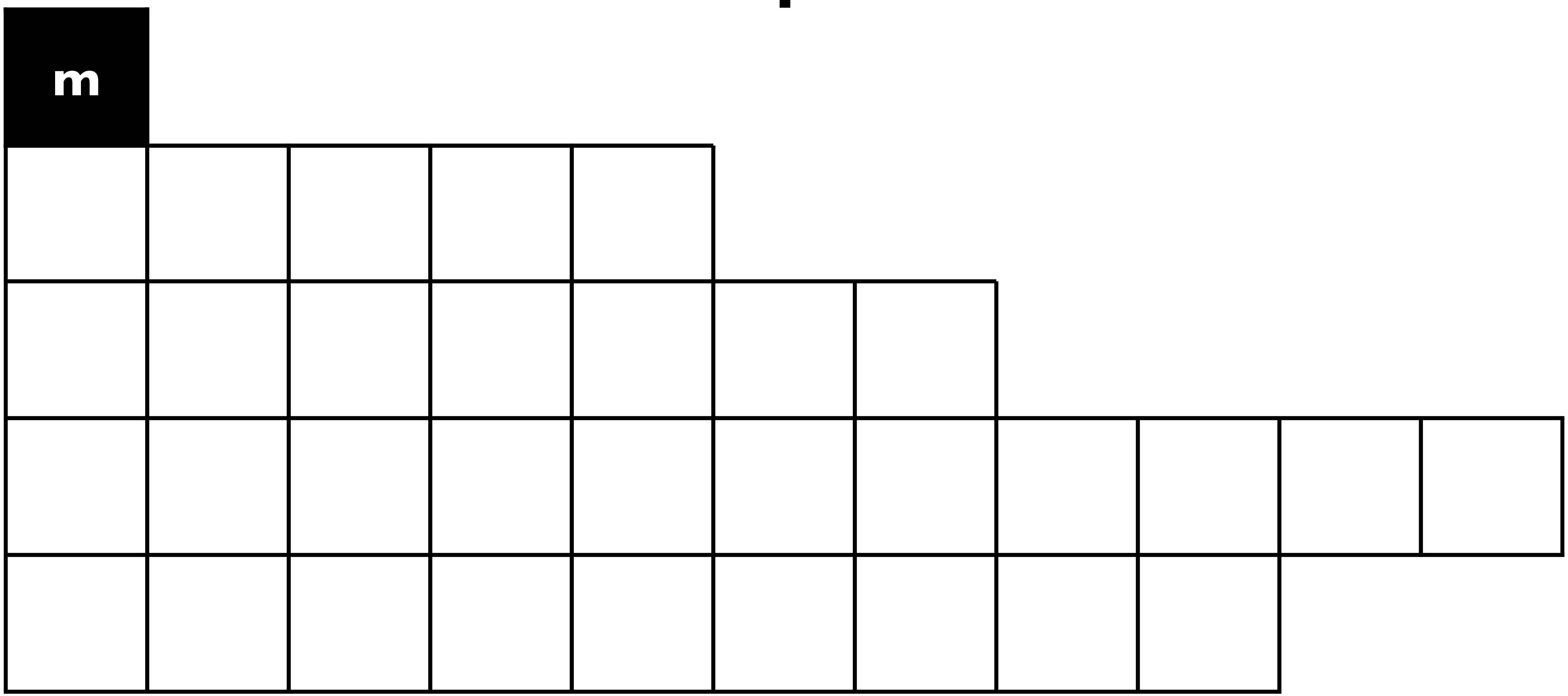

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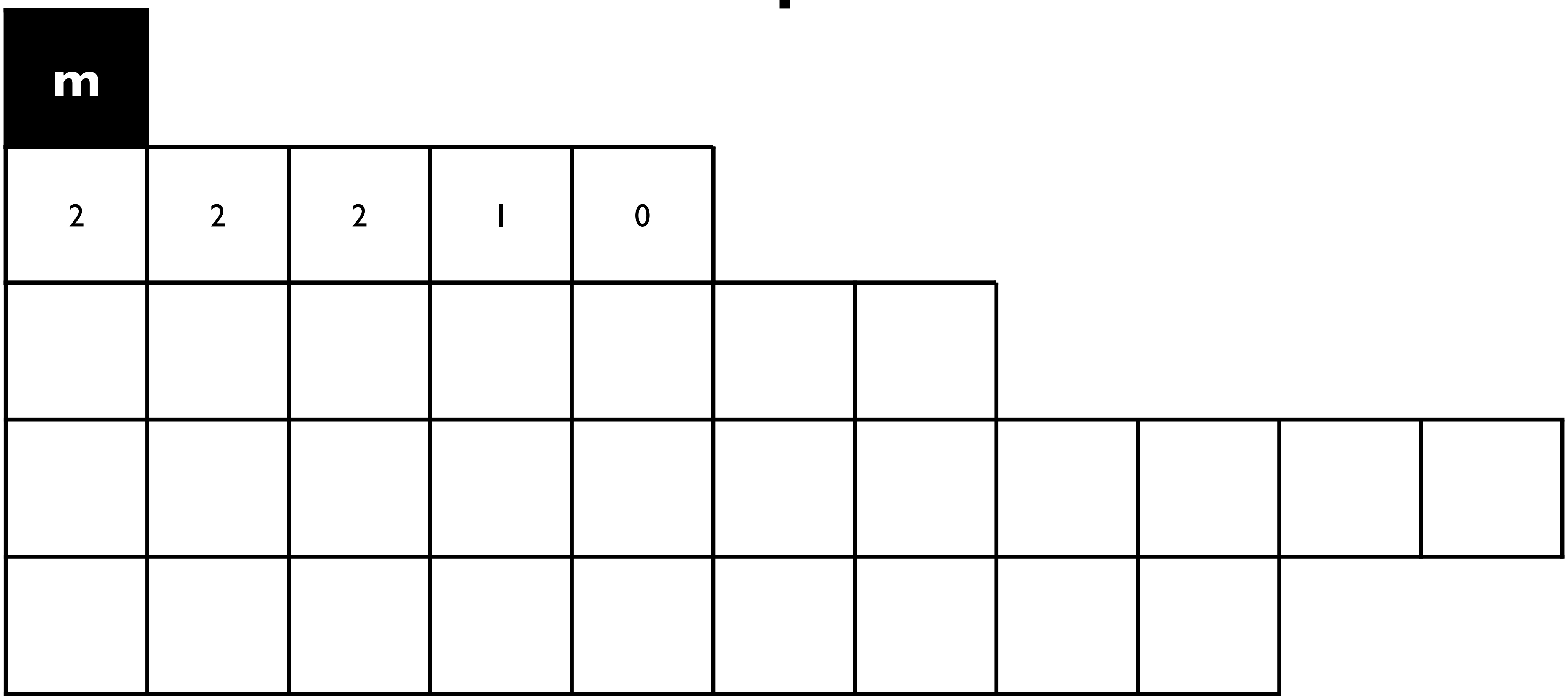

Sample Values

m										
2	2	2	1	0						
3	3	3	3	2	1	0				

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m										
2	2	2	1	0						
3	3	3	3	2	1	0				
4	4	26	41	60	83	109	139	...	11327 (96th term)	...

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m										
2	2	2	1	0						
3	3	3	3	2	1	0				
4	4	26	41	60	83	109	139	...	11327 (96th term)	...
5	15	$\sim 10^{13}$	$\sim 10^{155}$	$\sim 10^{2185}$	$\sim 10^{36306}$	10^{695975}	$10^{15151337}$	...		

This sequence actually goes to zero!

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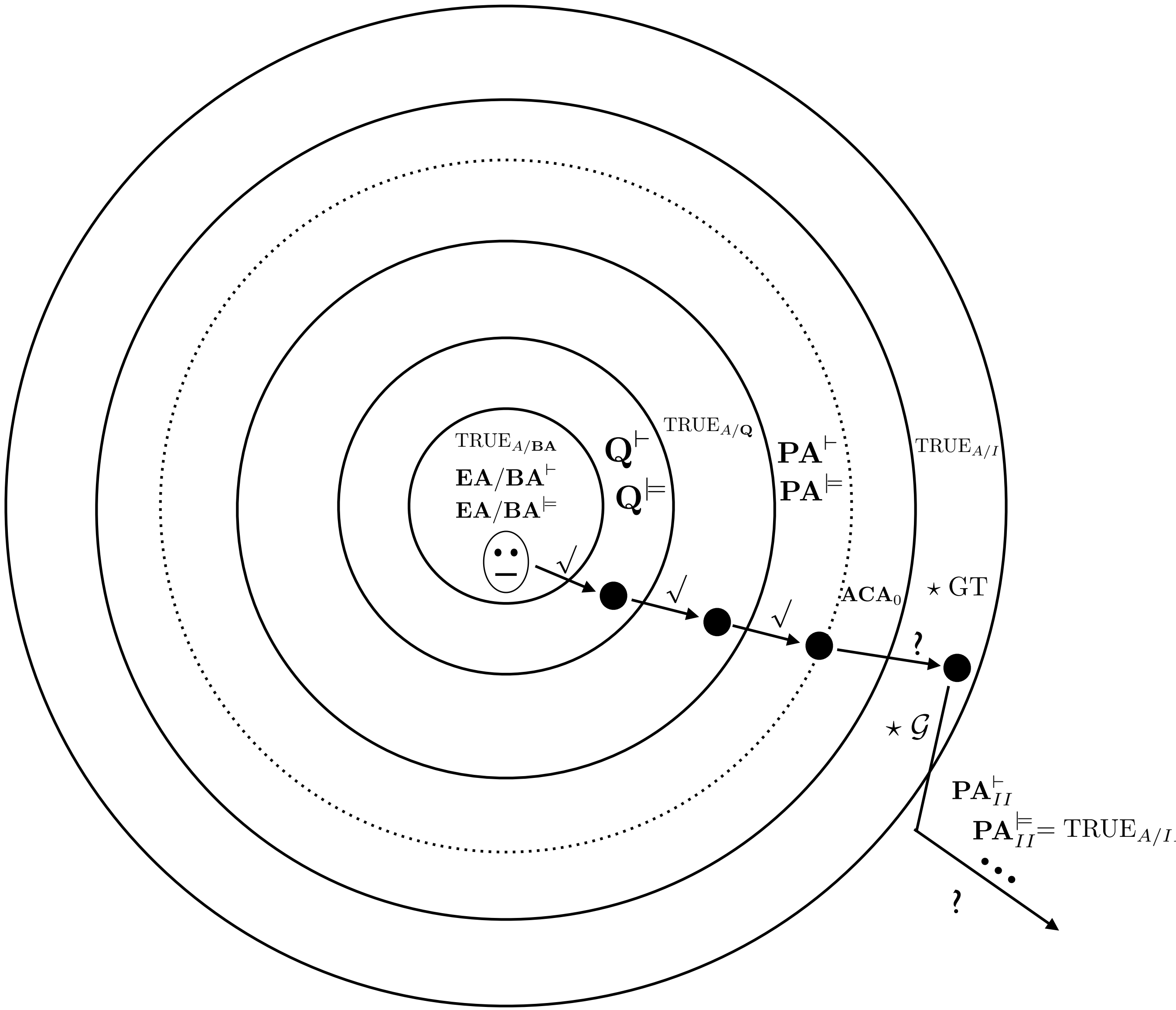

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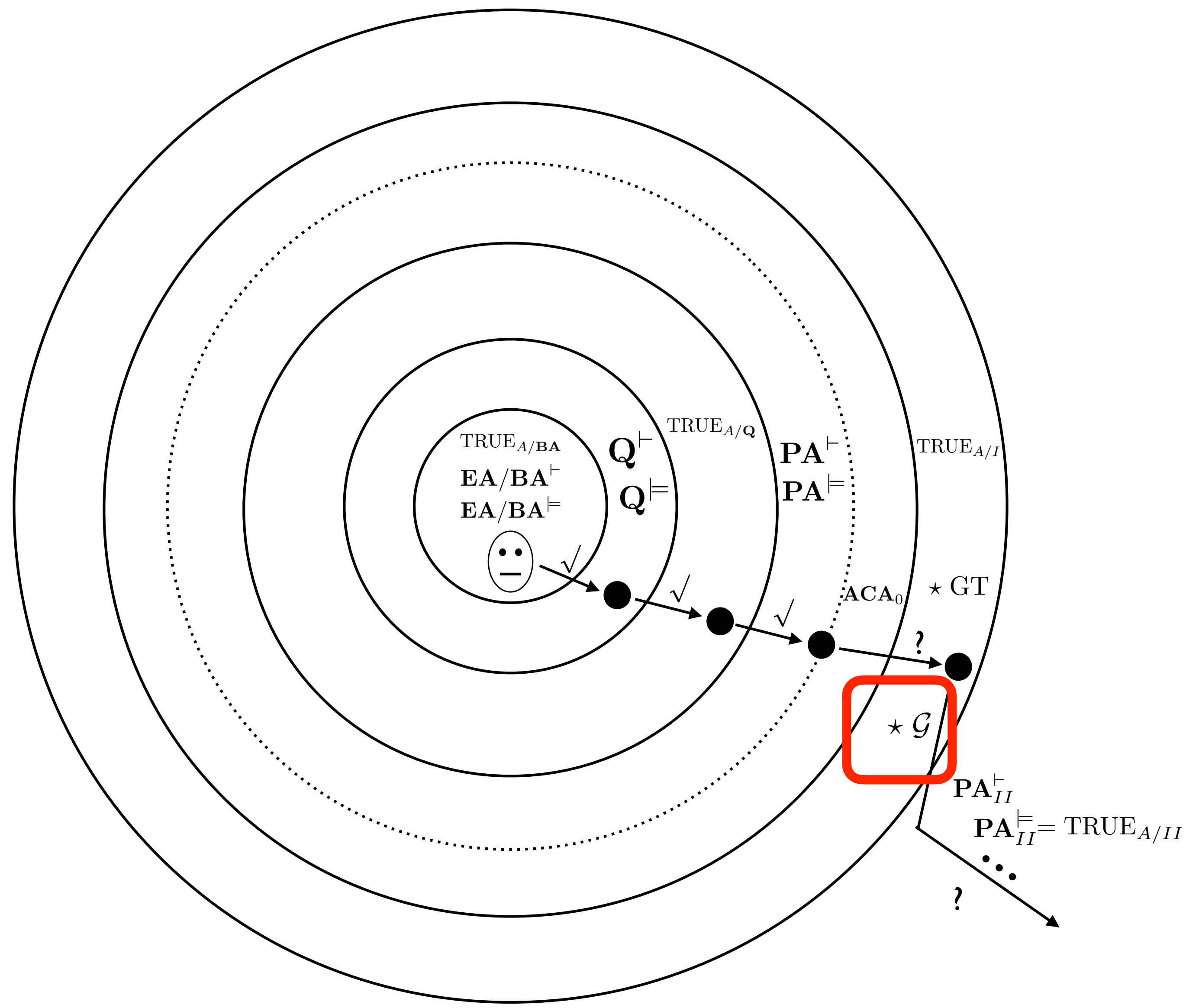

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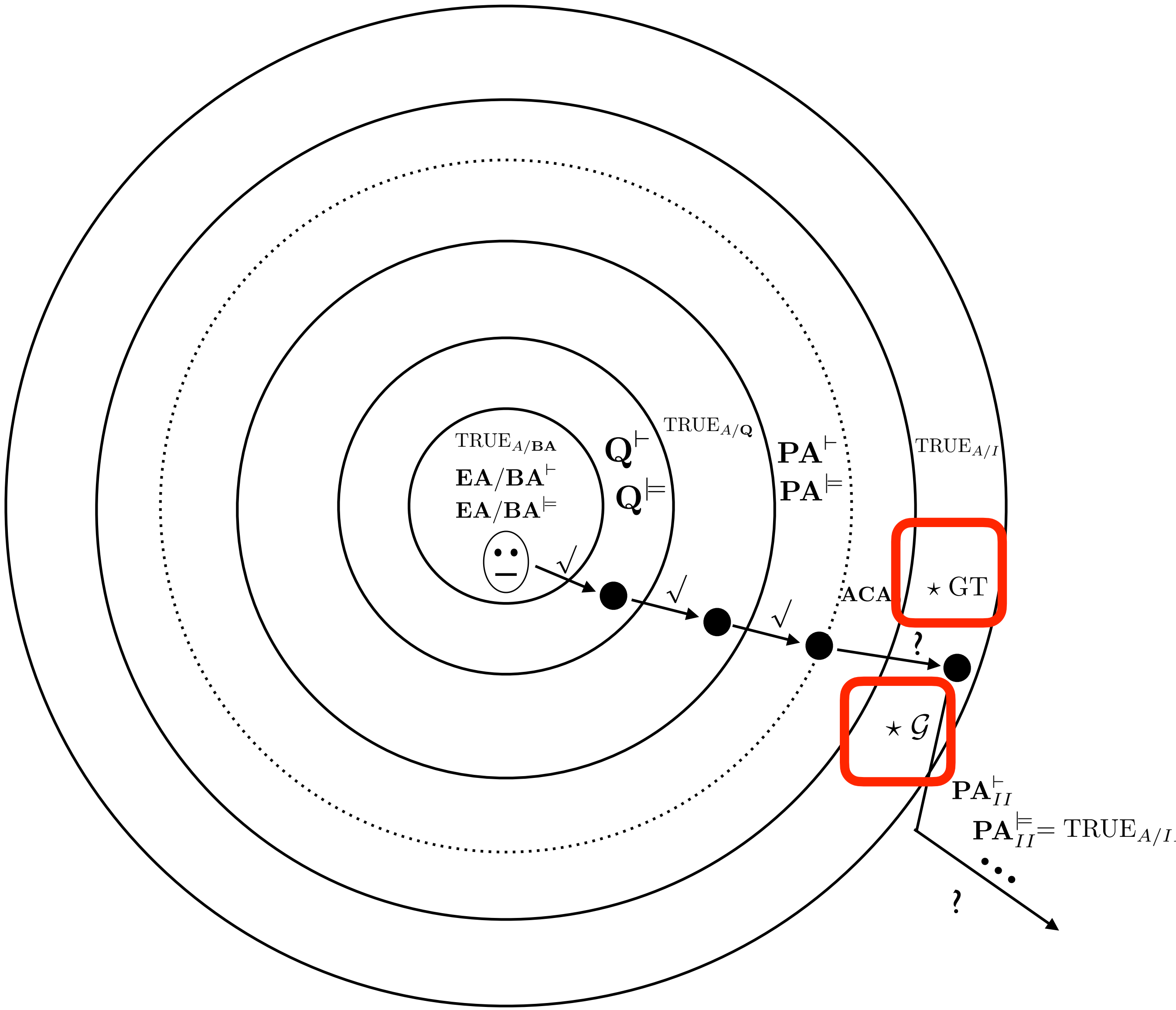

Could an AI Ever Match Gödel's G1 & G2?

Gödel's Great Theorems (OUP)

by Selmer Bringsjord

- Introduction ("The Wager")
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
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Med nok penger, kan logikk løse alle problemer.