

# Gödel's Greatest Theorem

(Gödel's "Silver Blaze" Theorem)

Selmer Bringsjord

IFLAI

Apr 17 2023

RPI

Troy NY USA



**AI in the news ...**



# A.I. Is Mastering Language. Should We Trust What It Says?

OpenAI's GPT-3 and other neural nets can now write original prose with mind-boggling fluency — a development that could have profound implications for the future.

By Steven Johnson Artwork by Nikita Iziev

Published April 15, 2022

Updated April 17, 2022, 10:53 a.m. ET

# Unprovability comes to machine learning

Scenarios have been discovered in which it is impossible to prove whether or not a machine-learning algorithm could solve a particular problem. This finding might have implications for both established and future learning algorithms.

Lev Reyzin 



During the twentieth century, discoveries in mathematical logic revolutionized our understanding of the very foundations of mathematics. In 1931, the logician Kurt Gödel showed that, in any system of axioms that is expressive enough to model arithmetic, some true statements will be unprovable<sup>1</sup>. And in the following decades, it was demonstrated that the continuum hypothesis – which states that no set of distinct objects has a size larger than that of the integers but smaller than that of the real numbers – can be neither proved nor refuted using the standard axioms of mathematics<sup>2–4</sup>. Writing in *Nature Machine Intelligence*, Ben-David *et al.*<sup>5</sup> show that the field of machine learning, although seemingly distant from mathematical logic, shares this limitation. They identify a machine-learning problem whose fate depends on the continuum hypothesis, leaving its resolution forever beyond reach

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STOP & REVIEW IF NEEDED!

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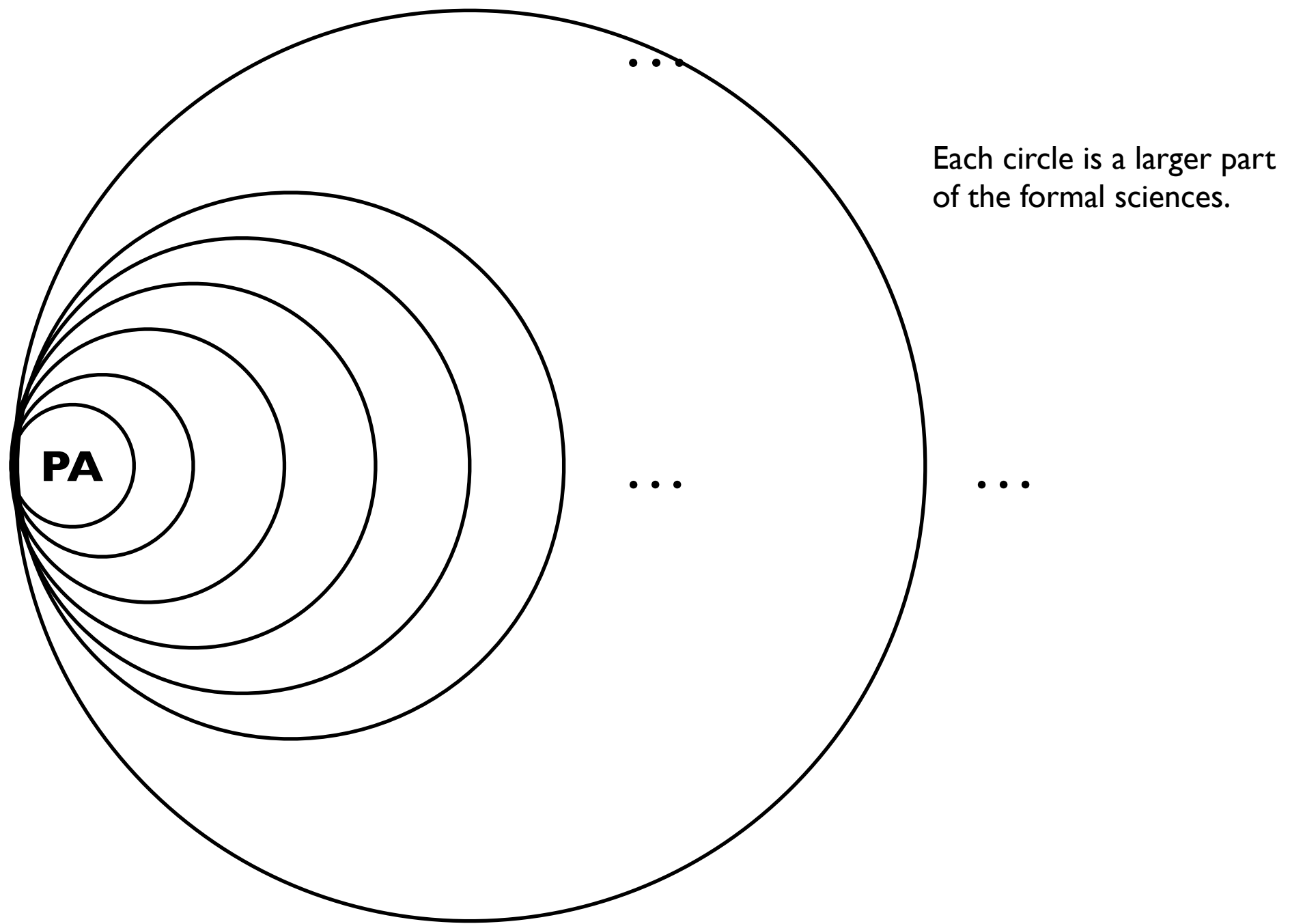
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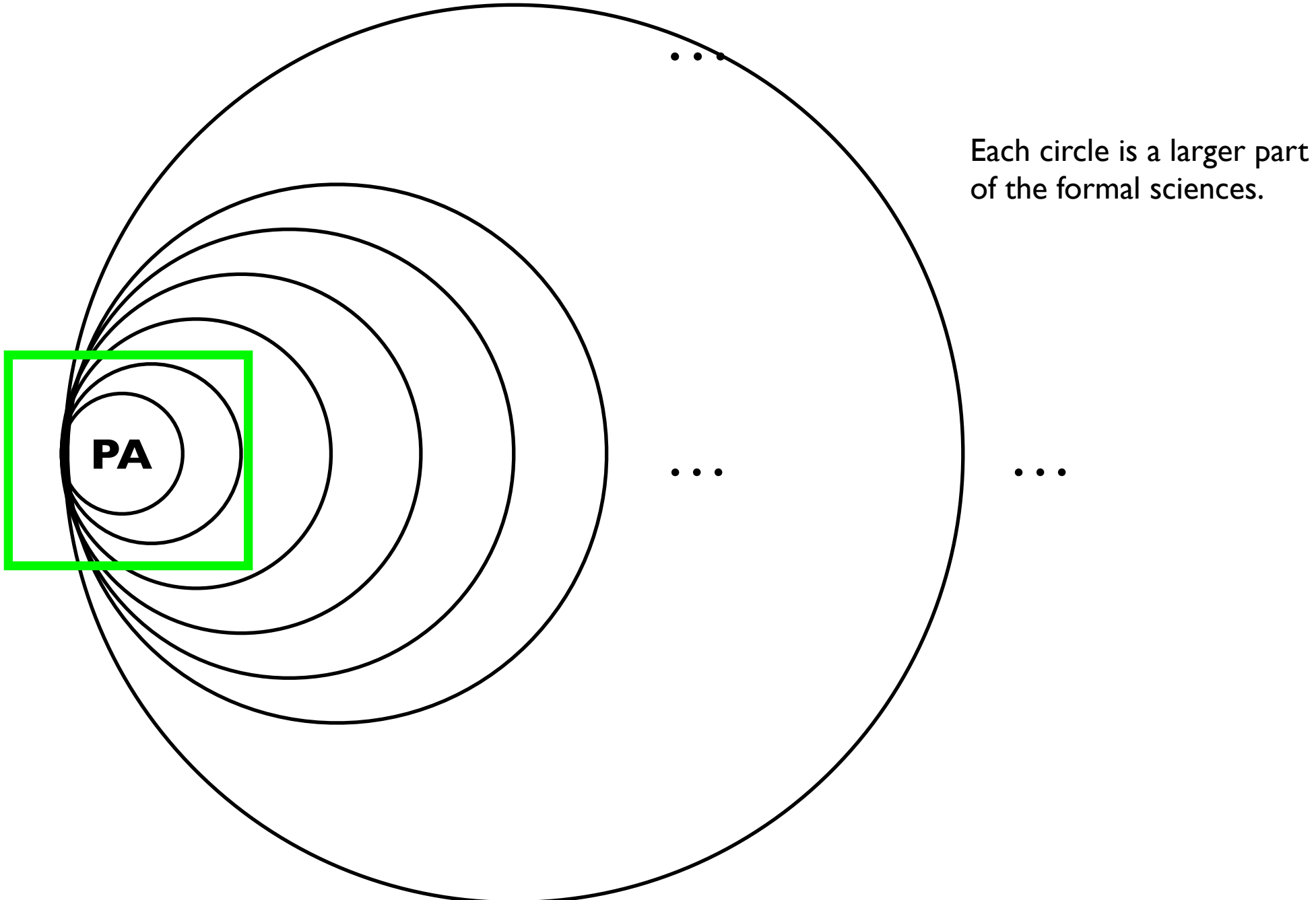
# Arithmetic is Part of All Things Sci/Eng/Tech!

and courtesy of Gödel: We can't even prove all truths of arithmetic!



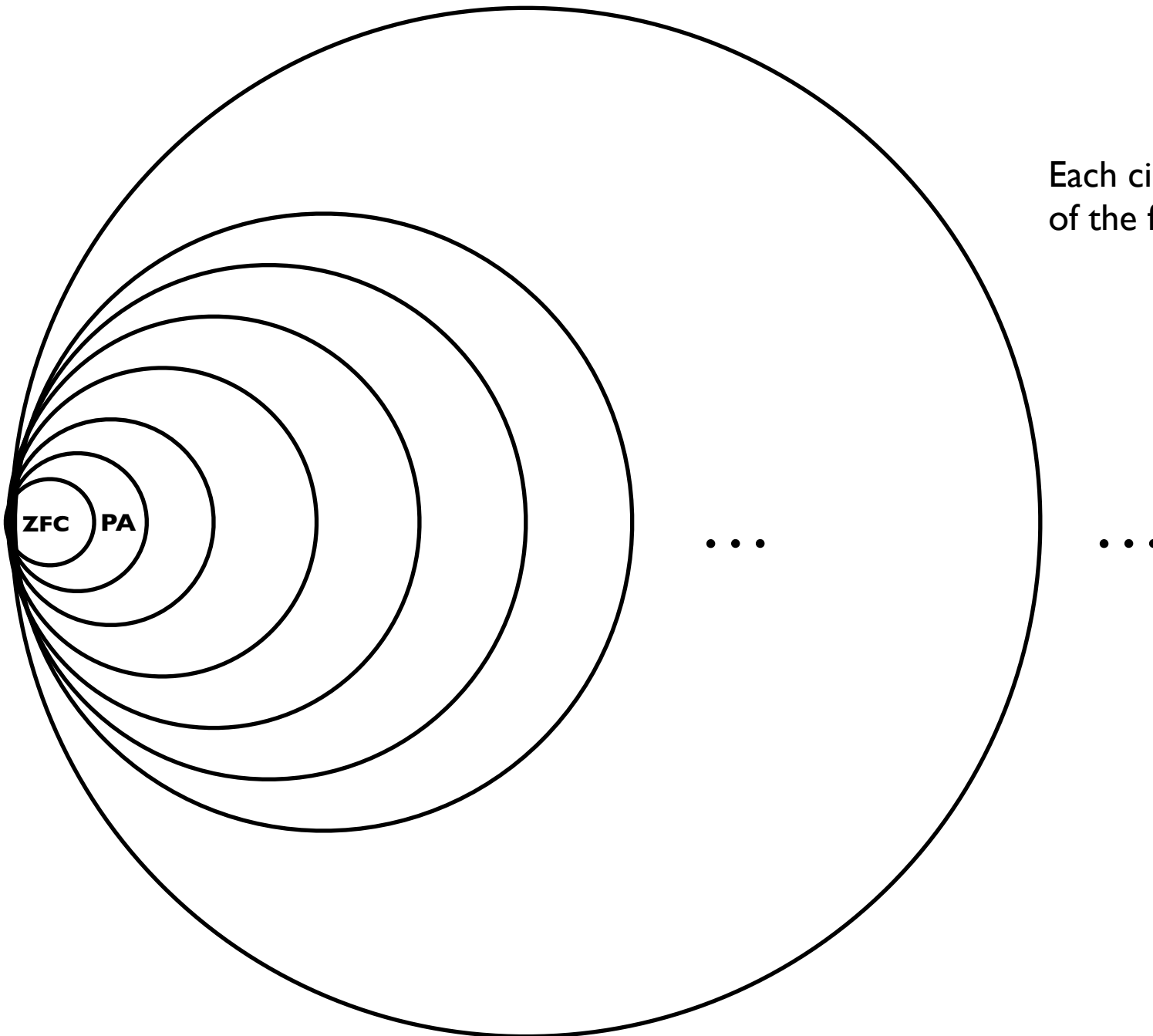
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Each circle is a larger part of the formal sciences.

# Actually, the true kernel is set theory!



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How do we know this????!???

# Continuum Hypothesis

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$$\text{CH} : \forall S[(S \subset \mathbb{R} \wedge \neg \mathbf{Fin}(S)) \rightarrow (S \sim \mathbb{N} \vee S \sim \mathbb{R})]$$

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Every infinite subset of the reals is either the same size as the natural numbers or the same size as the reals.

# *Generalized* Continuum Hypothesis

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For every infinite set  $S$ ,  $\mathcal{P}(S) > S$ .



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*Generalized* Continuum Hypothesis (GCH):

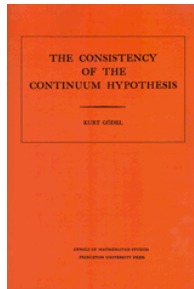
There's no set (size-wise) between  $S$  and  $P(S)$ .

# “Shorthand” History, Moral

Hilbert's #1 (1900): “very plausible theorem”: **CH**

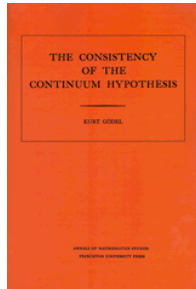
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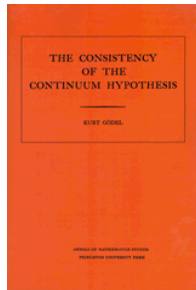
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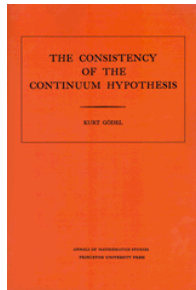
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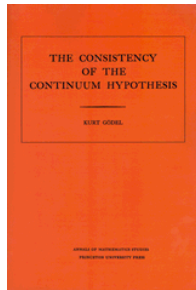


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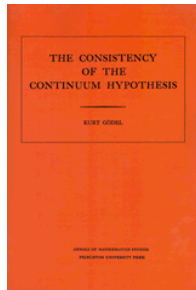
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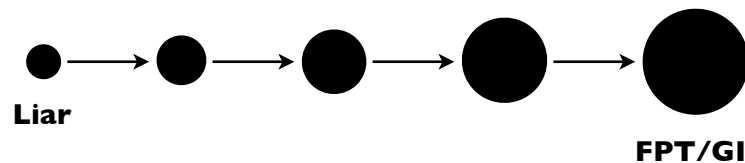
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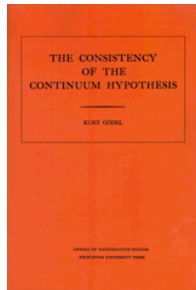
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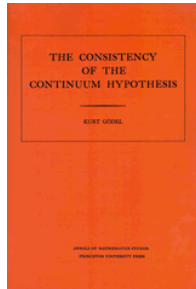
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Won’t “All-ly” work on *this* theorem of Gödel’s!

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# The Adventure of Silver Blaze

From Wikipedia, the free encyclopedia

*For the 1937 film, see [Silver Blaze \(1937 film\)](#). For the 1977 film, see [Silver Blaze \(1977 film\)](#).*

"**The Adventure of Silver Blaze**", one of the 56 [Sherlock Holmes](#) short stories written by Sir [Arthur Conan Doyle](#), is one of 12 in the cycle collected as *[The Memoirs of Sherlock Holmes](#)*. It was first published in *[The Strand Magazine](#)* in December 1892.<sup>[1]</sup>

Doyle ranked "Silver Blaze" 13th in a list of his 19 favourite Sherlock Holmes stories.<sup>[2]</sup> One of the most popular Sherlock Holmes short stories, "Silver Blaze" focuses on the disappearance of the eponymous [race horse](#) (a famous winner, owned by a Colonel Ross) on the eve of an important race and on the apparent murder of its trainer. The tale is distinguished by its atmospheric [Dartmoor](#) setting and late-Victorian sporting milieu. It also features some of Conan Doyle's most effective plotting, hingeing on the "curious incident of the dog in the night-time":

Gregory ([Scotland Yard](#) detective): Is there any other point to which you would wish to draw my attention?

Holmes: To the curious incident of the dog in the night-time.

Gregory: The dog did nothing in the night-time.

Holmes: That was the curious incident.

## Contents [\[hide\]](#)

- [Plot summary](#)
- [Publication history](#)
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  - [Film and television](#)
  - [Radio](#)
- [In popular culture](#)
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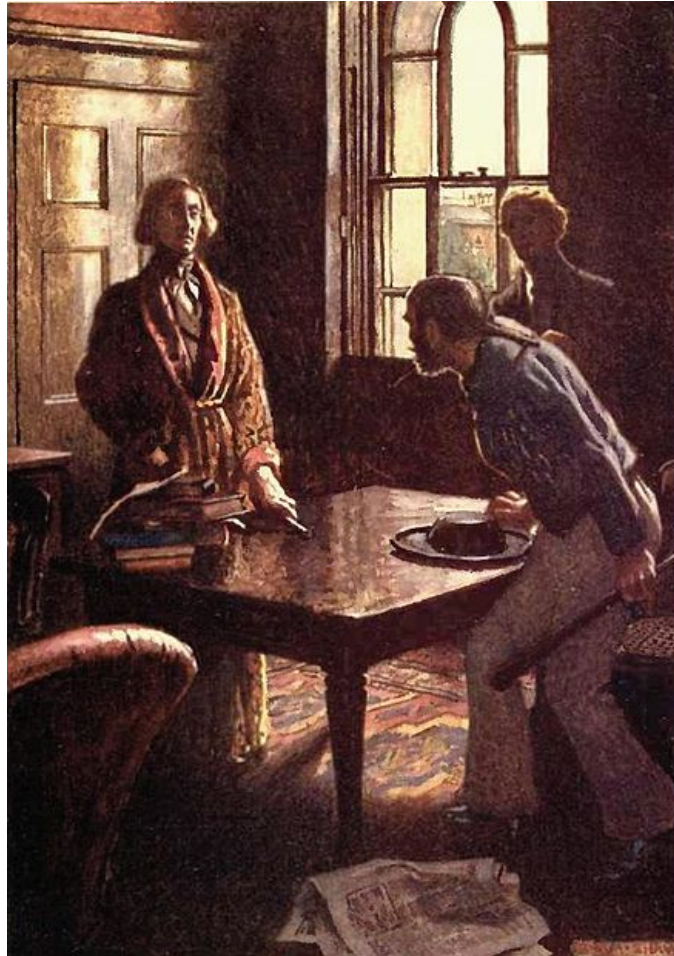
## "The Adventure of Silver Blaze"



1892 illustration by [Sidney Paget](#) in *[The Strand Magazine](#)*

<b>Author</b>	Arthur Conan Doyle
<b>Country</b>	Great Britain
<b>Language</b>	English
<b>Series</b>	<i>The Memoirs of Sherlock Holmes</i>
<b>Genre(s)</b>	Detective story
<b>Published in</b>	December 1892
<b>Preceded by</b>	" <a href="#">The Adventure of the Copper Beeches</a> "
<b>Followed by</b>	" <a href="#">The Adventure of the Cardboard Box</a> "

[



“What Would Poe Say About Today’s Social Robots?”

]

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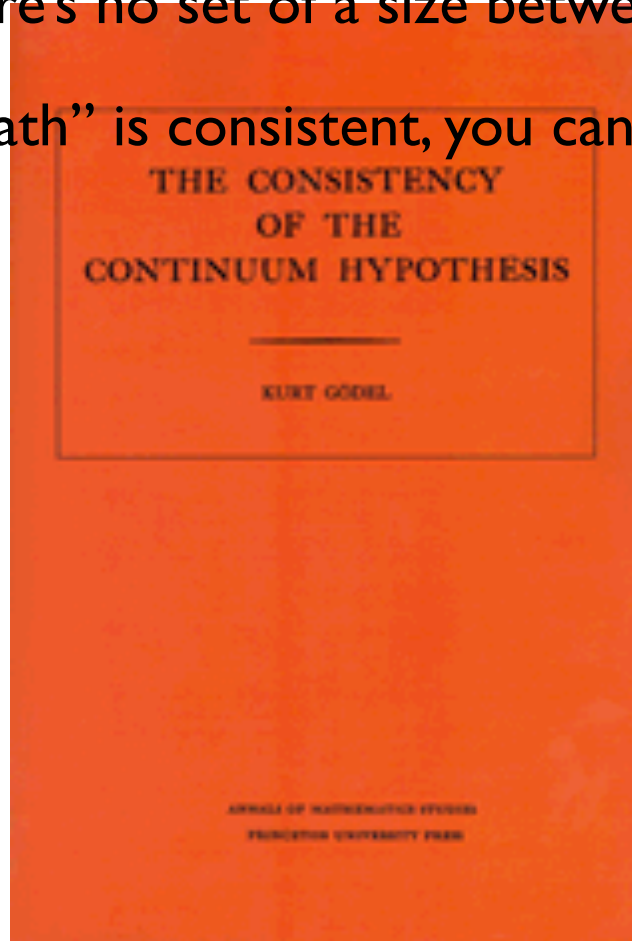


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CONTINUUM HYPOTHESIS

KURT GÖDEL

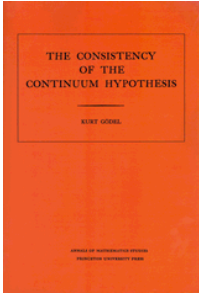
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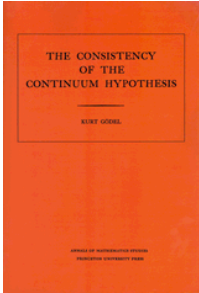
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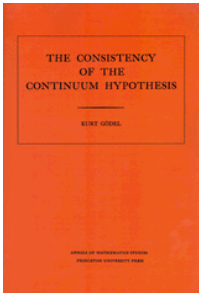
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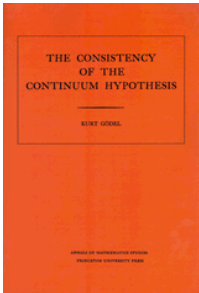
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“Silver Blaze”



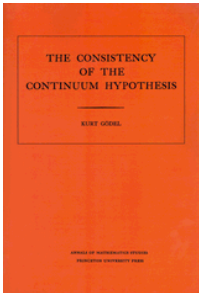
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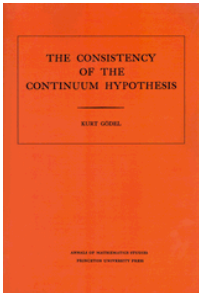
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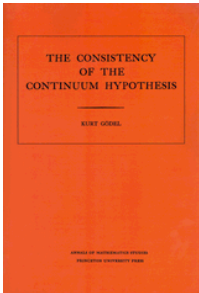
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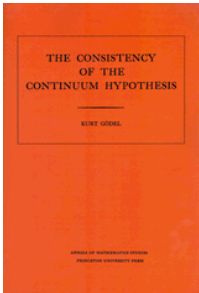
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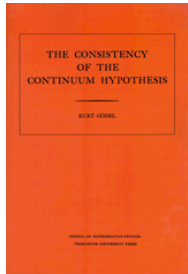
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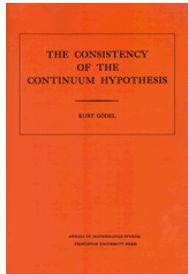
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Facts F ...

# Scenario G

## (in honor of Inspector Gregory)

"And yet," said I, "even now I fail to understand what the theory of the police can be."

"I am afraid that whatever theory we state has very grave objections to it," returned my companion. "The police imagine, I take it, that this Fitzroy Simpson, having drugged the lad, and having in some way obtained a duplicate key, opened the stable door and took out the horse, with the intention, apparently, of kidnapping him altogether. His bridle is missing, so that Simpson must have put this on. Then, having left the door open behind him, he was leading the horse away over the moor, when he was either met or overtaken by the trainer. A row naturally ensued. Simpson beat out the trainer's brains with his heavy stick without receiving any injury from the small knife which Straker used in self-defense, and then the thief either led the horse on to some secret hiding-place, or else it may have bolted during the struggle, and be now wandering out on the moors. That is the case as it appears to the police, and improbable as it is, all other explanations are more improbable still. However, I shall very quickly test the matter when I am once upon the spot, and until then I cannot really see how we can get much further than our present position."

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G = Inspector Gregory's scenario

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Holmes: "Gregory's claim is  $G$  (Simpson is guilty, & other details re. what he did). We have  $F$ , the facts of the case, disputed by no one. The question is:  $F \vdash G$ ? The answer is clearly No, for my scenario  $H$ , which entails  $\neg G$ , is consistent with the facts [ $\text{Con}(F \cup H)$ ], and  $H$  entails  $\neg G$ . Here's the proof:

**Proof:** Suppose for *reductio* that  $F \vdash G$ . Then  $F \cup H$  are inconsistent (since we can use  $F$  to prove  $G$  by our supposition; and we have  $H \vdash \neg G$  from me). But then we have that  $F \cup H$  is both consistent and inconsistent: contradiction! Hence by indirect proof  $F \not\vdash G$ . **QED**

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**Proof-Sketch:** Gödel created a scenario  $\mathcal{S}_G$  in which *all* the real numbers are present, and everything about them entailed by ZFC holds. I.e.,

$$(1) \quad \text{Con}(\mathcal{S}_G \cup \text{ZFC}).$$

In addition, Gödel also shows that given his scenario, CH can be proved; i.e.

$$(2) \quad \mathcal{S}_G \vdash \text{CH}.$$

Therefore, by parallel to Holmes's reasoning, CH can never be disproved from ZFC! Here's the reasoning: Suppose for *reductio* that  $\text{ZFC} \vdash \neg\text{CH}$ . Therefore  $\mathcal{S}_G$  and ZFC together are inconsistent. (Why? Because with the "combo" set  $\mathcal{S}_G \cup \text{ZFC}$  we can enlist (2) to prove CH, and our assumption for contradiction gives us  $\neg\text{CH}$ .) But this contradicts (1). **QED**

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(Of course, in “Silver Blaze,” Sherlock did proceed to show that his scenario was true. The analogue to that doesn't necessarily hold in the Gödel story —yet some think it should!!)



*slutten*