Rebuilding the Foundations of Math via (the “Theory”) ZFC; ZFC to Axiomatized Arithmetic (the “Theories” BA and PA)

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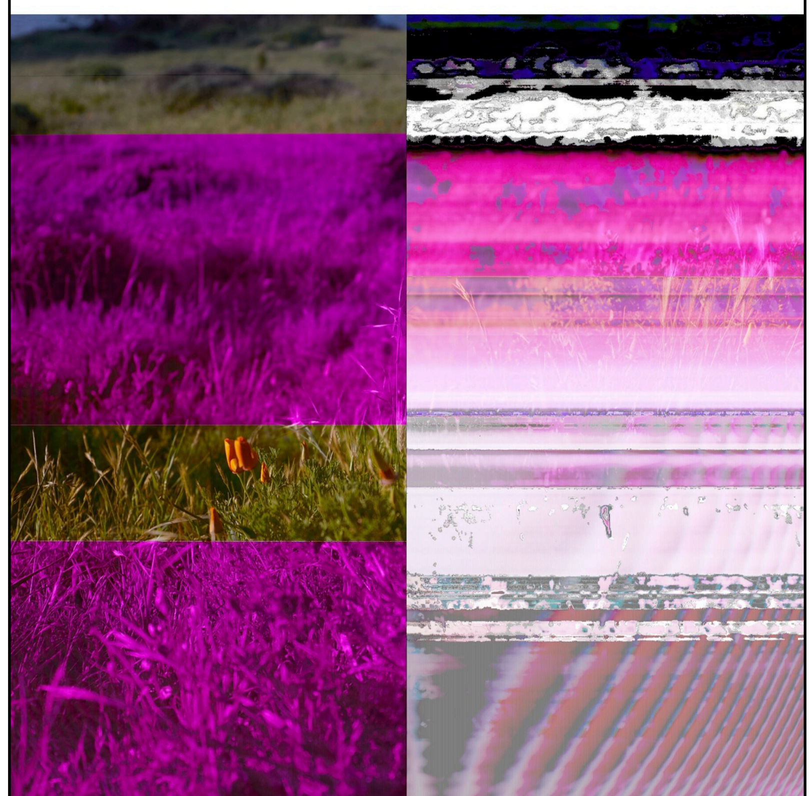
IFLAI1
3/13/2023

RAIR
Rensselaer AI and Reasoning Lab

AI & The News as We Head Toward the Topic of “Killer Robots” …

This Changes Everything

March 12, 2023, 6:00 a.m. ET 7 MIN READ



Ondrea Barbe/Trunk

By Ezra Klein
Opinion Columnist

Test 2 release postponed till March 20.

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New Required.. problems published on HyperGrader …

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Second-order quantification and axiom system for $\mathscr{L}_{pc}$ …

Reviewing the situation

…

Types of Paradoxes

- Deductive Paradoxes
- Inductive Paradoxes — coming (e.g. The Lottery Paradox & The St Petersburg Paradox)

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https://www.megamillions.com

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https://www.megamillions.com

1 in 302,575,350

Friday's Hill, Haslemere, 16 June 1902

Dear colleague,

For a year and a half I have been acquainted with your *Grundgesetze der Arithmetik*, but it is only now that I have been able to find the time for the thorough study I intended to make of your work. I find myself in complete agreement with you in all essentials, particularly when you reject any psychological element [[Moment]] in logic and when you place a high value upon an ideography [[Begriffsschrift]] for the foundations of mathematics and of formal logic, which, incidentally, can hardly be distinguished. With regard to many particular questions, I find in your work discussions, distinctions, and definitions that one seeks in vain in the works of other logicians. Especially so far as function is concerned (§ 9 of your *Begriffsschrift*), I have been led on my own to views that are the same even in the details. There is just one point where I have encountered a difficulty. You state (p. 17 [[p. 23 above]]) that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer its opposite follows. Therefore we must conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [[Menge]] does not form a totality.

I am on the point of finishing a book on the principles of mathematics and in it I should like to discuss your work very thoroughly.[1] I already have your books or shall buy them soon, but I would be very grateful to you if you could send me reprints of your articles in various periodicals. In case this should be impossible, however, I will obtain them from a library.

The exact treatment of logic in fundamental questions, where symbols fail, has remained very much behind; in your works I find the best I know of our time, and therefore I have permitted myself to express my deep respect to you. It is very regrettable that you have not come to publish the second volume of your *Grundgesetze*; I hope that this will still be done.

Very respectfully yours,

BERTRAND RUSSELL

The above contradiction, when expressed in Peano's ideography, reads as follows:

$$w = \mathrm{cls} \cap x \ni (x \sim\varepsilon\, x) .\supset: w\, \varepsilon\, w .=. w \sim\varepsilon\, w.$$

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Russell’s Theorem

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$$\vdash \neg\exists x \forall y (y \in x \leftrightarrow y \notin y)$$

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☑ RussellsLetter2Frege

The challenge here is to prove that from Russell's instantiation of Frege's doomed Axiom V a contradiction can be promptly derived. The letter has of course been examined in some detail by S Bringsjord (in the Mar 16 2020 lecture in the 2020 lecture lineup); it, along with an astoundingly soft-spoken reply from Frege, can be found here. Put meta-logically, your task in the present problem is to build a proof that confirms this:

$$\{\exists x \forall y((y \in x) \rightarrow (y \notin y))\} \vdash \zeta \wedge \neg\zeta.$$

Make sure you understand that the given here is an instantiation of Frege's Axiom V; i.e. it's an instantiation of

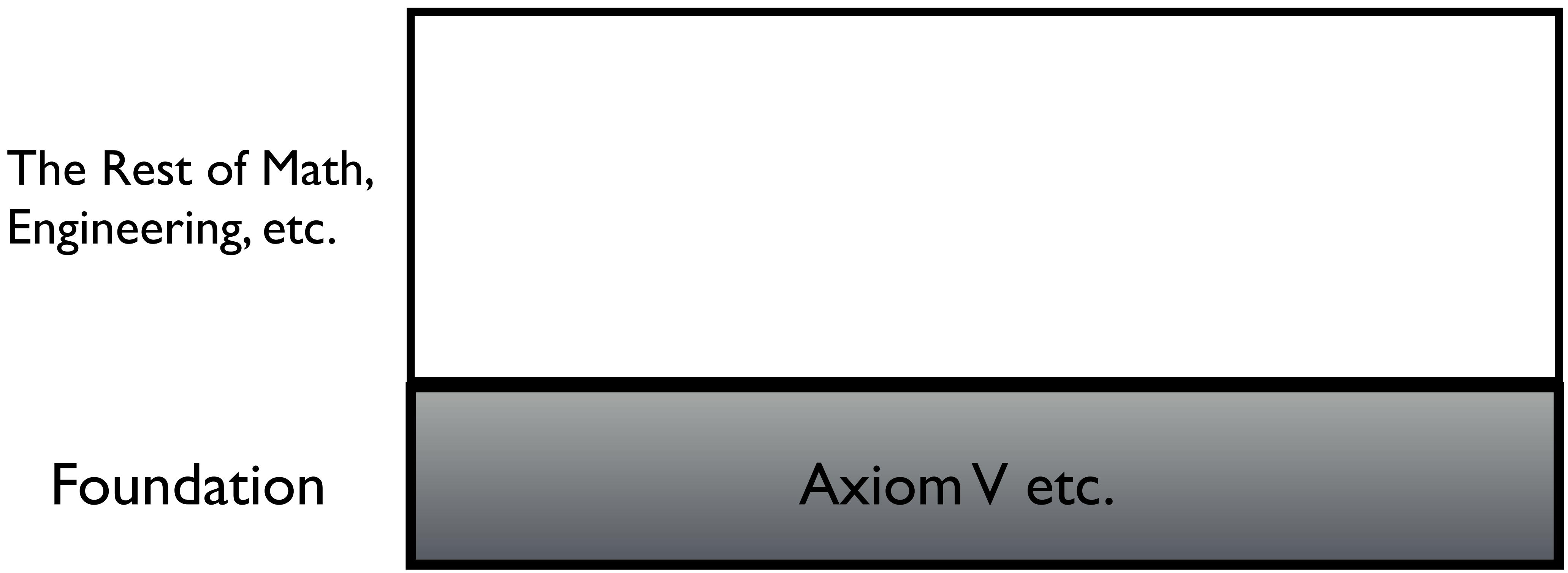
$$\exists x \forall y((y \in x) \rightarrow \phi(y)).$$

(The notation $\phi(y)$, recall, is the standard way in mathematical logic to say that y is free in ϕ.) **Note**: Your finished proof is allowed to make use the PC-provability oracle (but *only* that oracle).

(Now a brief remark on matters covered by in class by Bringsjord when second-order logic = $\mathscr{L}_2$ arrives on the scene: Longer term, and certainly constituting evidence of Frege's capacity for ingenius, intricate deduction, it has recently been realized that while Frege himself relied on Axiom V to obtain what is known as **Hume's Principle** (= HP), this reliance is avoidable. That from just HP we can deduce all of Peano Arithmetic (**PA**) (!) is a result Frege can be credited with showing; the result is known today as Frege's Theorem (= FT). Following the link just given will reward the reader with an understanding of HP, and how how to obtain **PA** from it.)

Solve

The Foundation Crumbles



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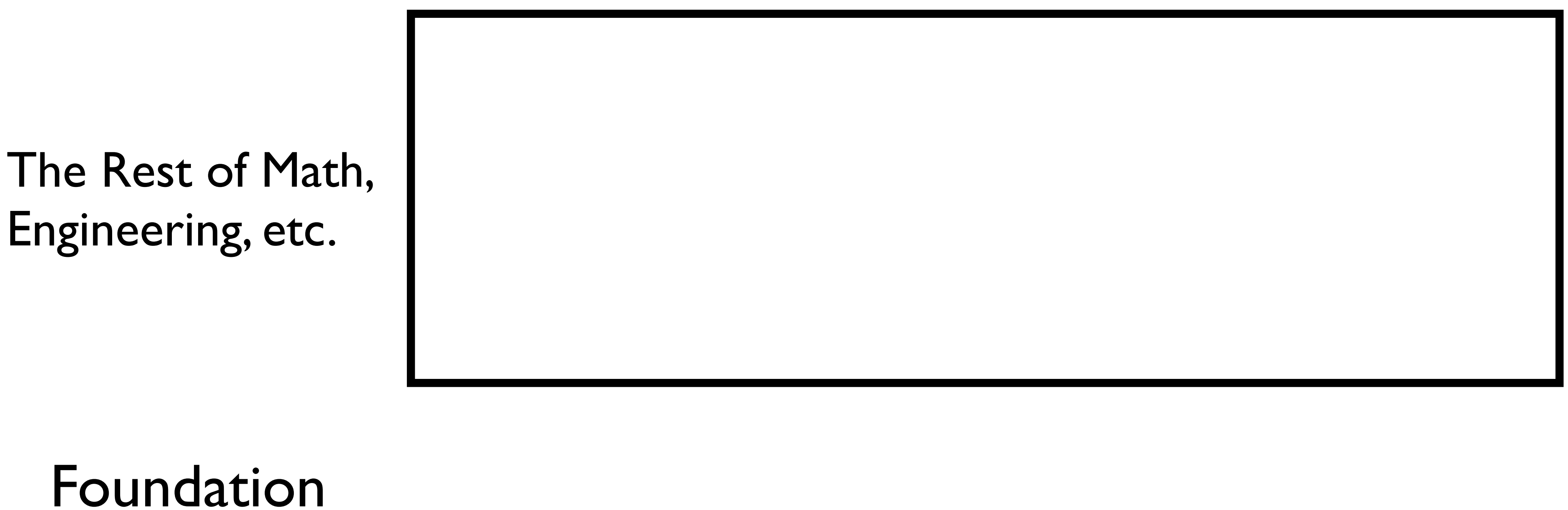

The Foundation Crumbles

The Rest of Math,
Engineering, etc.

Foundation

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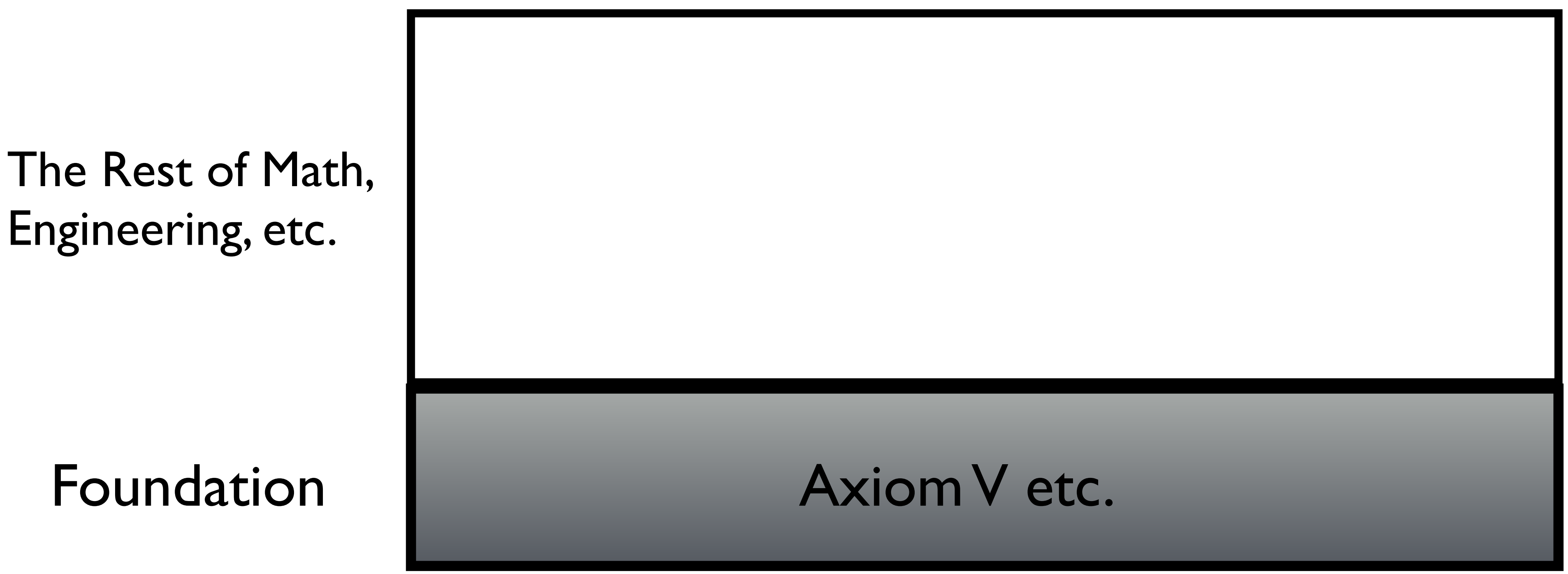

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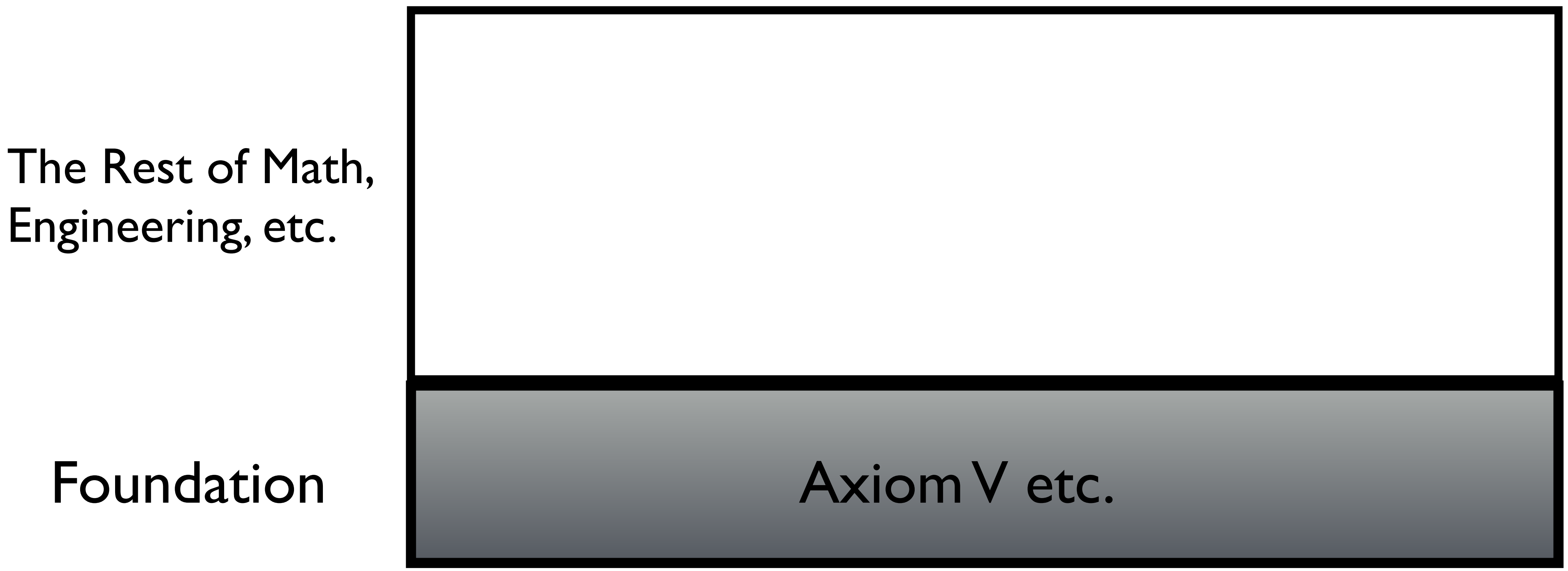

$$\text{Axiom V} \qquad \exists x \forall y [y \in x \leftrightarrow \phi(y)]$$

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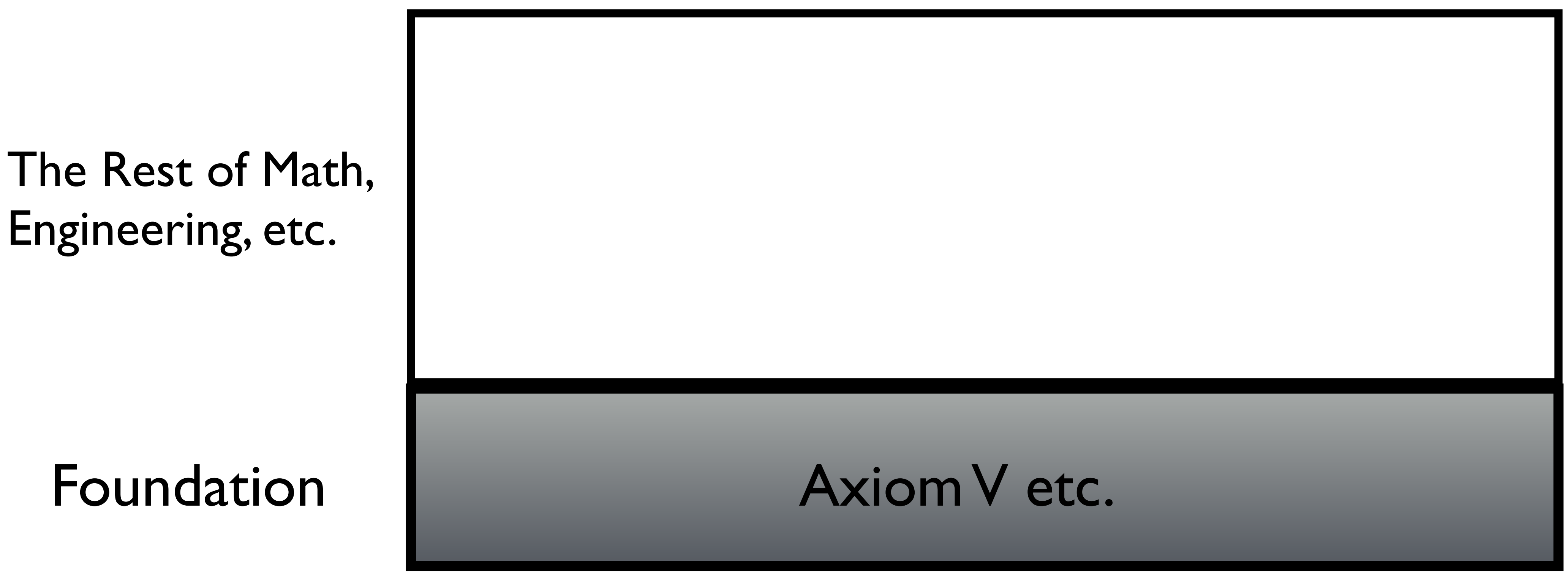
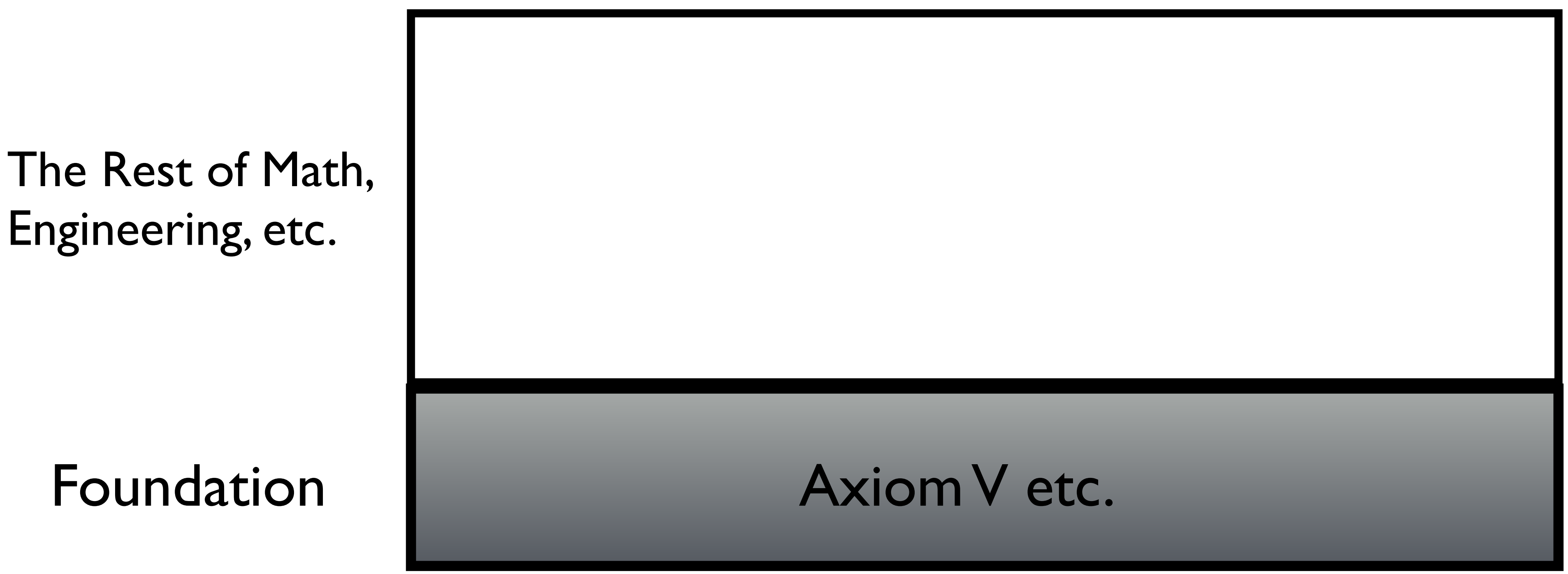


$$\text{Axiom V} \quad \exists x \forall y [y \in x \leftrightarrow \phi(y)]$$

a formula of arbitrary size in which the variable *y* is free; this formula ascribes a property to *y*

The Foundation Crumbles



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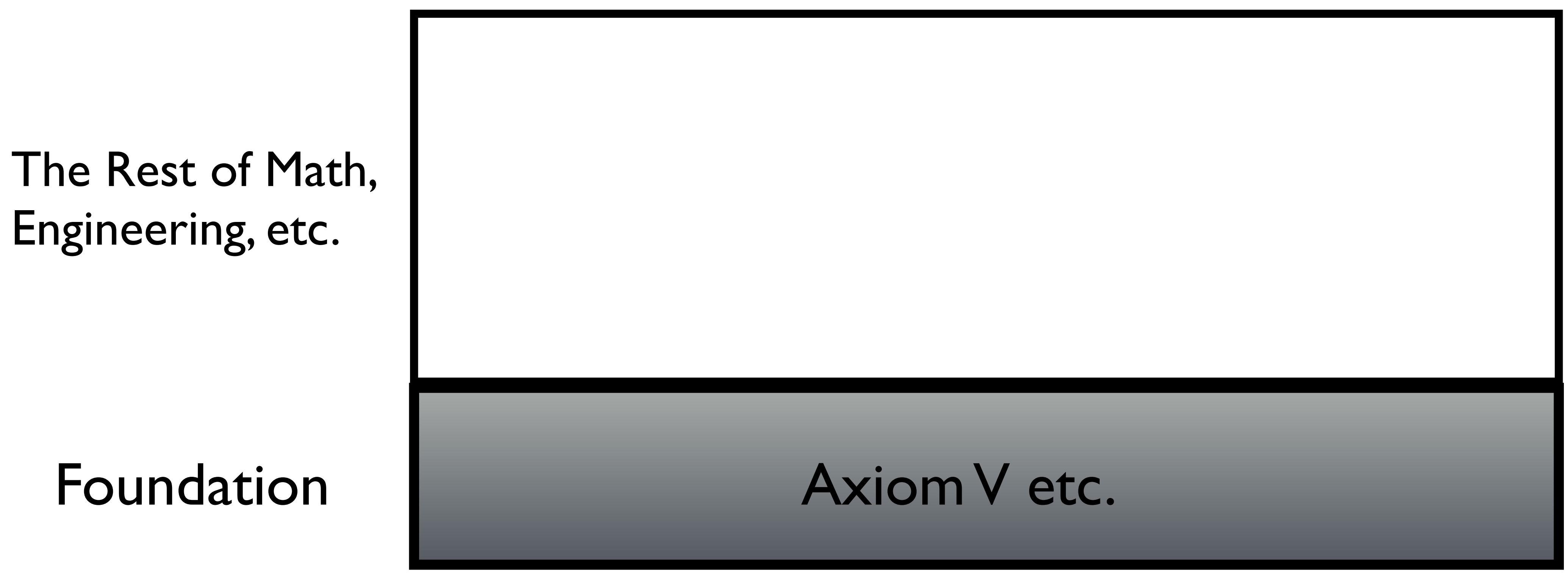

The Foundation Crumbles

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The Rest of Math,
Engineering, etc.

Foundation

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Foundation

It’s not just Russell’s Paradox that destroys naïve set theory:

It's not just Russell's Paradox that destroys naïve set theory:

Richard's Paradox …

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Doesn't define
a real number.

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Definition of Richard's *N*:

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Doesn't define a real number. ~~aaa~~
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E

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Doesn't define a real number. ~~aaa~~
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E

Definition of Richard's *N*:

"The real number whose whole part is zero, and whose *n*-th decimal is *p* plus one if the *n*-th decimal of the real number defined by the *n*-th member of *E* is *p* and *p* is neither eight nor nine, and is simply one if this *n*-th decimal is eight or nine."

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Doesn't define a real number. ~~aaa~~
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E

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Proof: *N* is defined by a finite string taken from the English alphabet, so *N* is in the sequence *E*. But on the other hand, by definition of *N*, for every *m*, *N* differs from the *m*-th element of *E* in at least one decimal place; so *N* is not any element of *E*. Contradiction! **QED**

a
b
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ab
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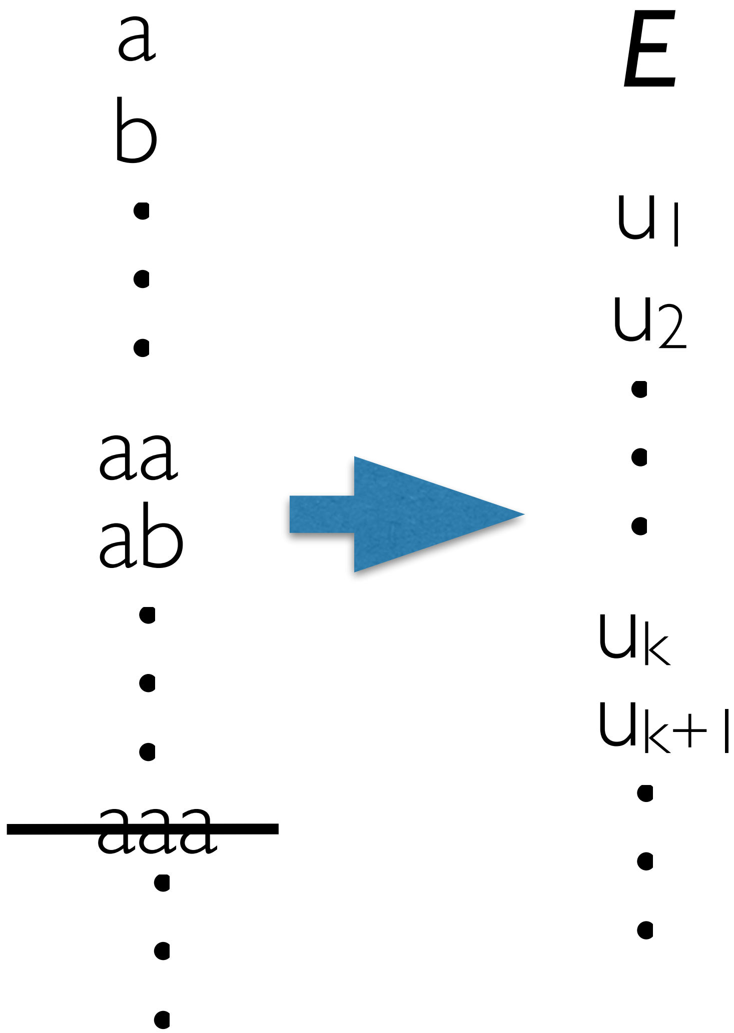

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E

u_1
u_2
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u_k
u_{k+1}
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Suppose *N* is u_m.

a
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→

E

u_1
u_2
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Suppose *N* is u_m.

Suppose u_m is 0.xxxx…xxx…

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E

u_1
u_2
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Suppose *N* is u_m.

Suppose u_m is 0.xxxx…xxx…

↑ mth

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b

⋮

aa

ab

⋮

~~aaa~~

⋮

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E

u_1

u_2

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u_k

u_{k+1}

⋮

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Suppose *N* is u_m.

Suppose u_m is 0.xxxx…xxx…

↑ mth

Suppose u_m is 0.xxxx…8xx…

↑ mth

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↑
mth

Suppose u_m is 0.xxxx…8xx…

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mth

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E

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Suppose *N* is u_m.

Suppose u_m is 0.xxxx…xxx…

↑ mth

Suppose u_m is 0.xxxx…8xx…

↑ mth

Then *N* is 0.xxxx…1xx…

↑ mth

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Suppose *N* is u_m.

Suppose u_m is 0.xxxx…xxx…

↑ mth

Suppose u_m is 0.xxxx…8xx…

↑ mth

Then *N* is 0.xxxx…1xx…

↑ mth

Since 8 ≠ 1, *N* can't be u_m!

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E

u_1
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Suppose u_m is 0.xxxx…xxx…

↑ mth

Suppose u_m is 0.xxxx…8xx…

↑ mth

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↑ mth

Then *N* is 0.xxxx…1xx…

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E

u_1
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Suppose *N* is u_m.

Suppose u_m is 0.xxxx...xxx... (↑ mth)

Suppose u_m is 0.xxxx...8xx... (↑ mth)

Suppose u_m is 0.xxxx...5xx... (↑ mth)

Then *N* is 0.xxxx...1xx... (↑ mth)

Since 8 ≠ 1, *N* can't be u_m!

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E

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Suppose *N* is u_m.

Suppose u_m is 0.xxxx…xxx…

mth

Suppose u_m is 0.xxxx…8xx…

mth

Then *N* is 0.xxxx…1xx…

mth

Since 8 ≠ 1, *N* can't be u_m!

Suppose u_m is 0.xxxx…5xx…

mth

Then *N* is 0.xxxx…6xx…

mth

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E

u_1
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Suppose *N* is u_m.

Suppose u_m is 0.xxxx…xxx…

↑ mth

Suppose u_m is 0.xxxx…8xx…

↑ mth

Then *N* is 0.xxxx…1xx…

↑ mth

Since 8 ≠ 1, *N* can't be u_m!

Suppose u_m is 0.xxxx…5xx…

↑ mth

Then *N* is 0.xxxx…6xx…

↑ mth

Since 5 ≠ 6, *N* can't be u_m!

The Foundation Rebuilt

The Rest of Math,
Engineering, etc.

New Foundation

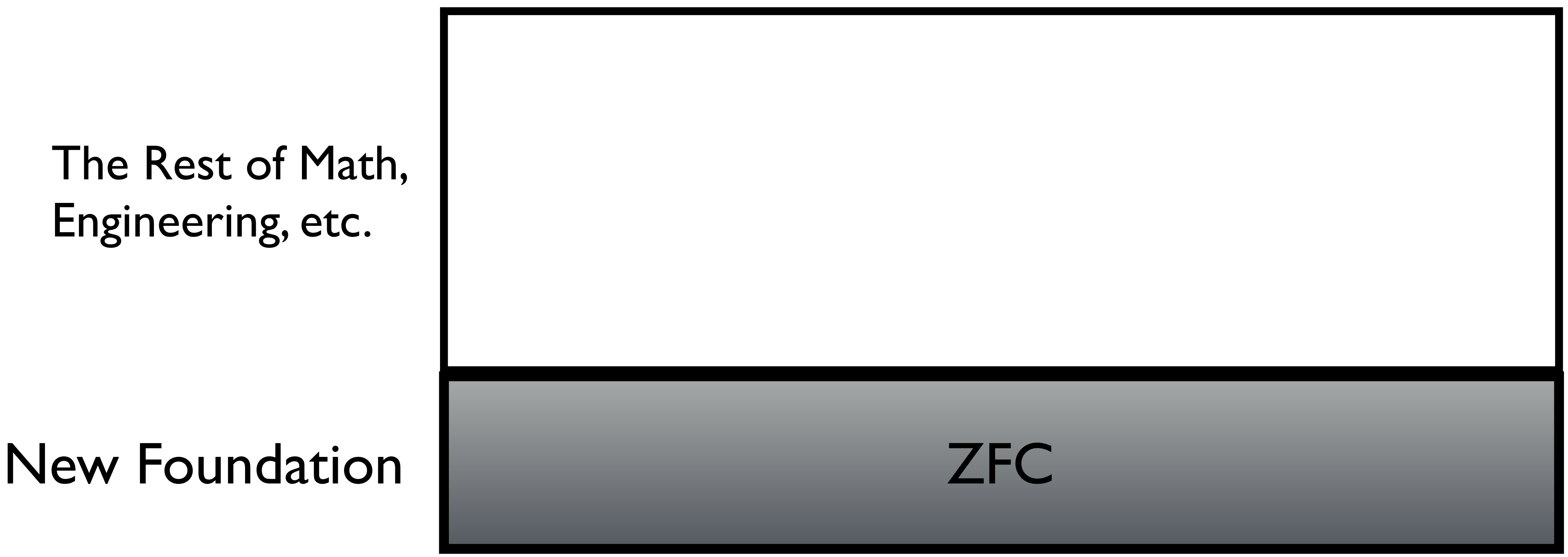
The Foundation Rebuilt

The Rest of Math,
Engineering, etc.

New Foundation

ZFC

The Foundation Rebuilt



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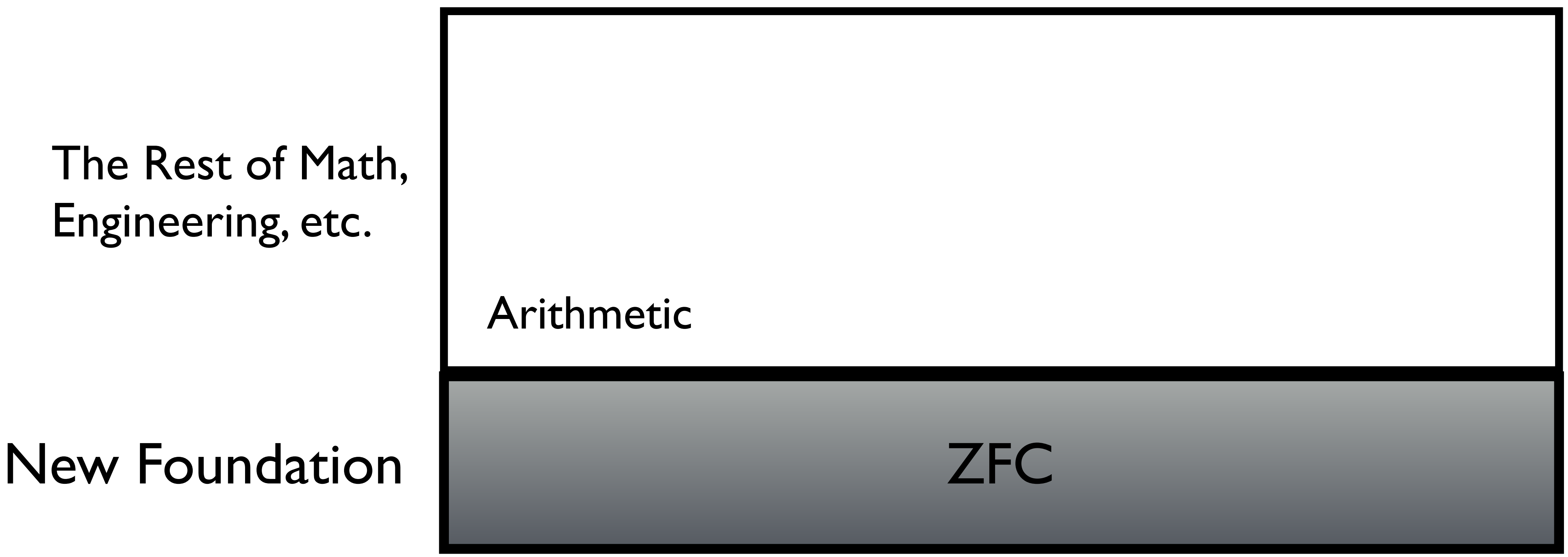

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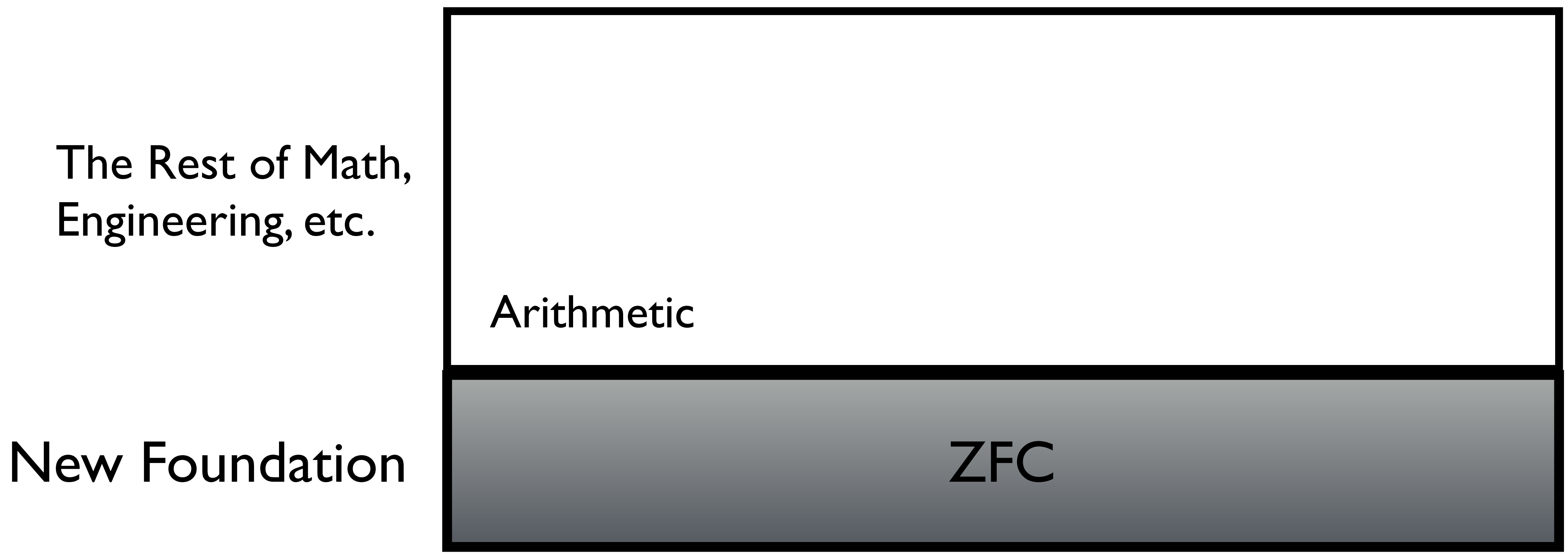
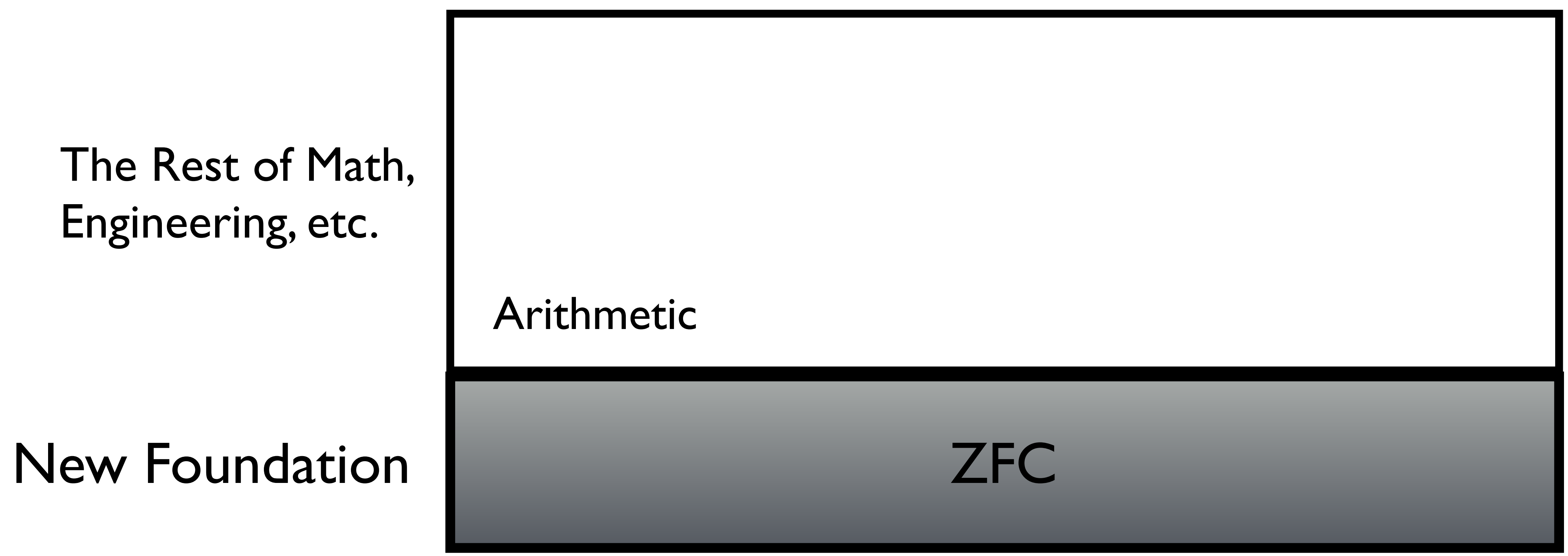


So what are the axioms in ZFC?

The Foundation Rebuilt



So what are the axioms in ZFC?

Axiom *Schema* of Separation (SEP)

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SEP

$$\forall x_1 \ldots \forall x_k \forall x \exists y \forall z[z \in y \leftrightarrow (z \in x \wedge \phi(z, x_1, \ldots, x_k))]$$

where x and y are distinct, and are both distinct from z and the x_i; and, as usual for us now, ϕ expresses a property using $\in$.

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"Given *beforehand* some set *x* and property $\mathscr{P}$ captured by a formula ϕ that uses $\in$ for its relation, the set *y* composed of $\{z \in x : \mathscr{P}(z)\}$ exists."

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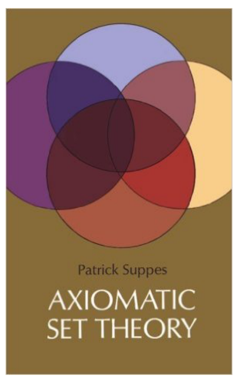
How does this neutralize Russell's letter to Frege?

Russell's Paradox … to ZFC

$$\vdash \neg \exists x \forall y (y \in x \leftrightarrow y \notin y)$$

(Russell's Theorem; poor Frege!)

http://plato.stanford.edu/entries/russell-paradox/#HOTP



Supplant Cantor's/Frege's Axiom V with the Axiom Schema of Separation (& put on our thinking caps …) and try to show Theorem 1 from Suppes:

$$\vdash \forall x (x \notin \emptyset)$$

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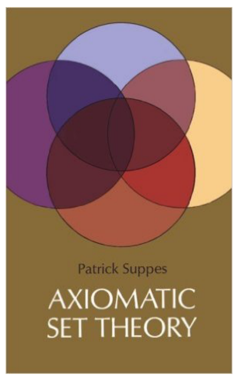

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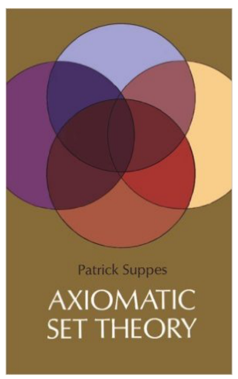

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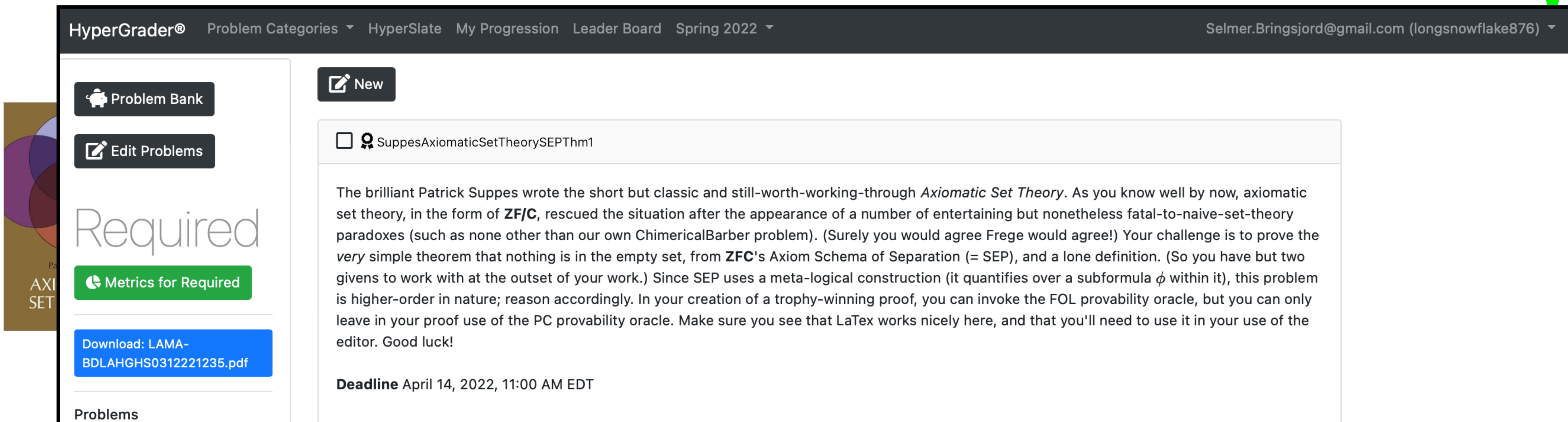

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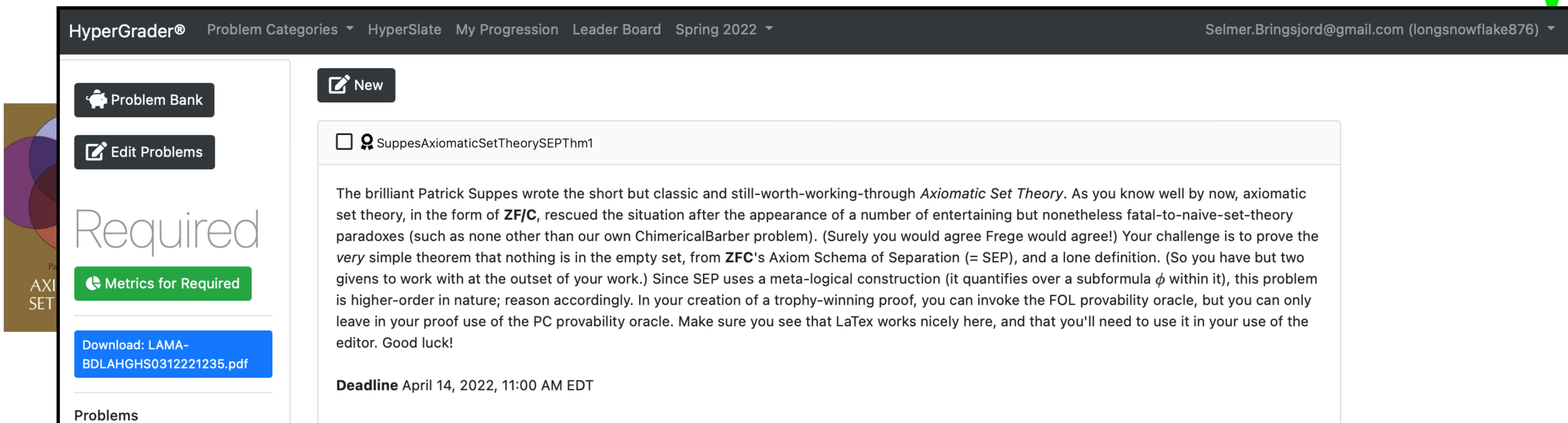

Try a second "Suppesian" theorem in ZFC:

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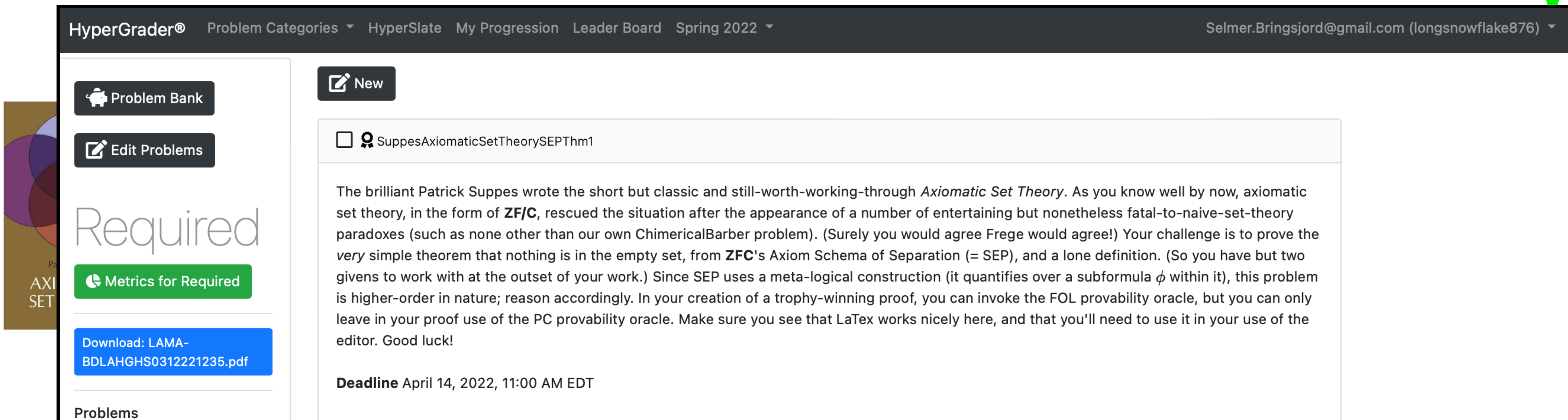

Try a second "Suppesian" theorem in ZFC:

$$\vdash \forall x [(\forall z (z \notin x)) \rightarrow x = \emptyset]$$

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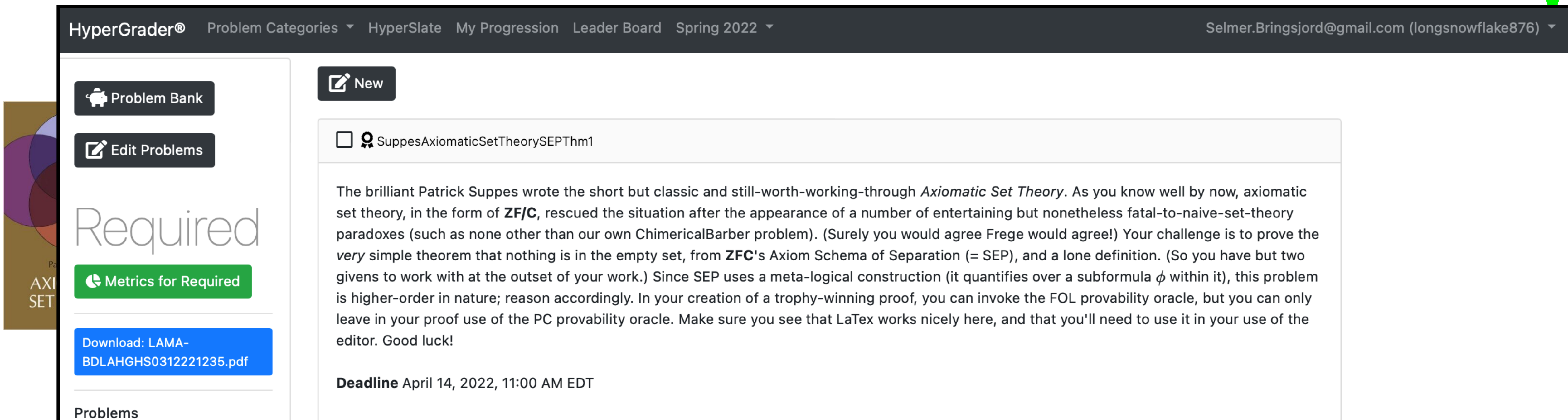

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Now let's add the Definition of Subset to ZFC:

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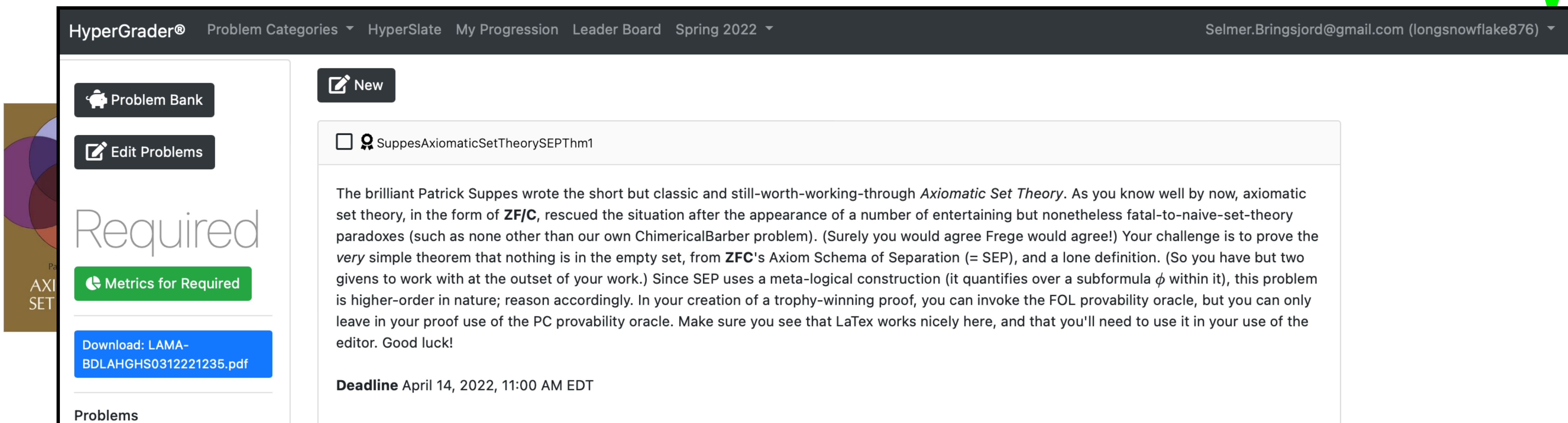

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$$\forall x \forall y[x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y)]$$

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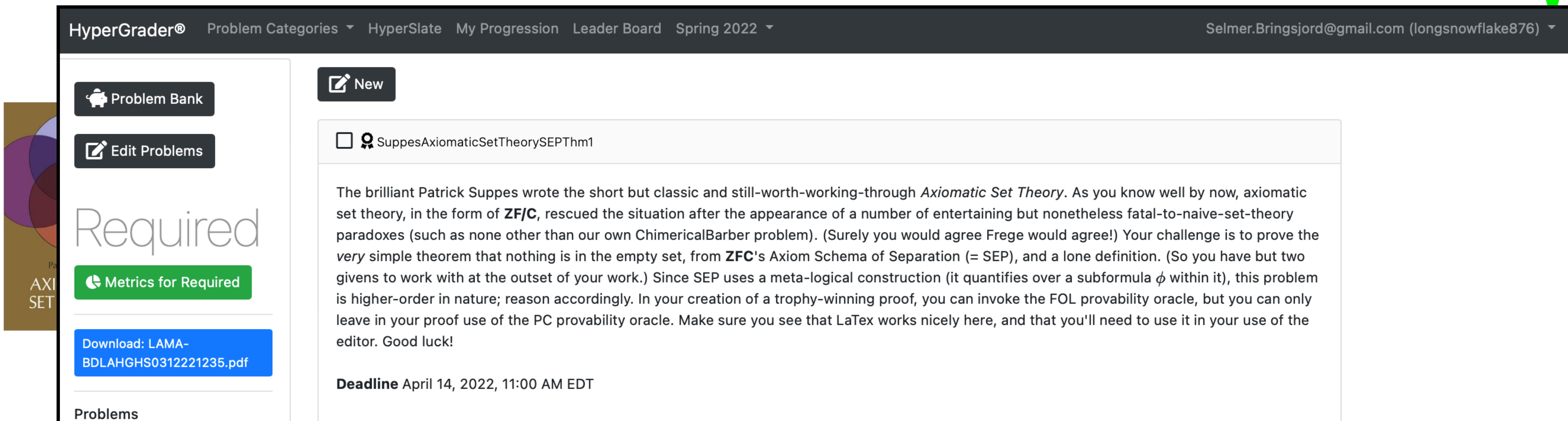

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With this definition, can you prove (Theorem 3) that every set is a subset of itself?

ZFC Completed

formulated with an eyes-wide-open understanding that paradoxes can rise up and threaten unreflective use of set-theoretic concepts. There are a number of different possibilities for specifying an axiomatic set theory. We turn now to the dominant one, known by the label 'ZFC.'

6.4.1 ZFC

The Zermelo-Fraenkel Axioms for Set Theory, or just 'ZFC' for short, include the following nine axioms.[3,4]

Axiom of Extensionality

$$\forall x \forall y (\forall z (z \in x \longleftrightarrow z \in y) \to x = y)$$

Axiom Schema of Separation

$$\forall x_0 \ldots \forall x_{n-1} \forall x \exists y \forall z (z \in y \longleftrightarrow (z \in x \wedge \phi(z, x_0, \ldots, x_{n-1})))$$

Pair Set Axiom

$$\forall x \forall y \exists z \forall w (w \in z \longleftrightarrow (w = x \vee w = y))$$

Sum Set Axiom

$$\forall x \exists y \forall z (z \in y \longleftrightarrow \exists w (w \in x \wedge z \in w))$$

Power Set Axiom

$$\forall x \exists y \forall z (z \in y \longleftrightarrow \forall w (w \in z \to w \in x))$$

Axiom of Infinity

$$\exists x (\emptyset \in x \wedge \forall y (y \in x \to y \cup \{y\} \in x))$$

Axiom Schema of Replacement

$$\forall x_0 \ldots \forall x_{n-1} (\forall x \exists^{=1} y\, \phi(x, y, x_0, \ldots, x_{n-1}) \to \forall u \exists v \forall y (y \in v \longleftrightarrow \exists x (x \in u \wedge \phi(x, y, x_0, \ldots, x_{n-1}))))$$

Axiom of Choice

$$\forall x ((\emptyset \notin x \wedge \forall u \forall v ((u \in x \wedge v \in x \wedge u \neq v) \to u \cap v = \emptyset)) \to \exists y \forall w (w \in x \to \exists^{=1} z\, z \in w \cap y))$$

6.4.1.1 Exercises

1. The Axiom Schema of Separation was the replacement for Axiom V. Show that Russell's reasoning fails when the attempt is made to apply it to the Axiom Schema of Separation.

2. Provide for each axiom of ZFC one clear English sentence that expresses the axiom.

[3] While it's obvious what the 'Z' and 'F' abbreviate in the label 'ZFC,' what about 'C'? This letter refers to one of the axioms that follow: the Axiom of **C**hoice. 'ZF' refers then to the following list of axioms, *without* the Axiom of Choice.

[4] Note that when we write '$\phi(x)$' we are saying that variable x appears free in formula ϕ. In the Axiom Schema of Separation, y does not occur free in '$\phi(z, x_0, \ldots, x_{n-1})$.'

Sum Set Axiom

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \land z \in w))$$

Power Set Axiom

$$\forall x \exists y \forall z (z \in y \leftrightarrow \forall w (w \in z \rightarrow w \in x))$$

Axiom of Infinity

$$\exists x (\emptyset \in x \land \forall y (y \in x \rightarrow y \cup \{y\} \in x))$$

Axiom Schema of Replacement

$$\forall x_0 \ldots \forall x_{n-1} (\forall x \exists^{=1} y \phi(x, y, x_0, \ldots, x_{n-1}) \rightarrow \forall u \exists v \forall y (y \in v \leftrightarrow \exists x (x \in u \land \phi(x, y, x_0, \ldots, x_{n-1}))))$$

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$$\forall x ((\emptyset \notin x \land \forall u \forall v ((u \in x \land v \in x \land u \neq v) \rightarrow u \cap v = \emptyset)) \rightarrow \exists y \forall w (w \in x \rightarrow \exists^{=1} z z \in w \cap y))$$

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With set theory well-founded, we can turn next to formal natural-number arithmetic …

PA (Peano Arithmetic)

A1 $\forall x(0 \neq s(x))$

A2 $\forall x \forall y(s(x) = s(y) \rightarrow x = y)$

A3 $\forall x(x \neq 0 \rightarrow \exists y(x = s(y)))$

A4 $\forall x(x + 0 = x)$

A5 $\forall x \forall y(x + s(y) = s(x + y))$

A6 $\forall x(x \times 0 = 0)$

A7 $\forall x \forall y(x \times s(y) = (x \times y) + x)$

And, every sentence that is the universal closure of an instance of

$$([\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x))] \rightarrow \forall x \phi(x))$$

where $\phi(x)$ is open wff with variable x, and perhaps others, free.

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We've already seen it in our coverage of ZFC.

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$$\exists y[s(s(0)) \times y = s(s(s(s(0))))]$$

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That 2 multiplied by some number yields 4.

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But this is very specific: the successor
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The open wff $\phi(x)$, where of course x is a free variable, can then be used to abbreviate the formula immediately above, and therefore expresses the arithmetic property 'even.'

A “Participation” Problem …

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Slutten