Propositional Calculus III:
Reductio ad Absurdum

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Intro to Logic
2/1/2024

R A I R
Rensselaer AI and Reasoning Lab

(Confessedly Redundant) Logistics …

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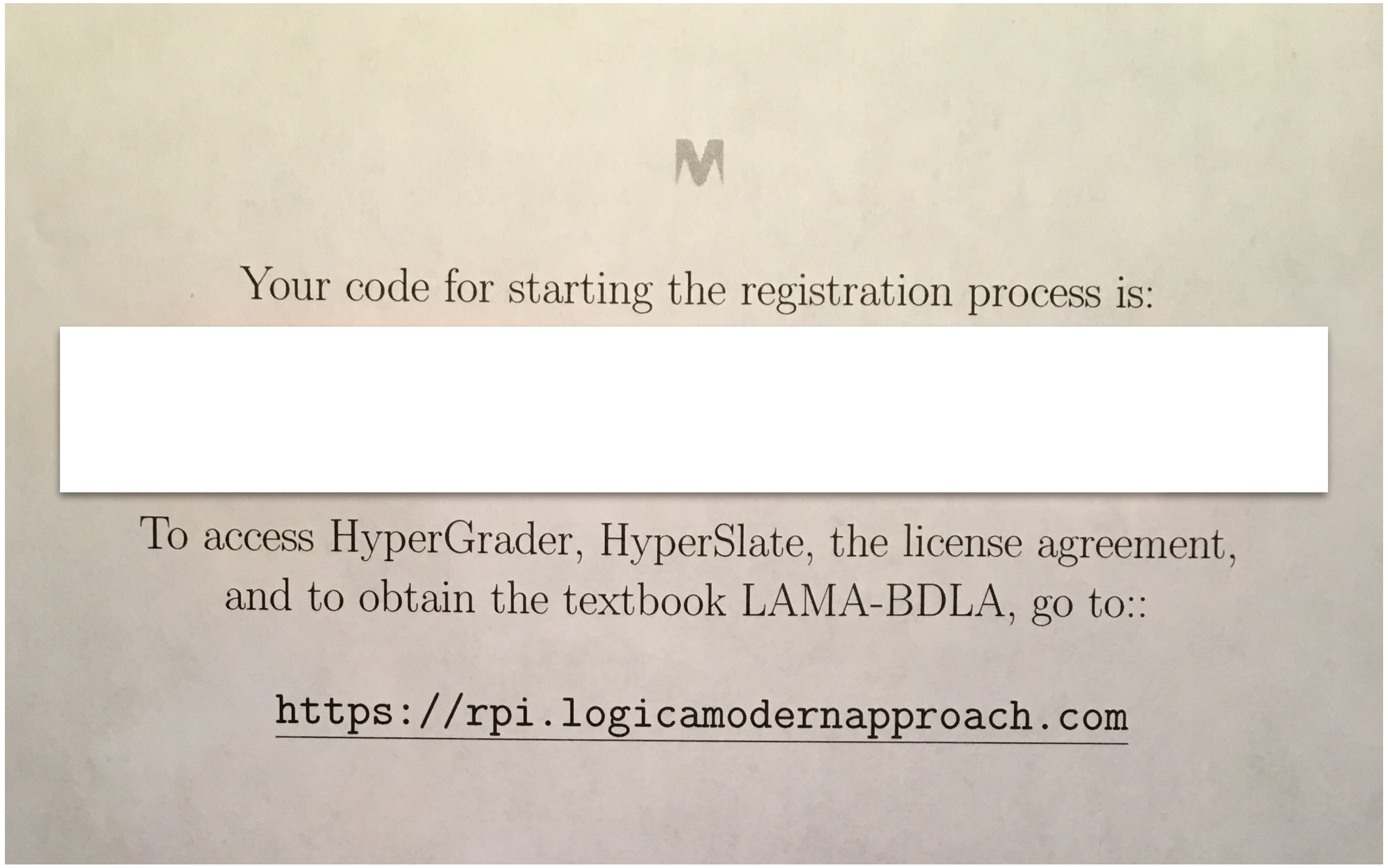
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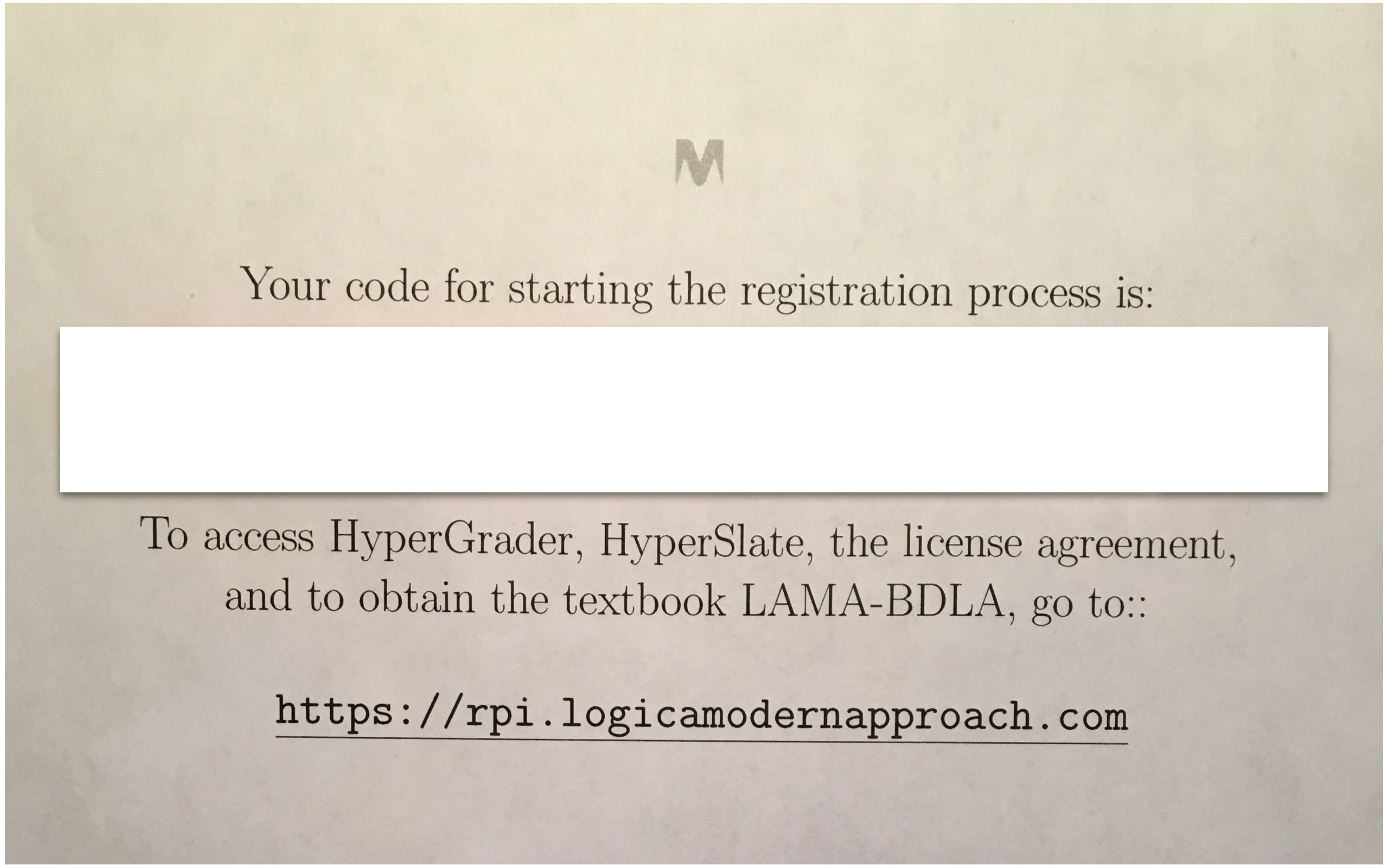

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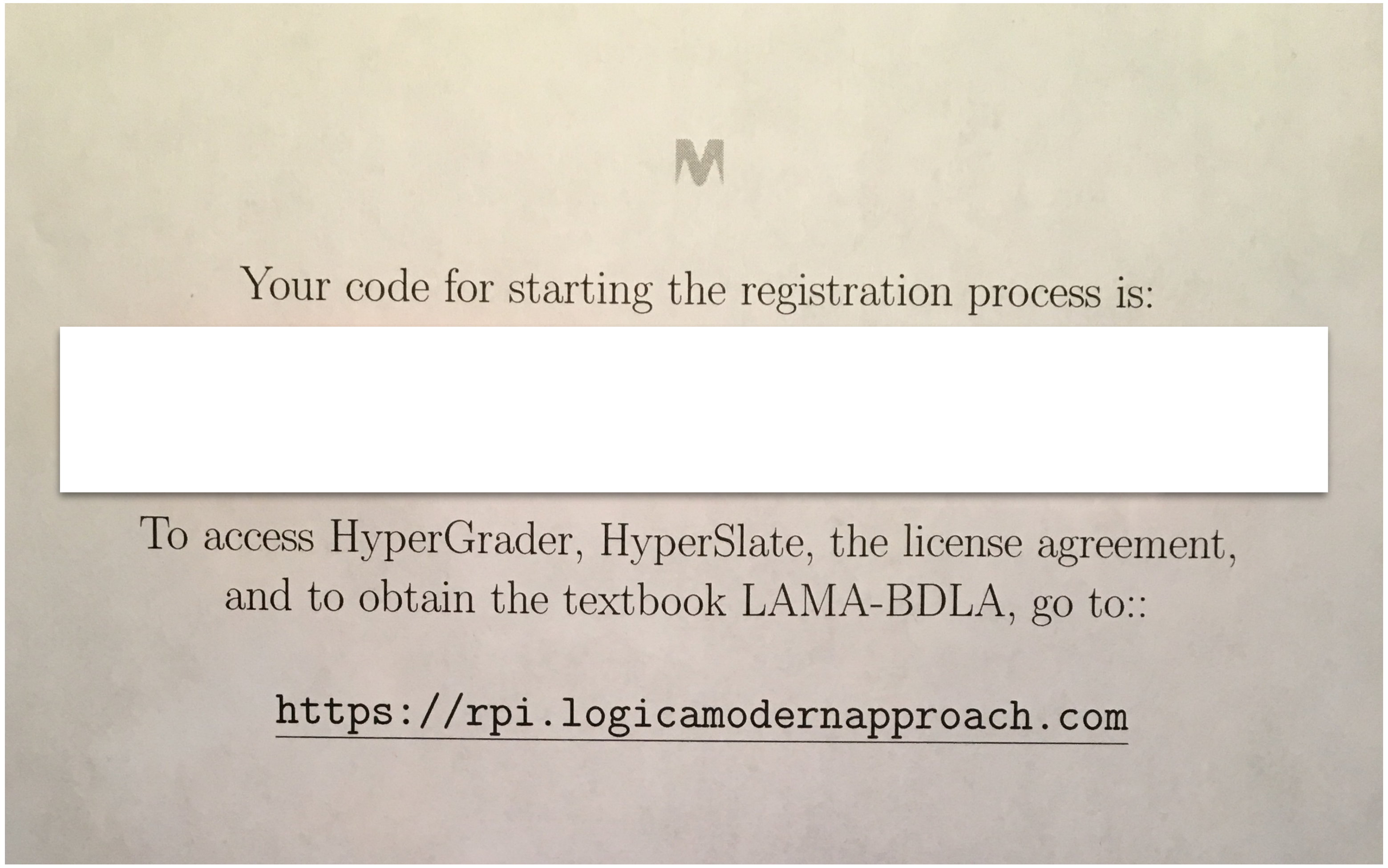

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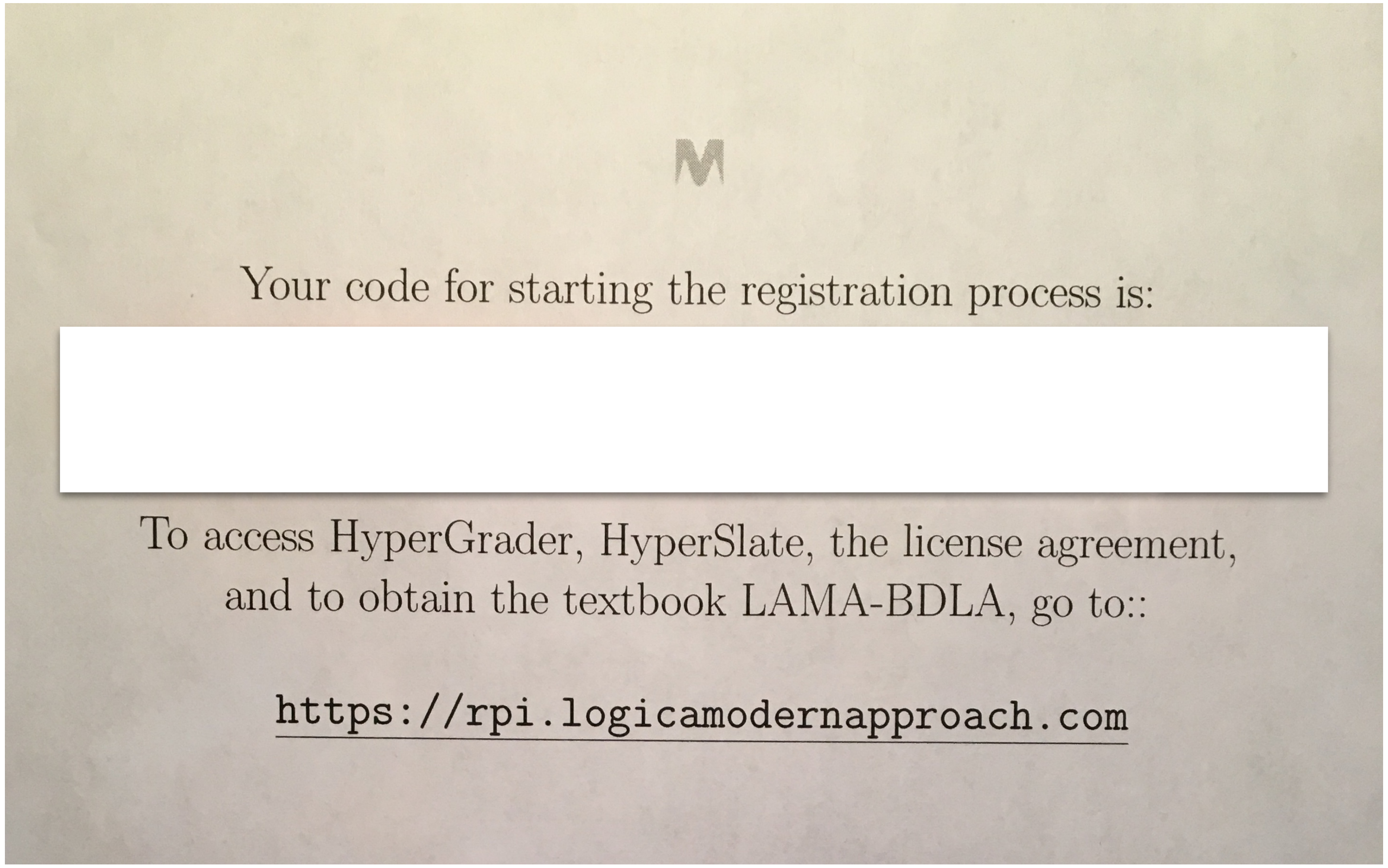
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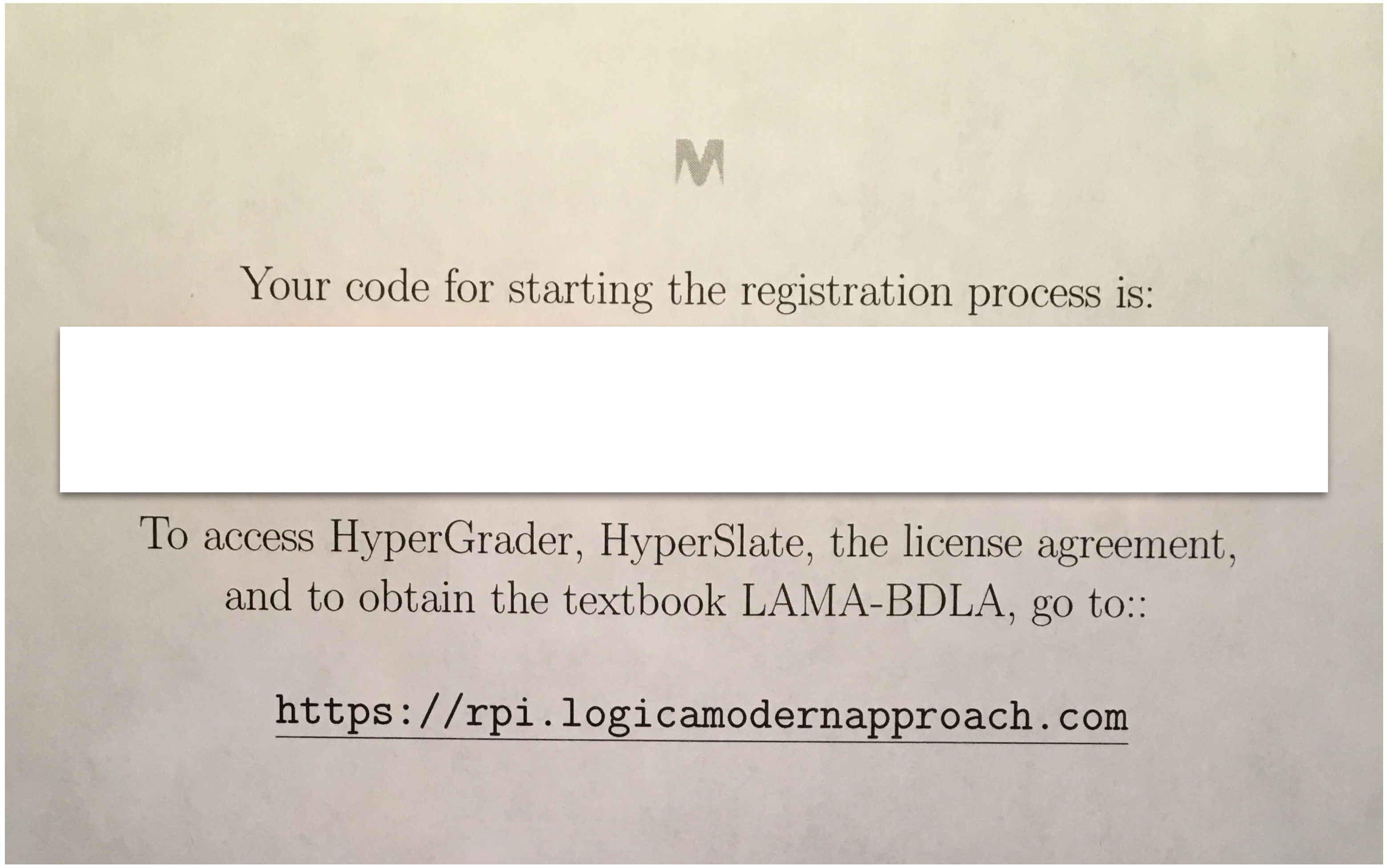

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Watch that the link emailed to you doesn't end up being classified as spam.

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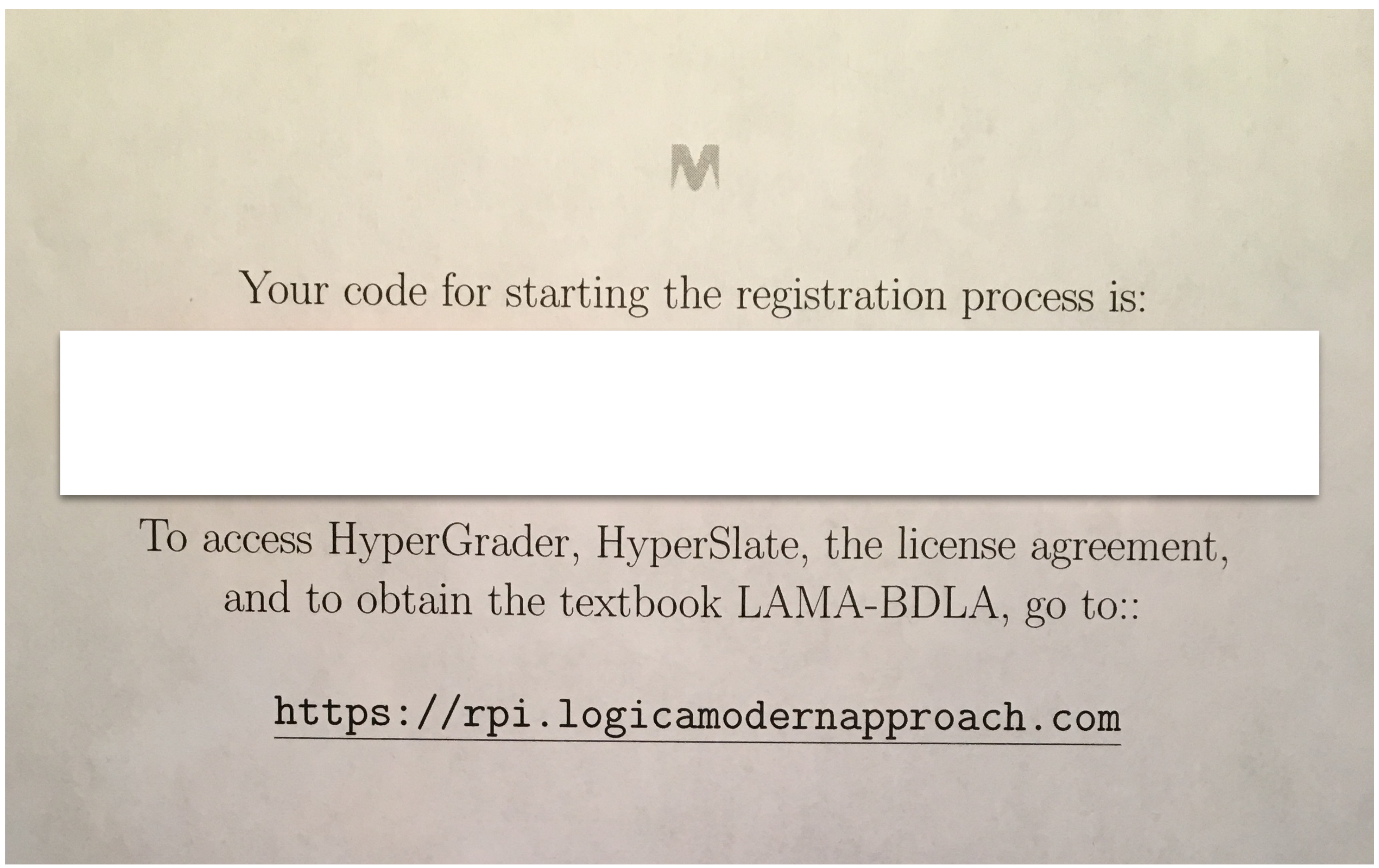
First prop. calc. (Exercise) Problem: `switching_conjuncts_fine`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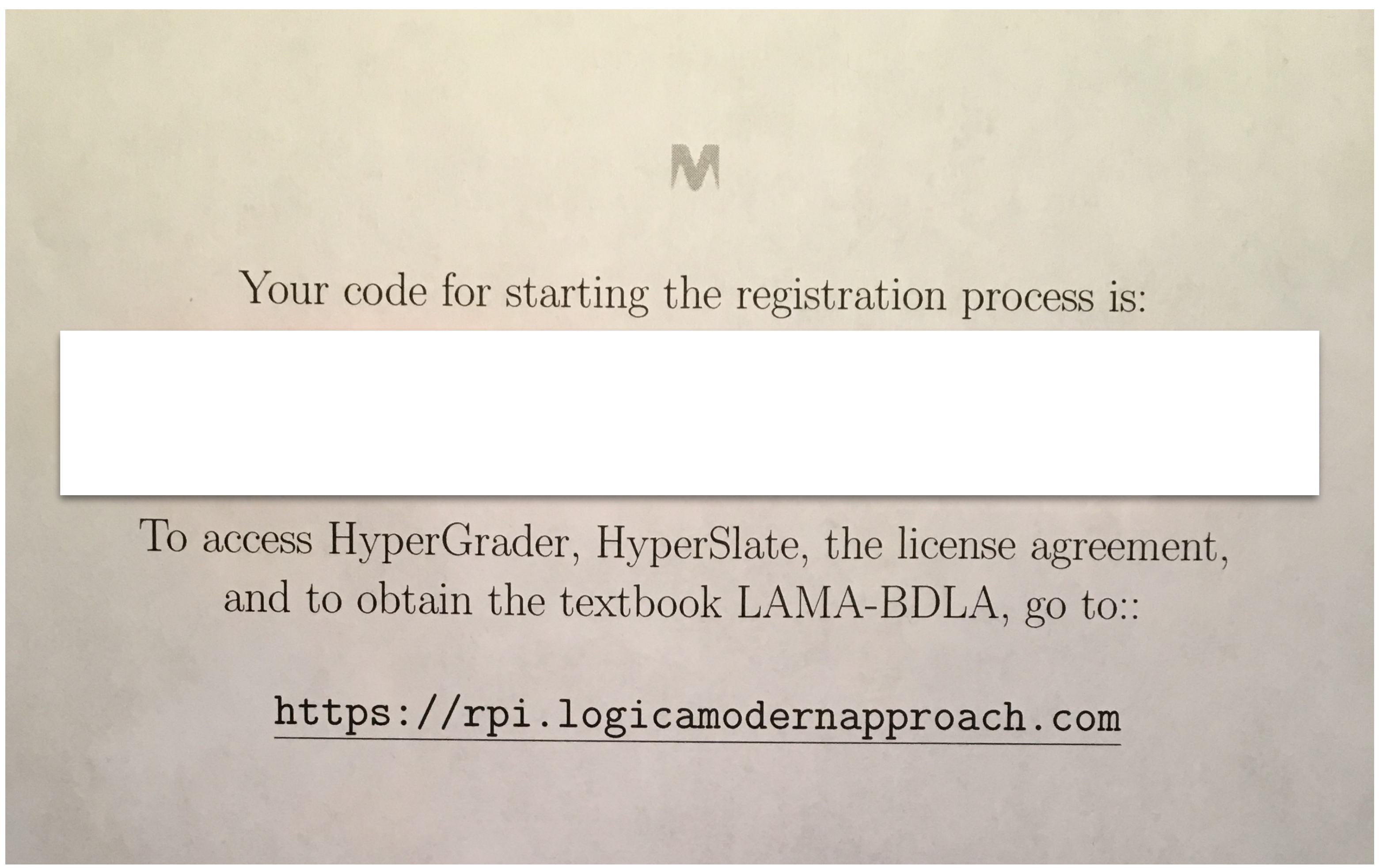
First prop. calc. (Exercise) Problem: `switching_conjuncts_fine`

Second prop. calc. (Exercise) Problem: `switching_disjuncts_fine`



E-Housekeeping Pts, Redux

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- Always work in the *same* browser window with multiple tabs; must do this with email and HyperGrader® & HyperSlate®.

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Reductio …

"Reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game."

–G. H. Hardy

A Greek-shocking Example …

$$\frac{p}{q}$$

What are rational numbers?

$$\frac{p}{q}$$

What are rational numbers?

- Any number that can be expressed in the form of $\frac{p}{q}$ such that we have a numerator p and a non-zero denominator.

- Rational numbers are a subset of real numbers!

- Examples of *ir*rational numbers?

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- Examples of *ir*rational numbers?

$$\sqrt{2}$$

Prove that:

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$$\sqrt{2}$$

Prove that:

$\sqrt{2}$ is irrational

(high-school textbook)

Suppose $\sqrt{2}$ is **rational.** That means it can be written as the ratio of two integers p and q

$$\sqrt{2} = \frac{p}{q} \tag{1}$$

where we may assume that p **and** q **have no common factors.** (If there are any common factors we cancel them in the numerator and denominator.) Squaring in (1) on both sides gives

$$2 = \frac{p^2}{q^2} \tag{2}$$

which implies

$$p^2 = 2q^2 \tag{3}$$

Thus p^2 is even. The only way this can be true is that p itself is even. But then p^2 is actually divisible by 4. Hence q^2 and therefore q must be even. So p and q are both even which is a contradiction to our assumption that they have no common factors. The square root of 2 cannot be rational!

Sizing Some Sets (see LAMA-BDLA)

?

$\mathbb{Q}^+$

$\mathbb{Z}^+$

$\mathbb{N}$

$\mathbb{R}^+$

Sizing Some Sets (see LAMA-BDLA)

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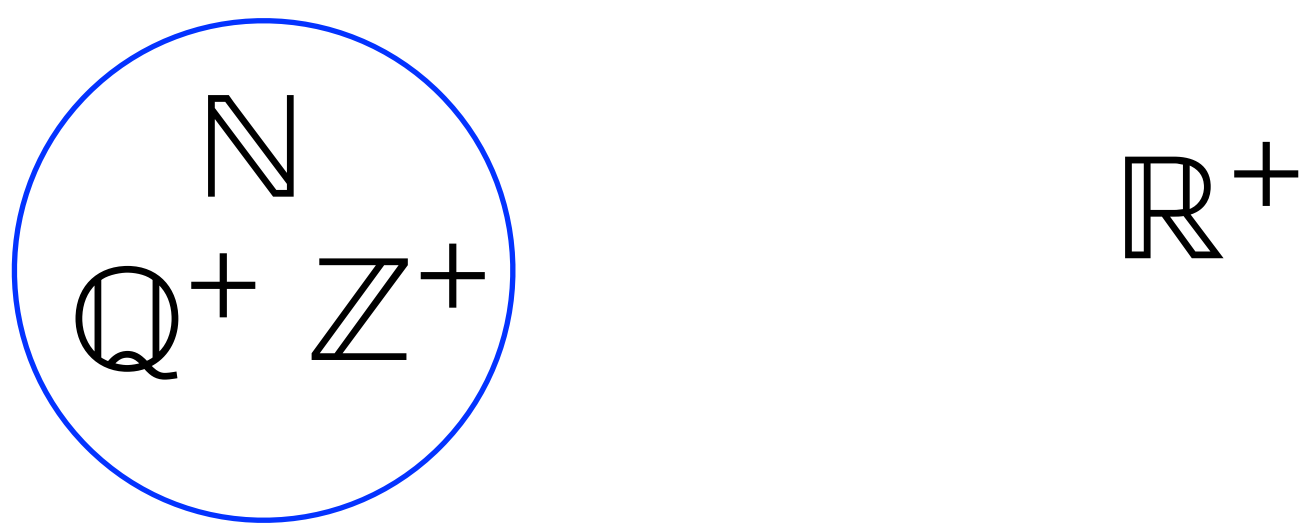

Sizing Some Sets (see LAMA-BDLA)

?

$$\{\mathbb{N},\ \mathbb{Q}^+,\ \mathbb{Z}^+\} \prec \mathbb{R}^+$$

Sizing Some Sets (see LAMA-BDLA)

$\mathbb{N}$

$\mathbb{Q}^+ \quad \mathbb{Z}^+ \quad \prec \quad \mathbb{R}^+$

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$\mathbb{N}$

$\mathbb{Q}^+ \quad \mathbb{Z}^+ \quad \prec \quad \mathbb{R}^+$

And now, what are *prime* numbers?

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- A number that can be divided by only two numbers, one and itself.
- Must be a whole number.
- Example: 2,3,5,7……

And recall: Euclidean "Magic"

Theorem: There are infinitely many primes.

Proof: We take an indirect route. Let $\Pi = p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_k$ be a finite, exhaustive consecutive sequence of prime numbers. Next, let $\mathbf{M}_\Pi$ be $p_1 \times p_2 \times \cdots \times p_k$, and set $\mathbf{M}'_\Pi$ to $\mathbf{M}_\Pi + 1$. Either $\mathbf{M}'_\Pi$ is prime, or not; we thus have two (exhaustive) cases to consider.

- C1 Suppose $\mathbf{M}'_\Pi$ is prime. In this case we immediately have a prime number beyond any in Π — contradiction!

- C2 Suppose on the other hand that $\mathbf{M}'_\Pi$ is *not* prime. Then some prime p divides $\mathbf{M}'_\Pi$. (Why?) Now, p itself is either in Π, or not; we hence have two sub-cases. Supposing that p is in Π entails that p divides $\mathbf{M}_\Pi$. But we are operating under the supposition that p divides $\mathbf{M}'_\Pi$ as well. This implies that p divides 1, which is absurd (a contradiction). Hence the prime p is outside Π.

Hence for *any* such list Π, there is a prime outside the list. That is, there are infinitely many primes. **QED**

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Notice that this proof uses two applications of *indirect proof*, and one application of *proof by cases*.

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(What is proof by cases in HyperSlate®?)

Study it word by word until you endorse it with your very soul — and consider variants on this theorem as conjectures that you yourself can attempt to settle!

Are we supposed to understand indirect proof = proof by contradiction = *reductio ad absurdum* = *reductio* in high school?

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Affirmative.

From Algebra 2 in High School …

(Pearson Common-Core Compliant Textbook)

A proof involving indirect reasoning is an **indirect proof.** Often in an indirect proof, a statement and its negation are the only possibilities. When you see that one of these possibilities leads to a conclusion that contradicts a fact you know to be true, you can eliminate that possibility. For this reason, indirect proof is sometimes called *proof by contradiction*.

TAKE NOTE Key Concept

Writing an Indirect Proof

Step 1 State as a temporary assumption the opposite (negation) of what you want to prove.

Step 2 Show that this temporary assumption leads to a contradiction.

Step 3 Conclude that the temporary assumption must be false and that what you want to prove must be true.

Problem 3 Writing an Indirect Proof

Proof

Given: $\triangle ABC$ is scalene.

Prove: $\angle A$, $\angle B$, and $\angle C$ all have different measures.

THINK	WRITE
Assume temporarily the opposite of what you want to prove.	Assume temporarily that two angles of $\triangle ABC$ have the same measure. Assume that $m\angle A = m\angle B$.
Show that this assumption leads to a contradiction.	By the Converse of the Isosceles Triangle Theorem, the sides opposite $\angle A$ and $\angle B$ are congruent. This contradicts the given information that $\triangle ABC$ is scalene.
Conclude that the temporary assumption must be false and that what you want to prove must be true.	The assumption that two angles of $\triangle ABC$ have the same measure must be false. Therefore, $\angle A$, $\angle B$, and $\angle C$ all have different measures.

SHOW SOLUTION

GOT IT?

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        Author = {R. C. Moore},
        Journal = {Educational Studies in Mathematics},
        Pages = {249–266},
        Title = {Making the Transition to Formal Proof},
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(A very important paper.)

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Can you create a formal proof in HyperSlate®?

SHOW SOLUTION

GOT IT?

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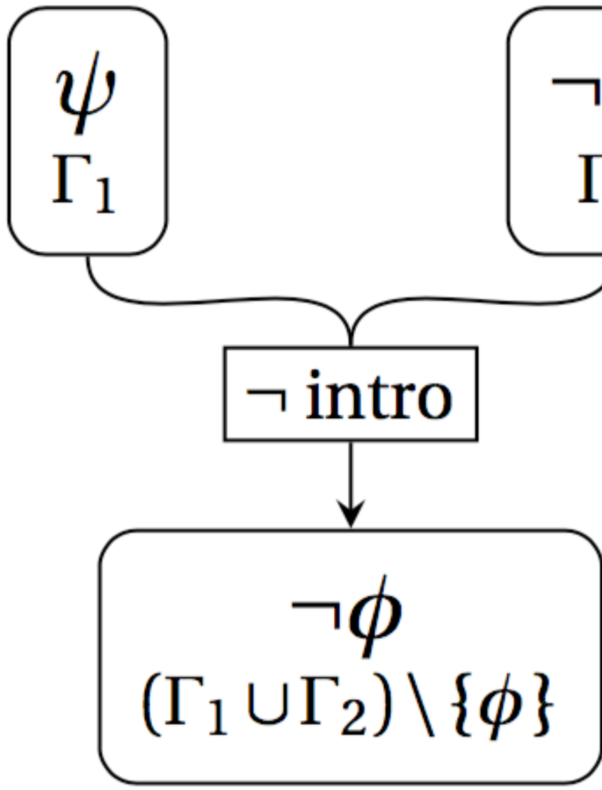
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Γ_1
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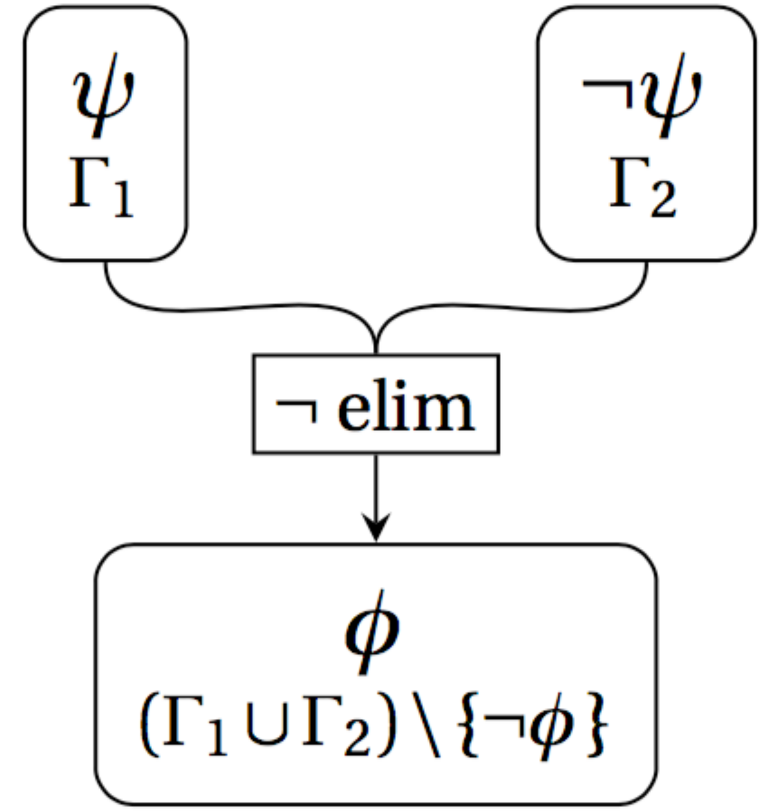
Γ_2
$\neg$ intro
$\neg\phi$
$(\Gamma_1 \cup \Gamma_2) \setminus \{\phi\}$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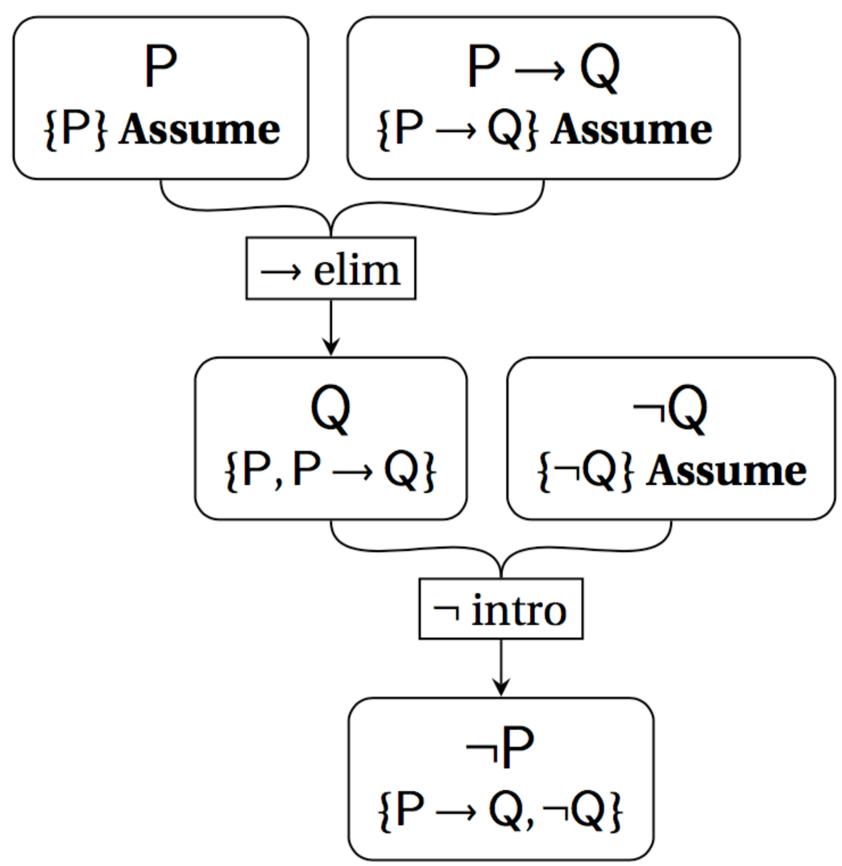
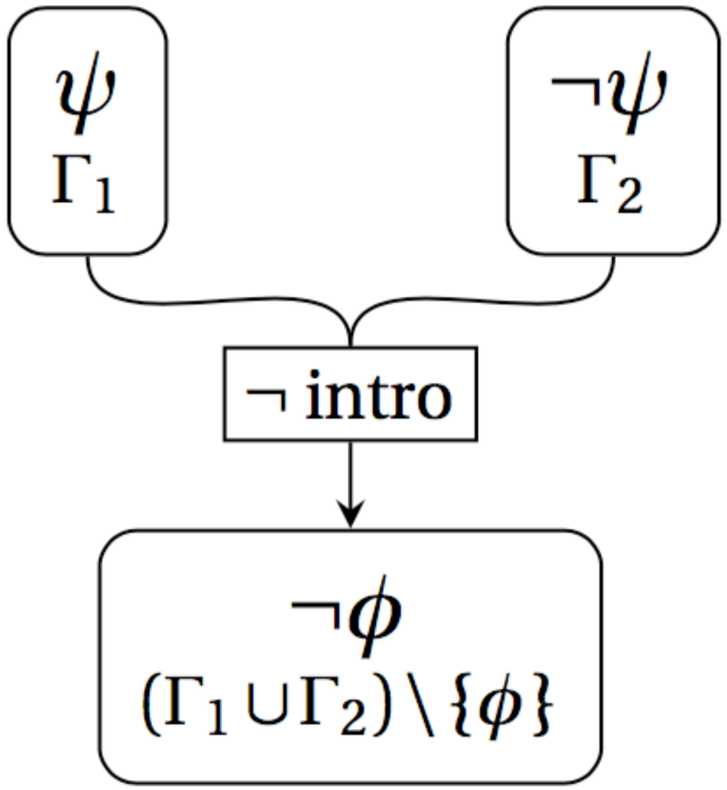
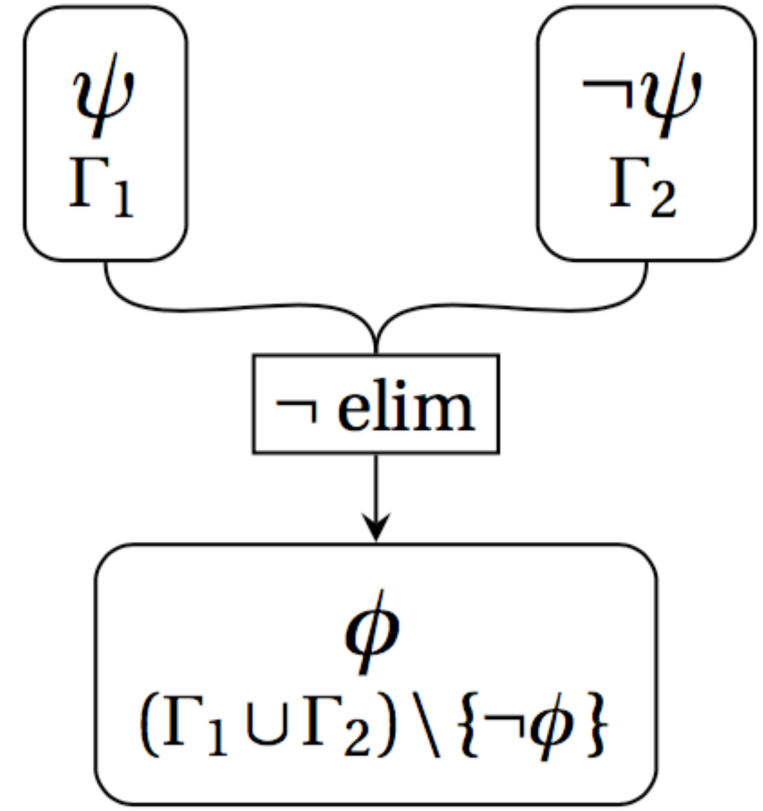
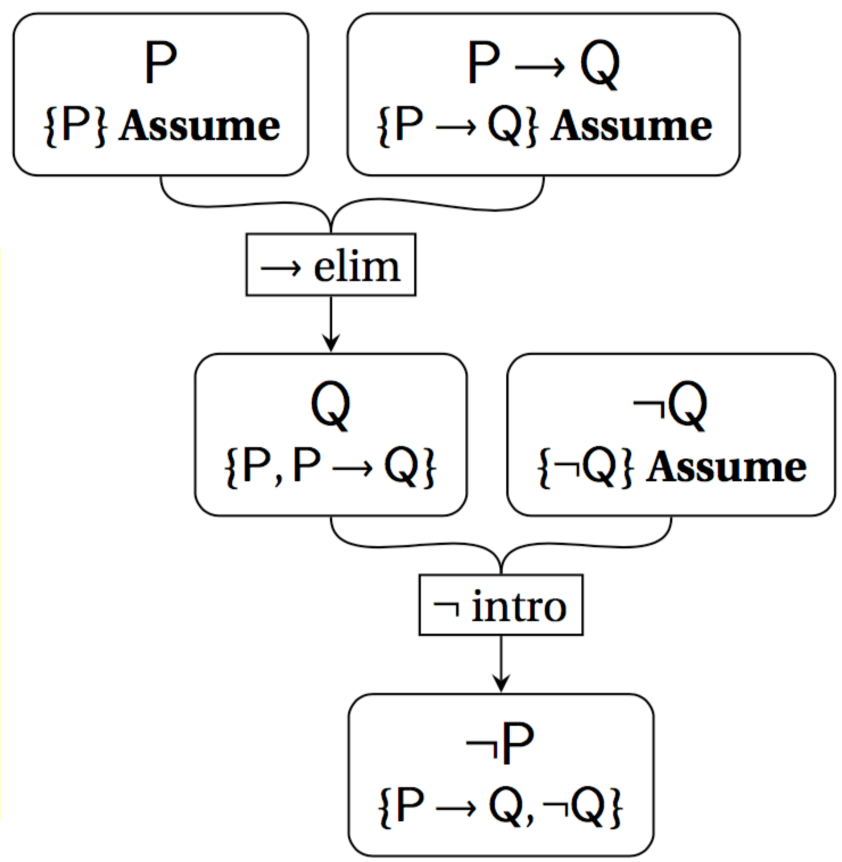

P
{P} **Assume**
P → Q
{P → Q} **Assume**
→ elim
Q
{P, P → Q}
¬Q
{¬Q} **Assume**
¬ intro
¬P
{P → Q, ¬Q}

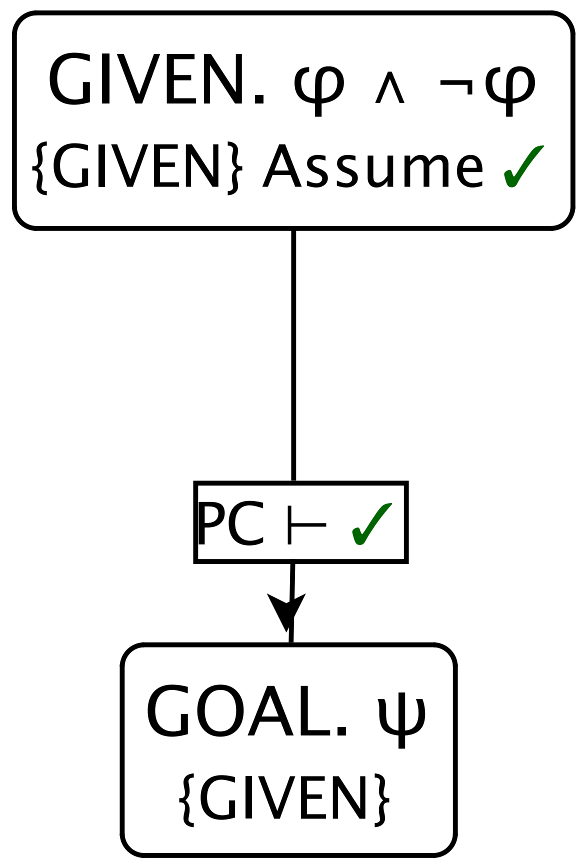




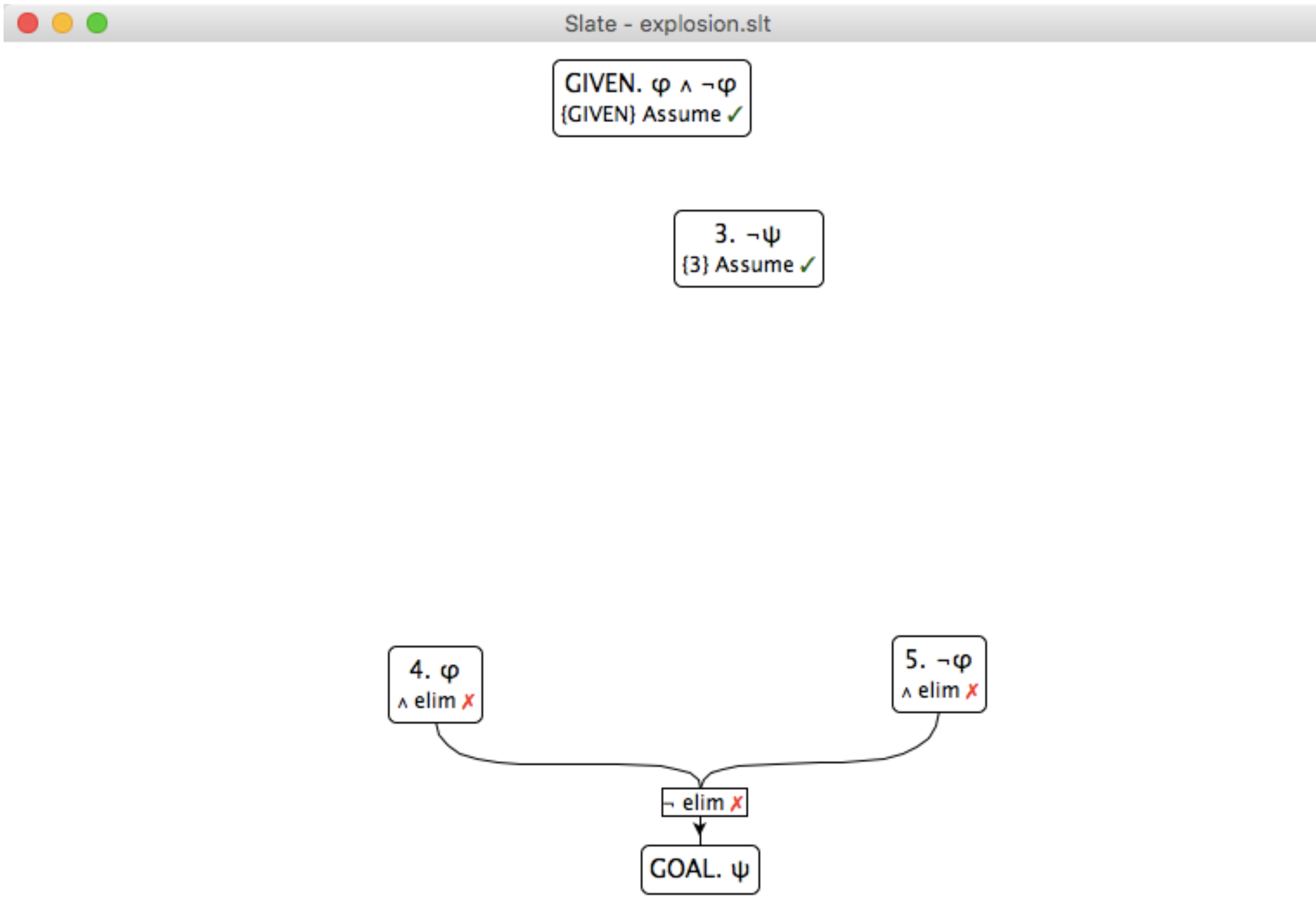


Make sure you understand every aspect of this slide! Ask questions if need be! Thx!

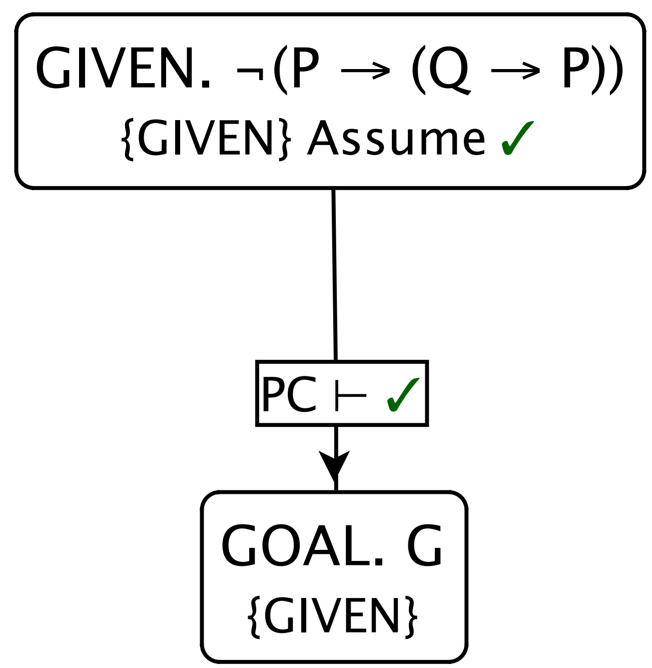
Explosion



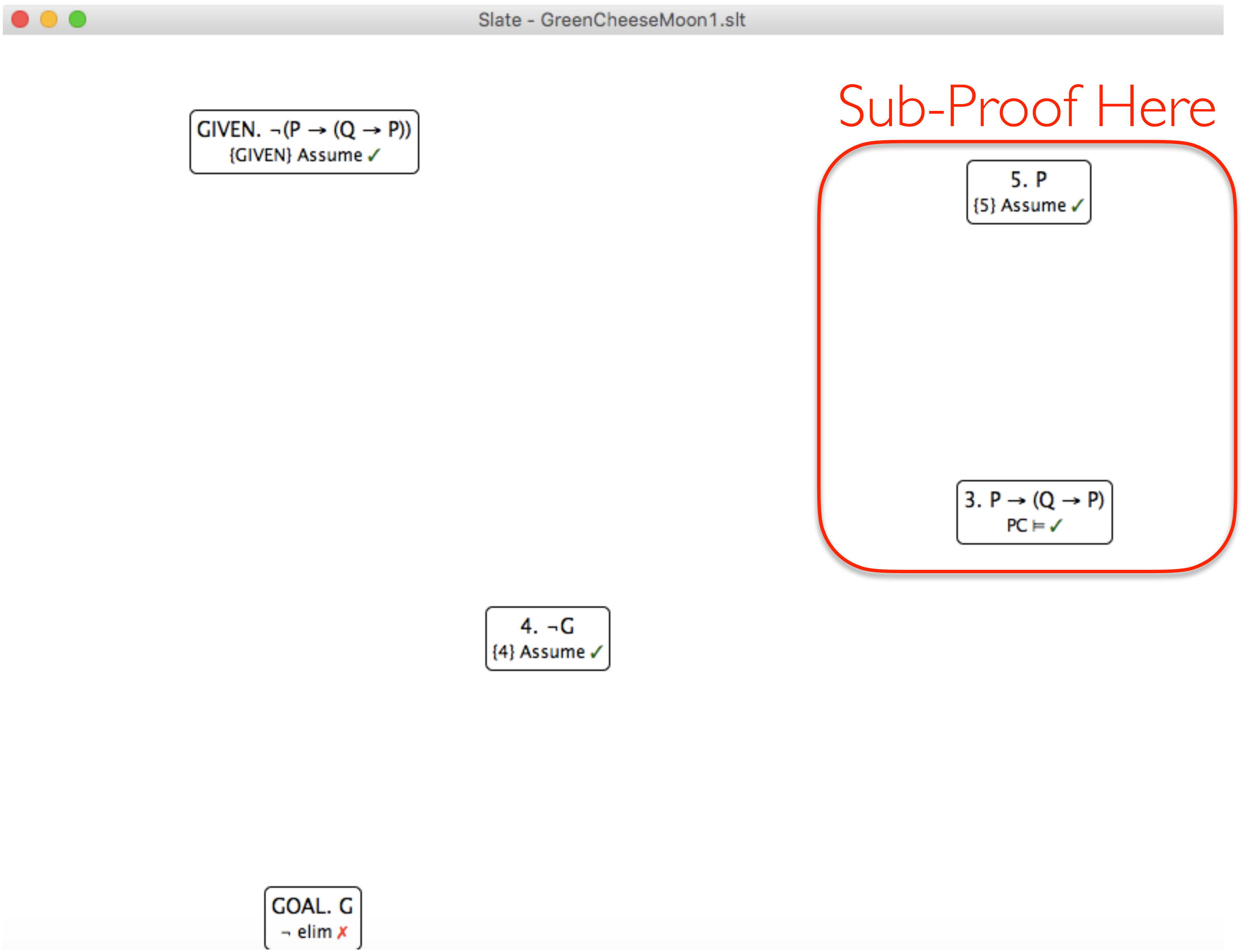
Explosion: Partial Proof Plan



GreenCheeseMoon1



GreenCheeseMoon1: Partial Proof Plan



Interlude:

HyperSlate® & High School Geometry: A Glimpse Ahead …

Geometry Reduced to Formal Logic

Proving Congruent Triangles

"… mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

– Bertrand Russell; *Recent Work on the Principles of Mathematics*, published in International Monthly, vol. 4

Do you have one of those friends who always doubts everything that you say? One of those people who always says, "Nuh-uh! No way!" (The person may even believe you, but your statement is so amazingly shocking that they can't believe that it could be true.) They can't believe your statement until they've seen it from a credible source or heard it from enough people.

Well, thank that person. Knowing things with certainty is a rare, rare occurrence.

Geometry Reduced to Formal Logic

SEC

Sect

It's getting harder and harder to determine what is true. People can manipulate pictures or sounds so well that it's near impossible to believe anything.

So if we can't be 100% certain by using physical objects, the only way to prove a fact with certainty is with pure logic. A proof is the logical structure of making an argument with clearly stated facts, and making logical conclusions with supporting reasons. If other people can understand your proof and agree that it is logically correct, then they can agree that the fact that you are stating must be true. This is the purpose of proof within mathematics: making a clear logical argument so that others can understand and agree with your claim.

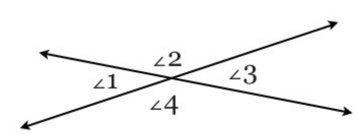
A proof is different from a definition. A definition is closer to just naming something. We can define a dog: a highly variable carnivorous domesticated mammal of the genus Canis (C. familiaris) closely related to the common wolf (Canis lupus). This is not *proving* what a dog is.

In geometry, we can define objects as well.

Definition: Vertical Angles are a pair of opposite angles formed by two intersecting lines.

$\angle 1$ and $\angle 3$ are vertical angles.

$\angle 2$ and $\angle 4$ are also vertical angles.

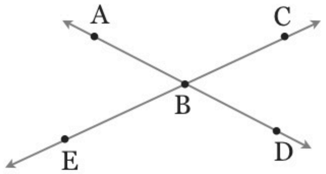


Suppose I make the statement, "If two angles are vertical angles, then they are congruent." This is making a claim about vertical angles. Someone may or may not agree with this claim. I would have to provide a proof to convince them that this statement is true.

Geometry Reduced to Formal Logic

One possible proof would be as follows:

∠ABD and ∠CBE are straight angles. We agreed that this can be assumed. Straight angles are equal to 180° by definition.



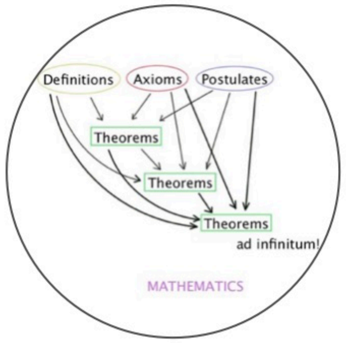
Since ∠ABE + ∠ABC = ∠CBE,
∠ABE + ∠ABC = 180°.

By subtraction, ∠ABE = 180° − ∠ABC.

Similarly, ∠ABC + ∠CBD = 180°,
so ∠CBD = 180° − ∠ABC.

Both ∠ABE and ∠CBD equal 180° − ∠ABC, so they are equal in measure and therefore congruent.

I hope that anyone could read this explanation and agree with each of the sentences and the flow of logic. This would prove the statement, "If two angles are vertical angles, they are congruent." Once a statement has been proved (and generally agreed as true by the mathematical community), it can be named a theorem.



Theorem: If two angles are vertical angles, then they are congruent.

Vert. ∠s → ≅

Theorems can be used with properties, axioms, postulates, and definitions to prove other statements. And then those theorems can be used to prove other statements. And so on, and so on, and so on...

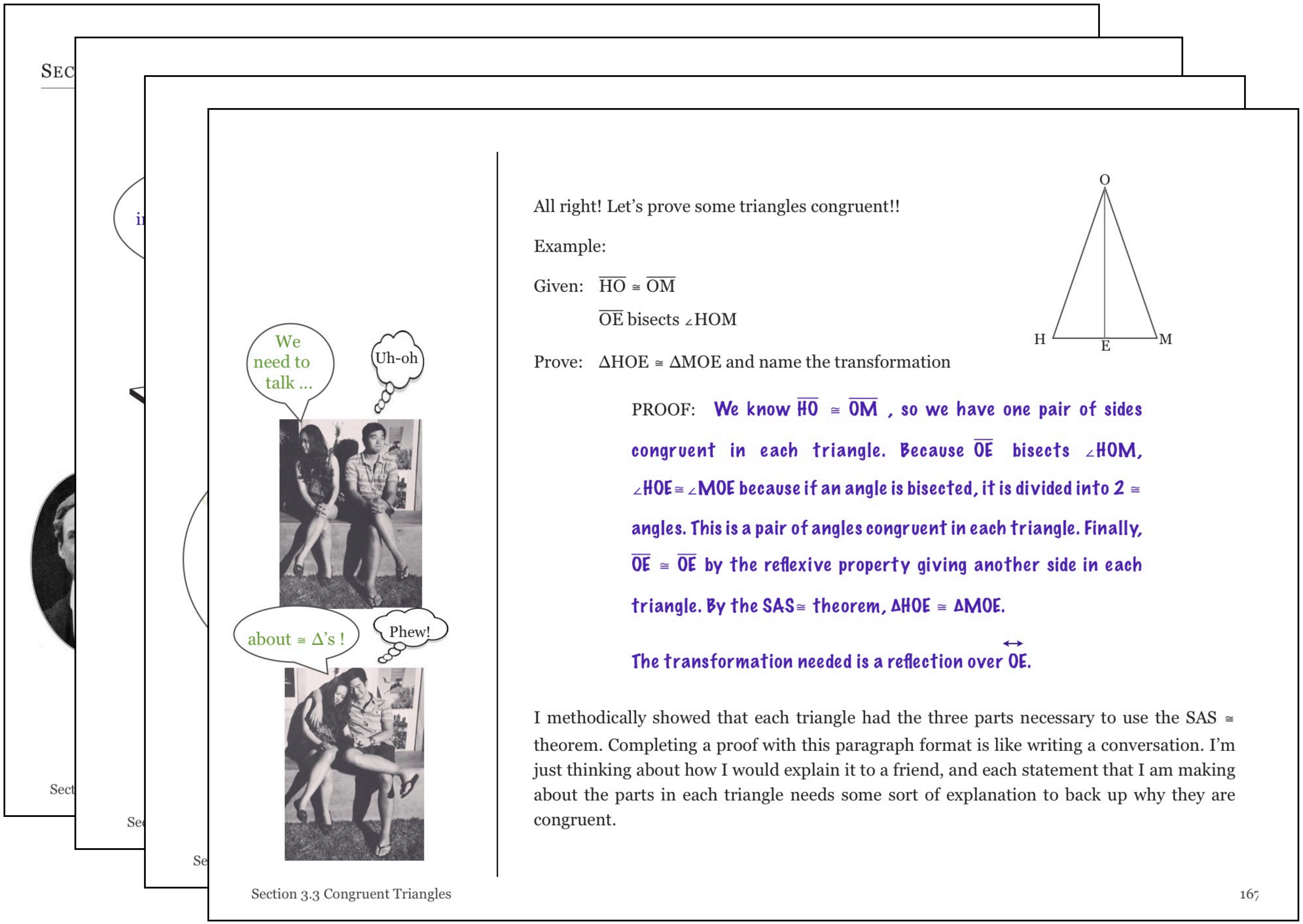
Two common sense properties:

Also known as the "Well, duh" theorem

Reflexive Property: Any object is congruent to itself.

Transitive Property: If 2 objects are congruent to the same object, then they are congruent to each other.

Geometry Reduced to Formal Logic



All right! Let's prove some triangles congruent!!

Example:

Given: $\overline{HO} \cong \overline{OM}$

$\overline{OE}$ bisects $\angle HOM$

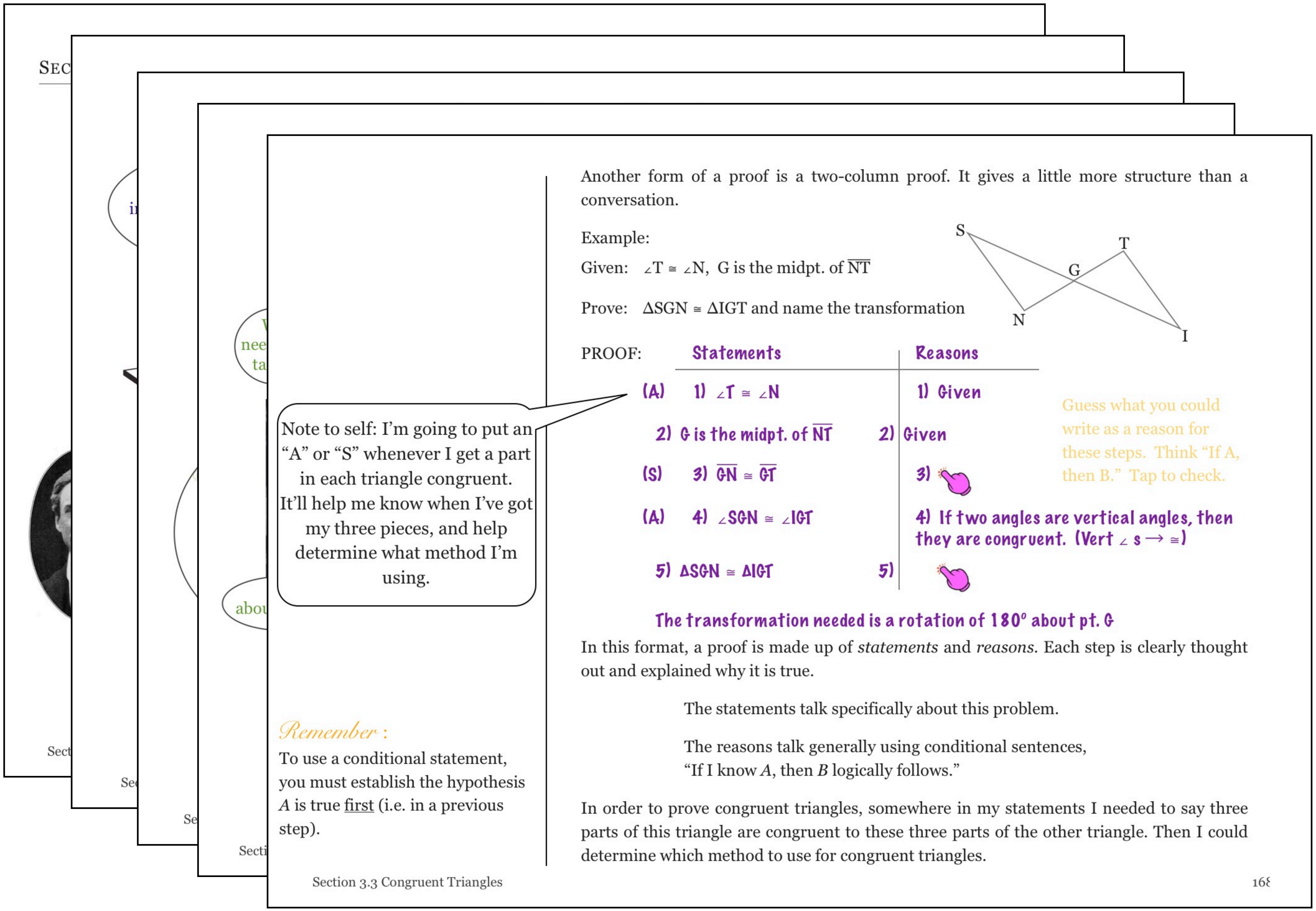
Prove: $\Delta HOE \cong \Delta MOE$ and name the transformation

PROOF: We know $\overline{HO} \cong \overline{OM}$, so we have one pair of sides congruent in each triangle. Because $\overline{OE}$ bisects $\angle HOM$, $\angle HOE \cong \angle MOE$ because if an angle is bisected, it is divided into 2 ≅ angles. This is a pair of angles congruent in each triangle. Finally, $\overline{OE} \cong \overline{OE}$ by the reflexive property giving another side in each triangle. By the SAS≅ theorem, $\Delta HOE \cong \Delta MOE$.

The transformation needed is a reflection over $\overleftrightarrow{OE}$.

I methodically showed that each triangle had the three parts necessary to use the SAS ≅ theorem. Completing a proof with this paragraph format is like writing a conversation. I'm just thinking about how I would explain it to a friend, and each statement that I am making about the parts in each triangle needs some sort of explanation to back up why they are congruent.

Geometry Reduced to Formal Logic


Another form of a proof is a two-column proof. It gives a little more structure than a conversation.

Example:

Given: $\angle T \cong \angle N$, G is the midpt. of $\overline{NT}$

Prove: $\Delta SGN \cong \Delta IGT$ and name the transformation

PROOF:

	Statements	Reasons
(A)	1) $\angle T \cong \angle N$	1) Given
	2) G is the midpt. of $\overline{NT}$	2) Given
(S)	3) $\overline{GN} \cong \overline{GT}$	3)
(A)	4) $\angle SGN \cong \angle IGT$	4) If two angles are vertical angles, then they are congruent. (Vert $\angle$ s $\to \cong$)
	5) $\Delta SGN \cong \Delta IGT$	5)

Guess what you could write as a reason for these steps. Think "If A, then B." Tap to check.

Note to self: I'm going to put an "A" or "S" whenever I get a part in each triangle congruent. It'll help me know when I've got my three pieces, and help determine what method I'm using.

The transformation needed is a rotation of 180° about pt. G

In this format, a proof is made up of *statements* and *reasons*. Each step is clearly thought out and explained why it is true.

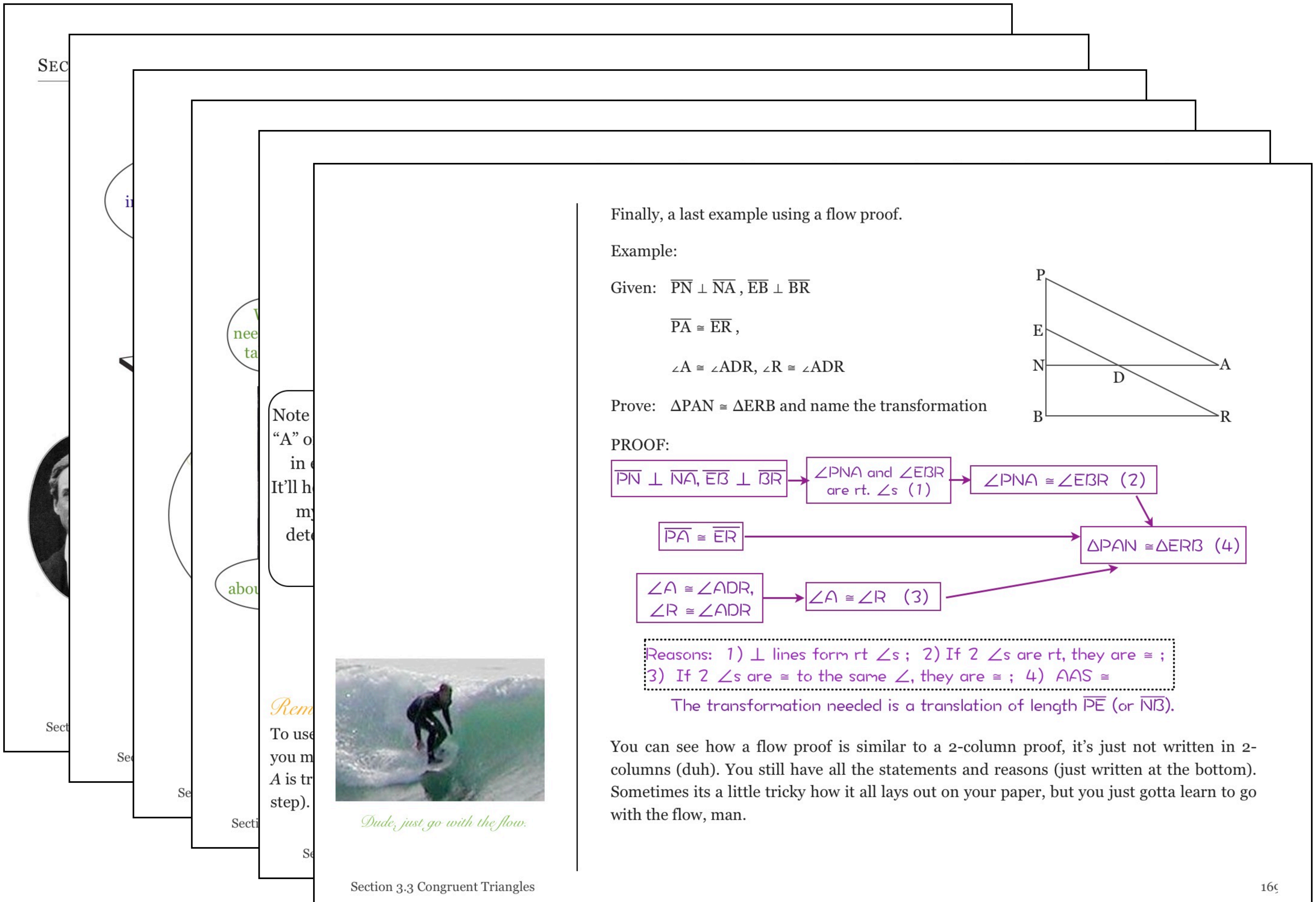
The statements talk specifically about this problem.

The reasons talk generally using conditional sentences, "If I know *A*, then *B* logically follows."

In order to prove congruent triangles, somewhere in my statements I needed to say three parts of this triangle are congruent to these three parts of the other triangle. Then I could determine which method to use for congruent triangles.

Remember :
To use a conditional statement, you must establish the hypothesis *A* is true first (i.e. in a previous step).

Geometry Reduced to Formal Logic


Finally, a last example using a flow proof.

Example:

Given: $\overline{PN} \perp \overline{NA}$, $\overline{EB} \perp \overline{BR}$

$\overline{PA} \cong \overline{ER}$,

$\angle A \cong \angle ADR$, $\angle R \cong \angle ADR$

Prove: $\Delta PAN \cong \Delta ERB$ and name the transformation

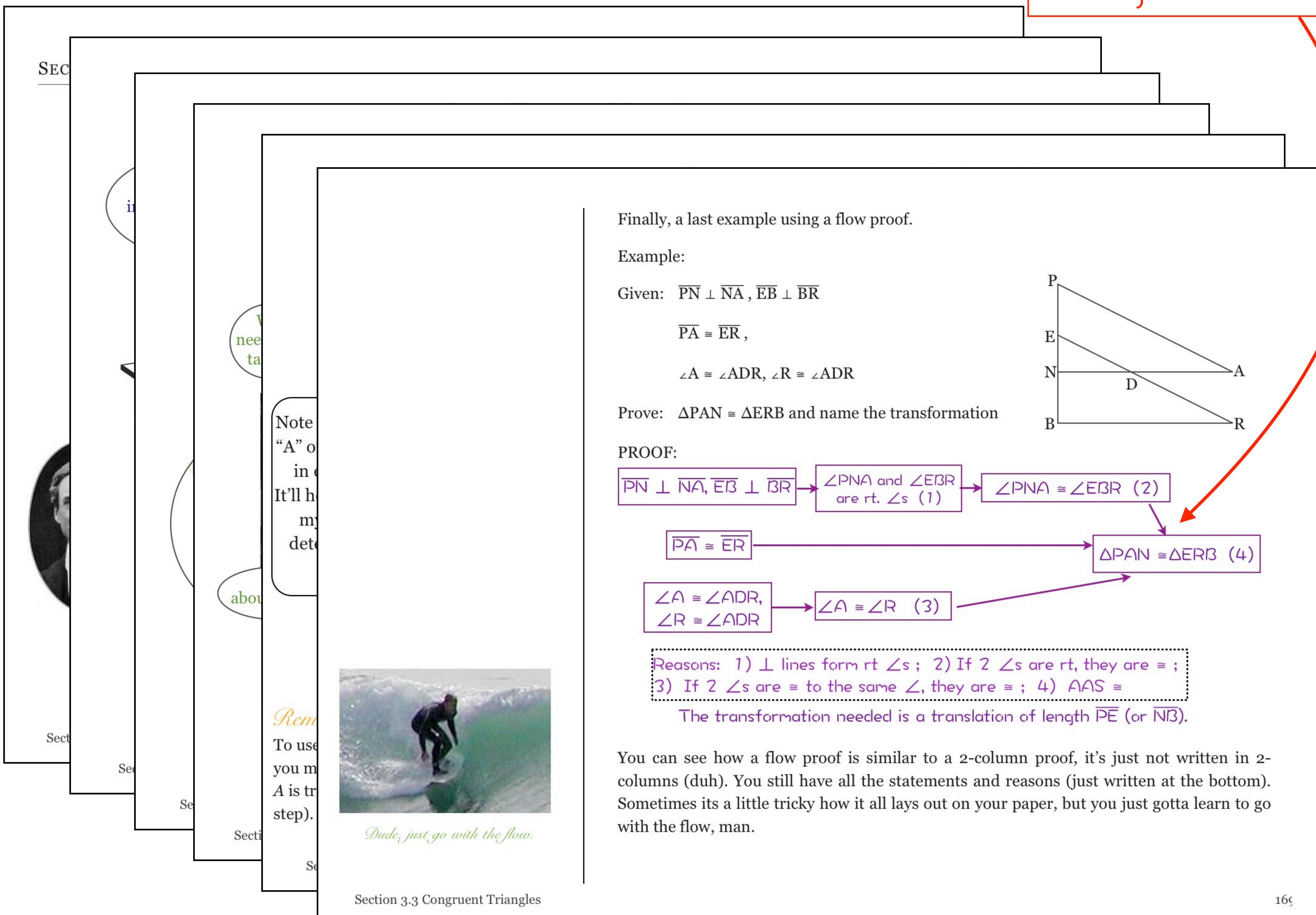
PROOF:

You can see how a flow proof is similar to a 2-column proof, it's just not written in 2-columns (duh). You still have all the statements and reasons (just written at the bottom). Sometimes its a little tricky how it all lays out on your paper, but you just gotta learn to go with the flow, man.

Dude, just go with the flow.

Geometry Reduced to Formal Logic

This is kooky, because what justifies an arrow?!



Finally, a last example using a flow proof.

Example:

Given: $\overline{PN} \perp \overline{NA}$, $\overline{EB} \perp \overline{BR}$

$\overline{PA} \cong \overline{ER}$,

$\angle A \cong \angle ADR$, $\angle R \cong \angle ADR$

Prove: $\Delta PAN \cong \Delta ERB$ and name the transformation

PROOF:

Reasons: 1) $\perp$ lines form rt $\angle$s ; 2) If 2 $\angle$s are rt, they are $\cong$; 3) If 2 $\angle$s are $\cong$ to the same $\angle$, they are $\cong$; 4) AAS $\cong$

The transformation needed is a translation of length $\overline{PE}$ (or $\overline{NB}$).

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Tarski's System(s) of Geometry

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TARSKI'S SYSTEM OF GEOMETRY

ALFRED TARSKI AND STEVEN GIVANT

Abstract. This paper is an edited form of a letter written by the two authors (in the name of Tarski) to Wolfram Schwabhäuser around 1978. It contains extended remarks about Tarski's system of foundations for Euclidean geometry, in particular its distinctive features, its historical evolution, the history of specific axioms, the questions of independence of axioms and primitive notions, and versions of the system suitable for the development of 1-dimensional geometry.

In his 1926–27 lectures at the University of Warsaw, Alfred Tarski gave an axiomatic development of elementary Euclidean geometry, the part of plane Euclidean geometry that is not based upon set-theoretical notions, or, in other words, the part that can be developed within the framework of first-order logic. He proved, around 1930, that his system of geometry admits elimination of quantifiers: every formula is provably equivalent (on the basis of the axioms) to a Boolean combination of basic formulas. From this theorem he drew several fundamental corollaries. First, the theory is complete: every assertion is either provable or refutable. Second, the theory is decidable—there is a mechanical procedure for determining whether or not any given assertion is provable. Third, there is a constructive consistency proof for the theory. Substantial simplifications in Tarski's axiom system and the development of geometry based on them were obtained by Tarski and his students during the period 1955-65. All of these various results were described in Tarski [41], [44], [45], and Gupta [5].

Aside from the importance of its metamathematical properties, Tarski's system of geometry merits attention because of the extreme elegance and simplicity of its set of axioms, especially in the final form that it achieved around 1965. Yet, until fairly recently, no systematic development of geometry based on his axioms existed. In the early 1960s Wanda Szmielew and Tarski began the project of preparing a treatise on the foundations of geometry developed within the framework of contemporary mathematical logic. A systematic development of Euclidean geometry based on Tarski's axioms was to constitute the first part of the treatise. The project made

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of the axiom schema As. 11. Thus it is an axiom set for elementary 2-dimensional Euclidean geometry. The possibility of modifying the dimension axioms Ax. $8^{(2)}$ and Ax. $9_1^{(2)}$ in order to obtain an axiom set for n-dimensional geometry is briefly mentioned. (The case $n = 1$ will be disregarded in this section and in Sections 3–5.) The passage to an axiom set for the full (non-elementary) Euclidean geometry, by replacing all instances of the axiom schema As. 11 with Ax. 11, is not mentioned explicitly.

The next version of the axiom set appeared in Tarski [41]. Since $=$ is treated there as a logical notion, Ax. 13 and Ax. 19 are easily derivable from the remaining axioms, and therefore have been omitted. Ax. 20 is replaced by a somewhat more concise variant, Ax. 20_1; we do not analyze this modification since Ax. 20 is dropped entirely in subsequent versions.

A rather substantial simplification of the axiom set in Tarski [41] was obtained in 1956–57 as a result of joint efforts by Eva Kallin, Scott Taylor, and Tarski (see Tarski [44], p. 20, footnote). First, four axioms, Ax. 5_1, Ax. 7_2, Ax. $9_1^{(2)}$, and Ax. 10, have been respectively replaced by equivalent formulations Ax. 5, Ax. 7_1, Ax. $9^{(2)}$, and Ax. 10_1. In the case of Ax. $9_1^{(2)}$ the new formulation differs essentially from the old one, in both its form and its mathematical content. In the remaining three cases the differences are very slight. Some remarks in the later discussion will throw light on the purpose of all these modifications. Next, in the modified axiom set six axioms, Ax. 12, Ax. 14, Ax. 16, Ax. 17, Ax. 20_1, and Ax. 21, are shown to be derivable from the remaining ones, and hence are omitted. Thus we arrive at the set consisting of twelve axioms: Ax. 1–Ax. 6, Ax. 7_1, Ax. $8^{(2)}$, Ax. $9^{(2)}$, Ax. 10_1, Ax. 15, Ax. 18, and all instances of the old axiom schema As. 11. This axiom set was discussed by Tarski in his course on the foundations of geometry given at the University of California, Berkeley, during the academic year 1956–57. It appeared in print in Tarski [44]. It was pointed out there that, by enriching the logical framework of our system of geometry and by replacing the axiom schema As. 11 with the (second-order) sentence Ax. 11, we arrive at an axiom set for the full (non-elementary) 2-dimensional Euclidean geometry. Also, it was mentioned that, by replacing Ax. $8^{(2)}$ and Ax. $9^{(2)}$ in either of the two above axiom sets with their n-dimensional analogues ($n = 3, 4, \dots$), which are explicitly listed in Section 1 above as Ax. $8^{(n)}$ and Ax. $9^{(n)}$, axiom sets for n-dimensional geometry are obtained.

Some general metamathematical results, published at about the same time in Scott [30] and Szmielew [37], show that the dimension axioms Ax. $8^{(n)}$ and Ax. $9^{(n)}$, and Euclid's axiom Ax. 10_1 can be equivalently replaced by a great variety of sentences. The results will be discussed in Section 4, in connection with the axioms involved. It does not seem that these results lead to any formal simplification of the axiom sets discussed here.

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Det er en ære å lære formell logikk!

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Part II of Class: Hands On