

Gödel's ...

First Incompleteness Theorem

(GI)

Selmer Bringsjord*

Rensselaer AI & Reasoning (RAIR) Lab
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Department of Computer Science
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4/11/2024



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*With Naveen Sundar G's introduction to Gödel numbering, and his depiction of GT growth, etc.

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Note: This is a version designed for those who have had at least one robust, proof-intensive university-level course in formal logic to the level of \mathcal{L}_2 .



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Background Context ...

Gödel's Great Theorems (OUP)

by Selmer Bringsjord

- Introduction (“The Wager”)
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
- The Continuum-Hypothesis Theorem
- The Time-Travel Theorem
- Gödel’s “God Theorem”
- Could a Finite Machine Match Gödel’s Greatness?



STOP & REVIEW IF NEEDED!

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A corollary of the First Incompleteness Theorem: *We cannot prove (in classical mathematics) that mathematics is consistent.*

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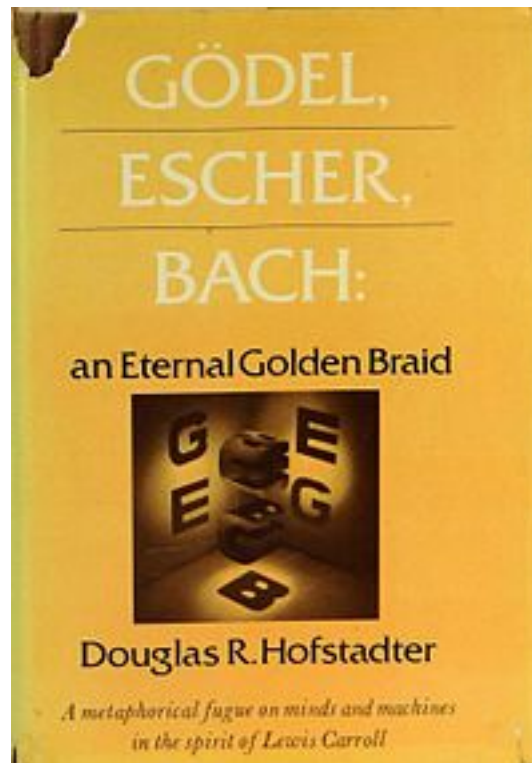
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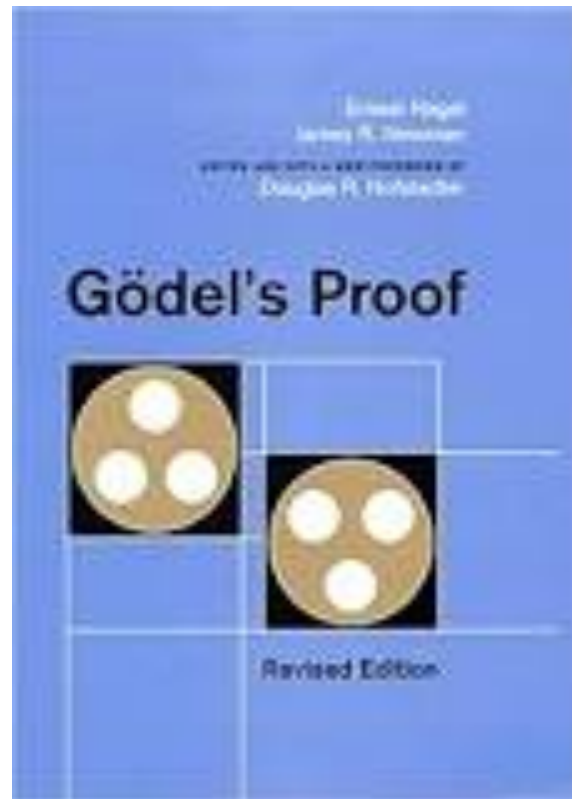
By far the greatest of GGT; Selm’s analysis based Sherlock Holmes’ mystery “Silver Blaze.”

Deficient; Beware

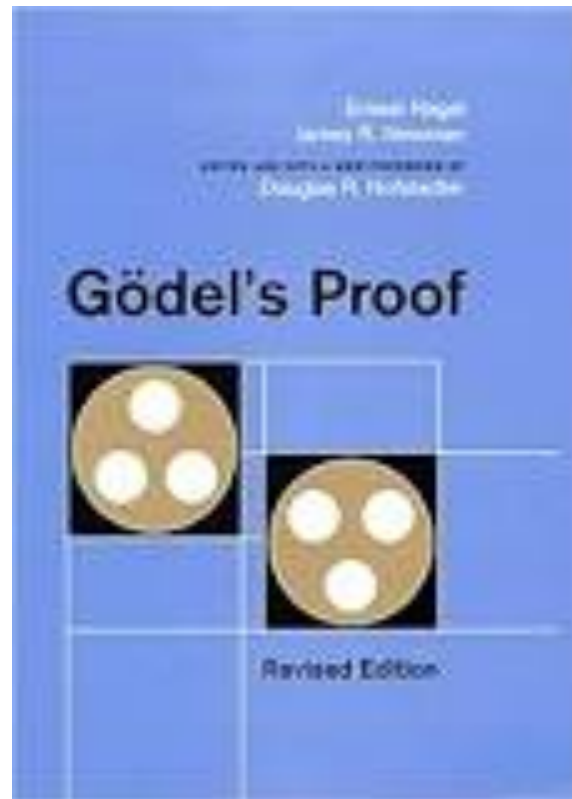
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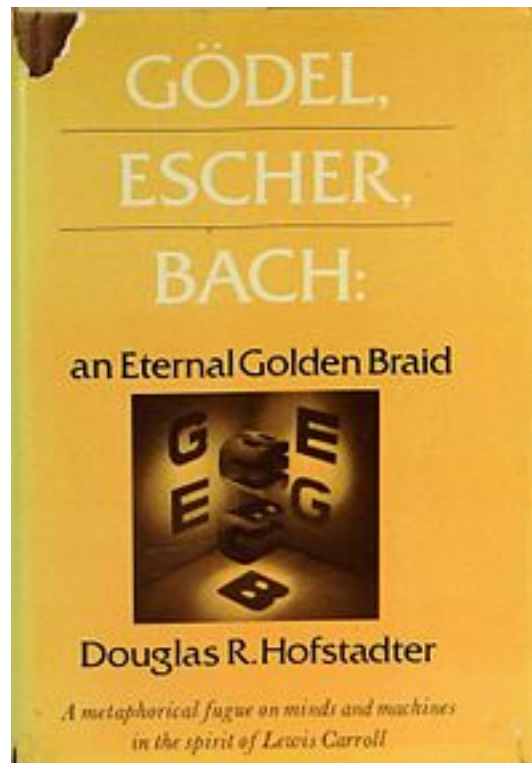
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Some Timeline Points

1978 Princeton NJ USA.



1940 Back to USA, for good.

1936 Schlick murdered; Austria annexed

1933 Hitler comes to power.

1930 Announces (First) *Incompleteness Theorem*

1929 Doctoral Dissertation: Proof of Completeness Theorem

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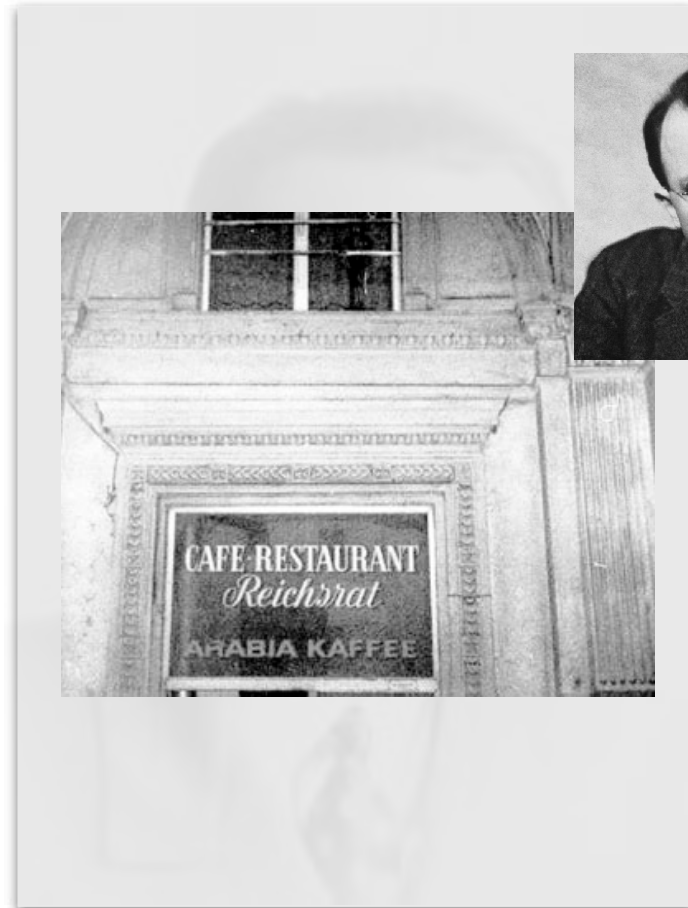
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“Well, uh, hmm, ...”



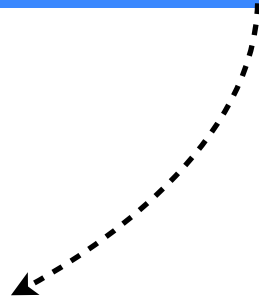
The “Liar Tree”

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The Liar Paradox

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The Liar Paradox

Pure Proof-Theoretic Route



The “Liar Tree”

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```
graph TD; A[The Liar Paradox] -.-> B(Pure Proof-Theoretic Route); A -.-> C[ ];
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Pure Proof-Theoretic Route

The “Liar Tree”

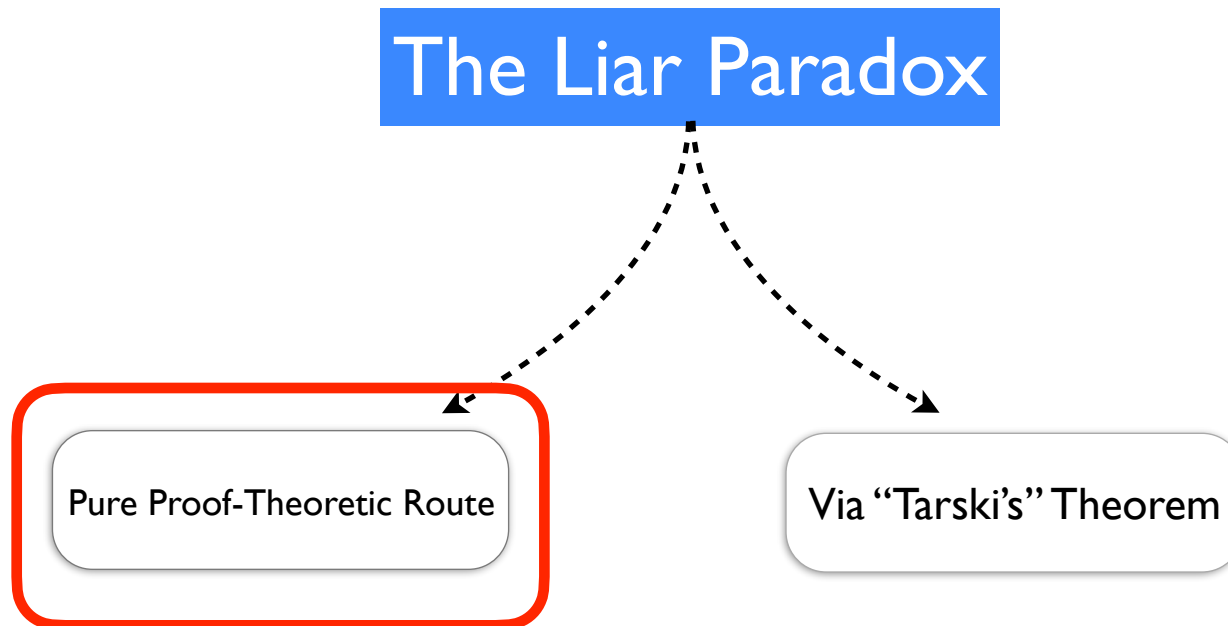
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graph TD; A[The Liar Paradox] -.-> B(Pure Proof-Theoretic Route); A -.-> C(Via “Tarski’s” Theorem)
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Pure Proof-Theoretic Route

Via “Tarski’s” Theorem

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The “Liar Tree”

The Liar Paradox

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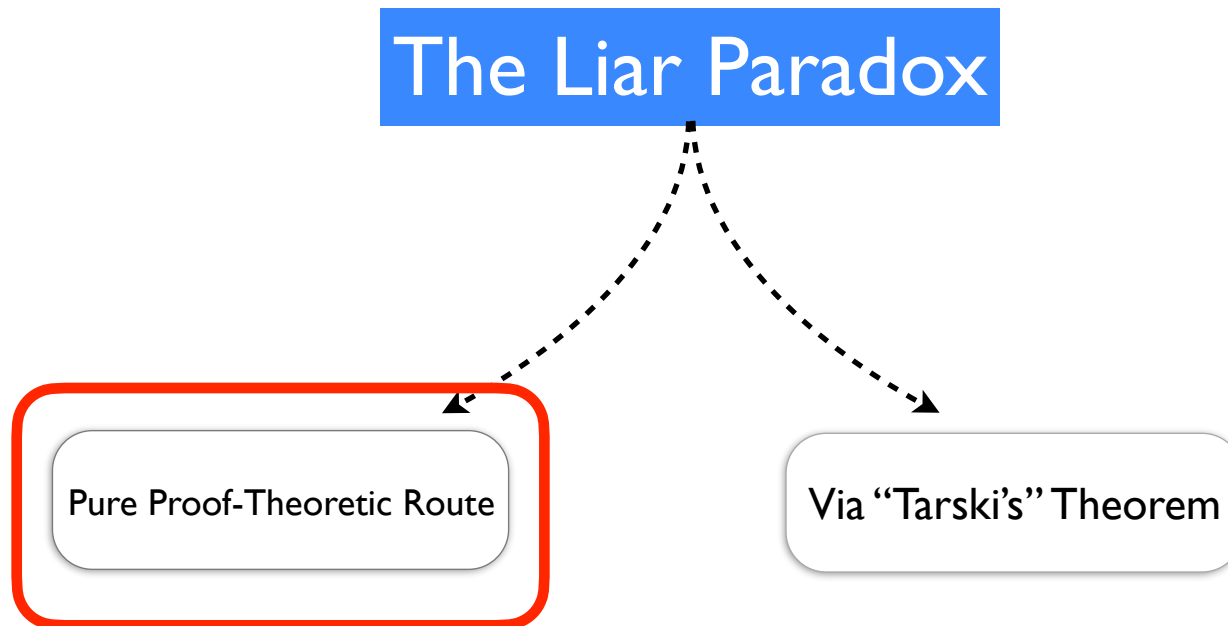


Paul Erdős



“The Book”

The “Liar Tree”



Ergo, step one: *What is LP?*



Paul Erdős



“The Book”

“The (Economical) Liar”

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Suppose that $T(\mathbf{L})$; then $\neg T(\mathbf{L})$.

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Hence: $T(\mathbf{L})$ iff (i.e., if & only if) $\neg T(\mathbf{L})$.

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Contradiction!

The “Gödelian” Liar (from me)

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\bar{P} : This sentence is unprovable.

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Suppose that \bar{P} is true. Then we can immediately deduce that \bar{P} is provable, because here is a proof: $\bar{P} \rightarrow \bar{P}$ is an easy theorem, and from it and our supposition we deduce \bar{P} by *modus ponens*. But since what \bar{P} says is that it's unprovable, we have deduced that \bar{P} is false under our initial supposition.

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Suppose on the other hand that \bar{P} is false. Then we can immediately deduce that \bar{P} is unprovable: Suppose for *reductio* that \bar{P} is provable; then \bar{P} holds as a result of some proof, but what \bar{P} says is that it's unprovable; and so we have contradiction. But since what \bar{P} says is that it's unprovable, and we have just proved that under our supposition, we arrive at the conclusion that \bar{P} is true.

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All of this is fishy; but
Gödel transformed it into
utterly precise, impactful,
indisputable reasoning ...

PA (Peano Arithmetic):

$$\text{A1} \quad \forall x(0 \neq s(x))$$

$$\text{A2} \quad \forall x \forall y (s(x) = s(y) \rightarrow x = y)$$

$$\text{A3} \quad \forall x (x \neq 0 \rightarrow \exists y (x = s(y)))$$

$$\text{A4} \quad \forall x (x + 0 = x)$$

$$\text{A5} \quad \forall x \forall y (x + s(y) = s(x + y))$$

$$\text{A6} \quad \forall x (x \times 0 = 0)$$

$$\text{A7} \quad \forall x \forall y (x \times s(y) = (x \times y) + x)$$

And, every sentence that is the universal closure of an instance of

$$([\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x)))] \rightarrow \forall x \phi(x))$$

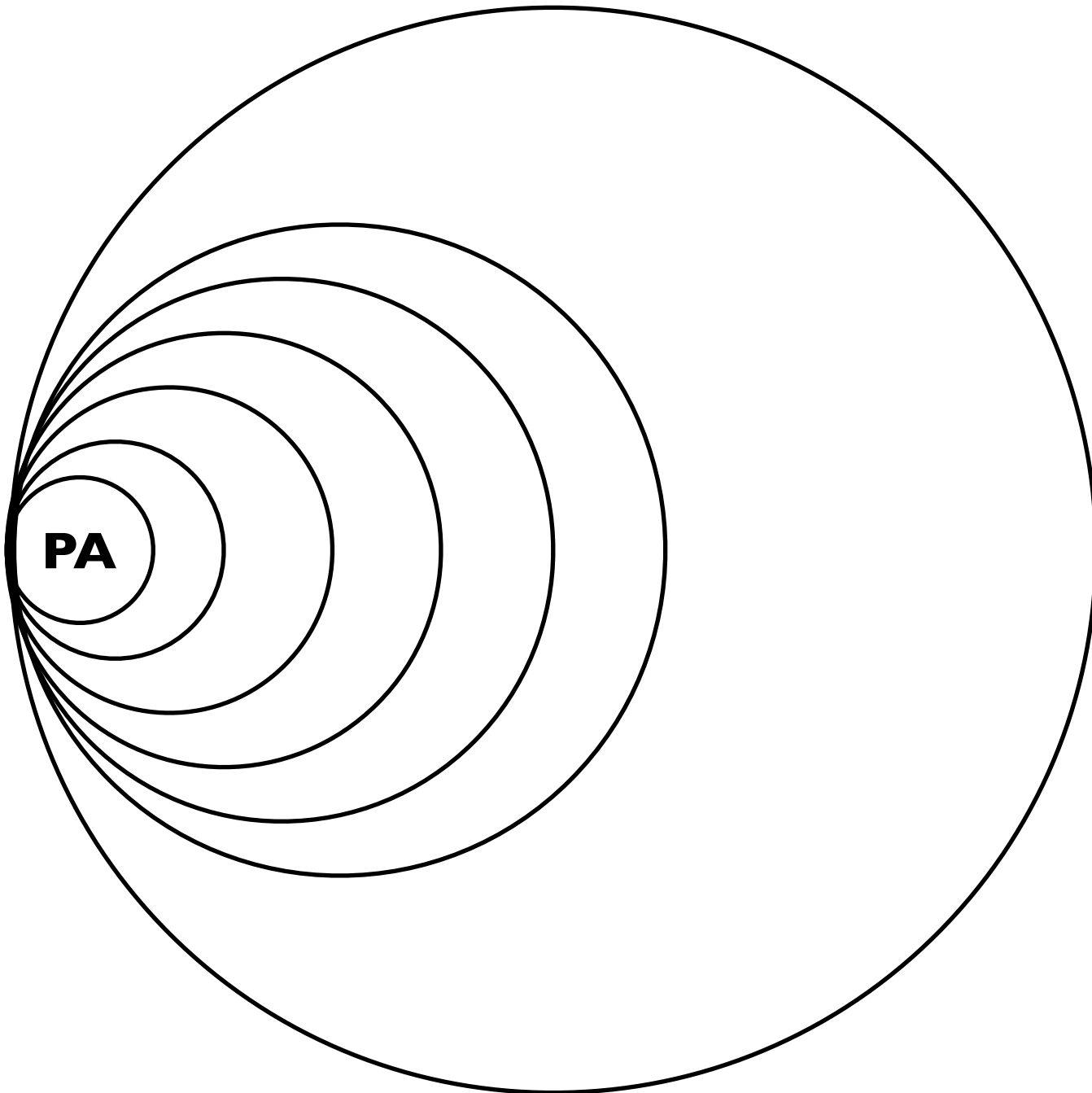
where $\phi(x)$ is open wff with variable x , and perhaps others, free.

Arithmetic Crucial Part of All Things Sci/Eng/Tech!

but alas, courtesy of Gödel: An infinite number of arithmetic propositions impossible to settle/decide.

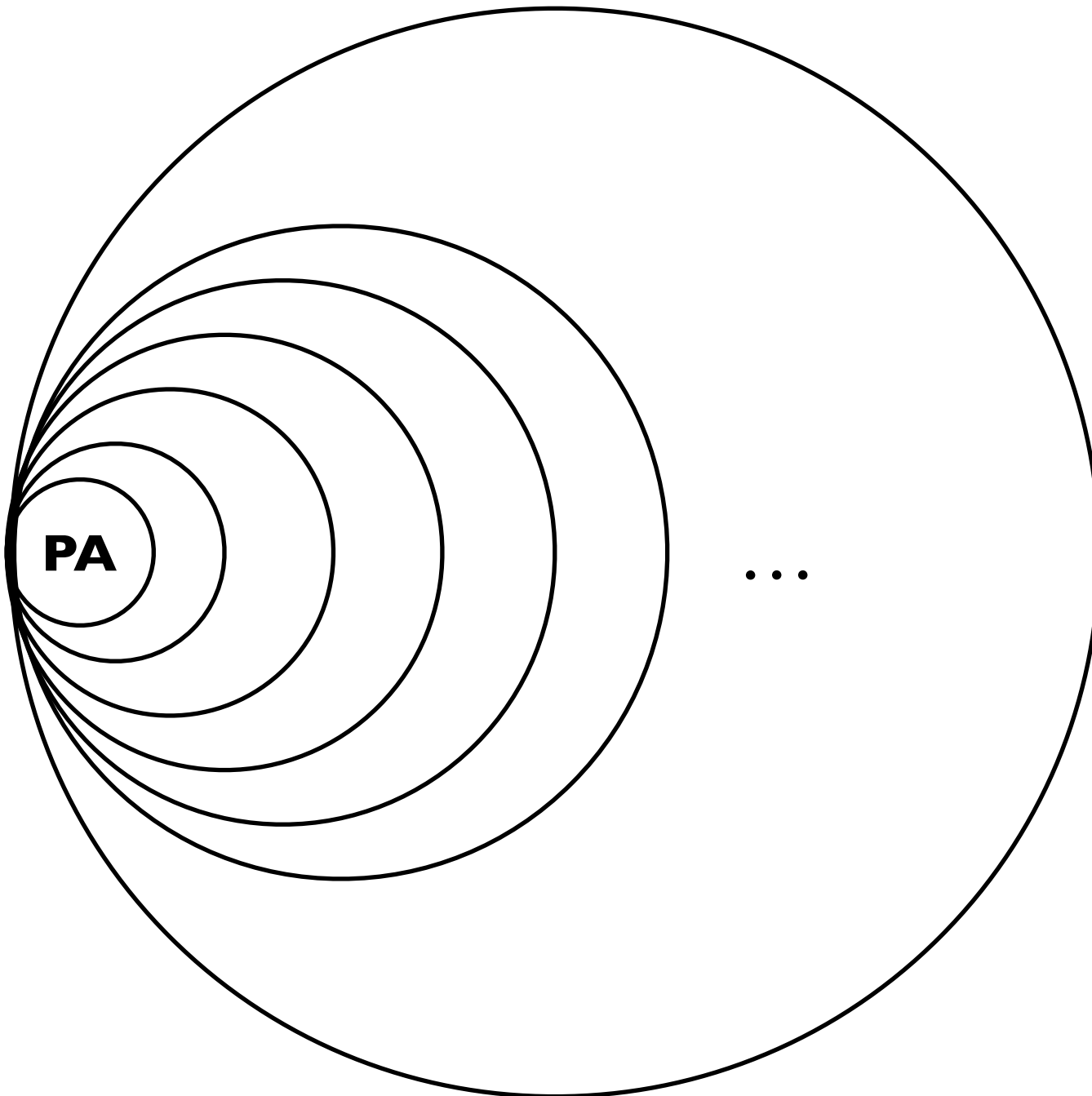
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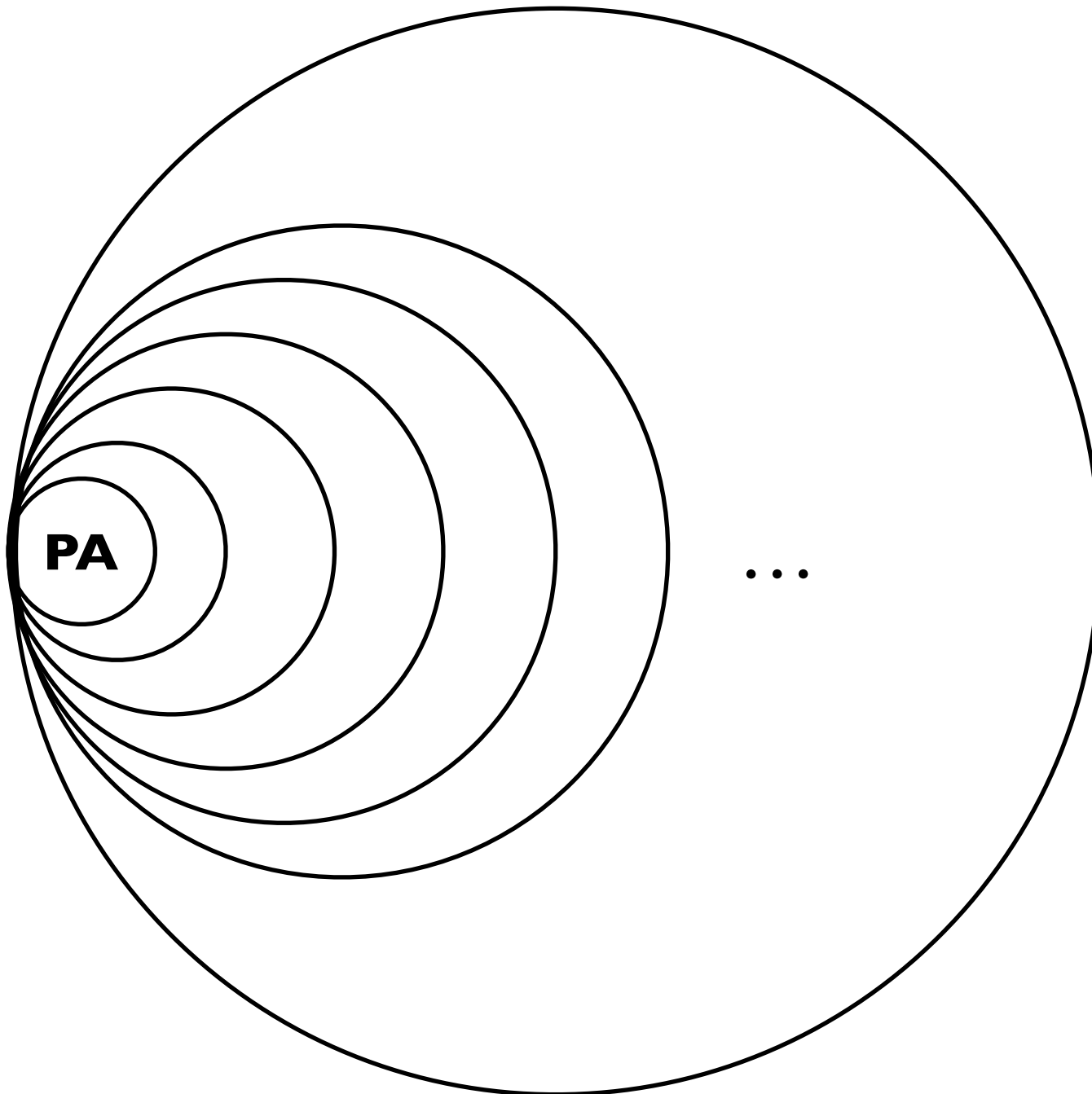
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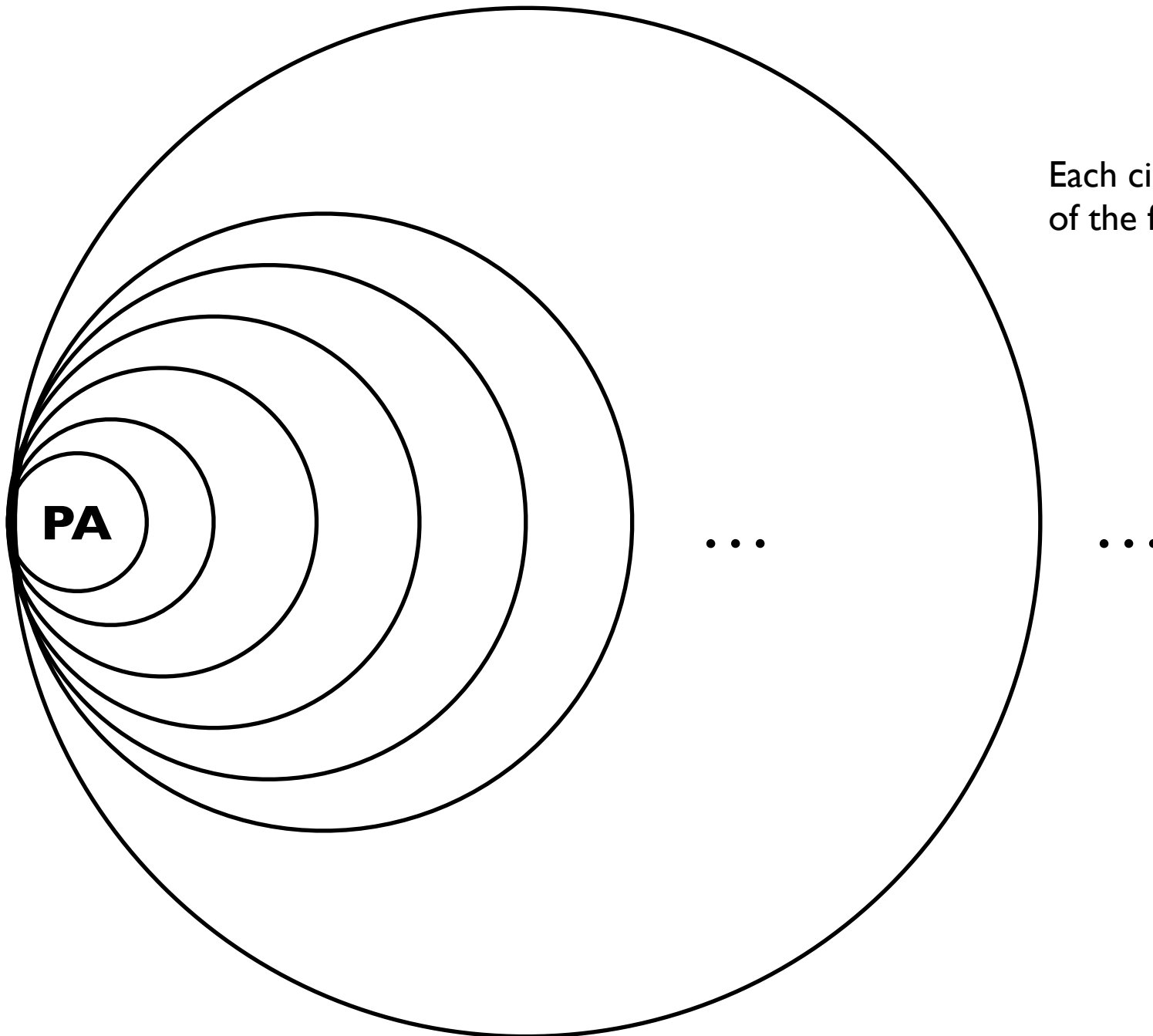
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of the formal sciences.

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Problem: How do we enable a formula to refer to other formulae and itself (and also other objects like proofs, terms etc.), in a perfectly consistent way?

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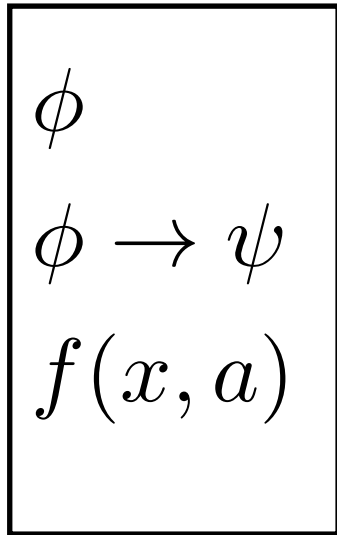
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in the language of \mathcal{L}_1

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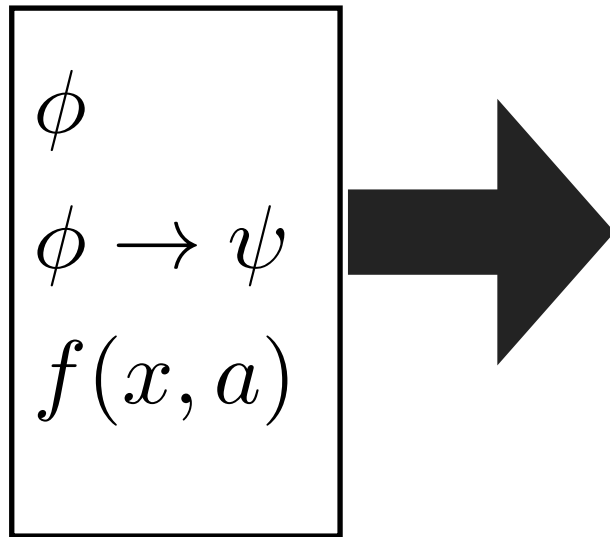


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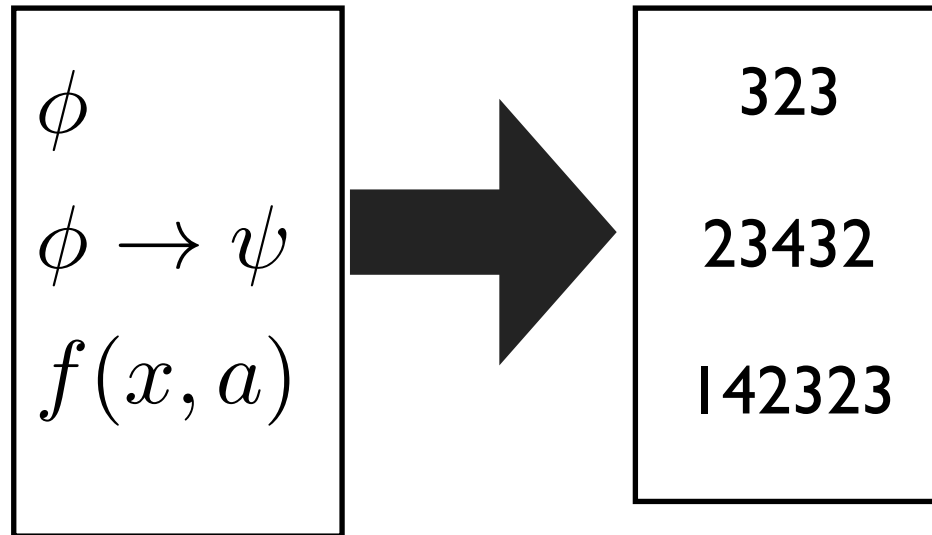


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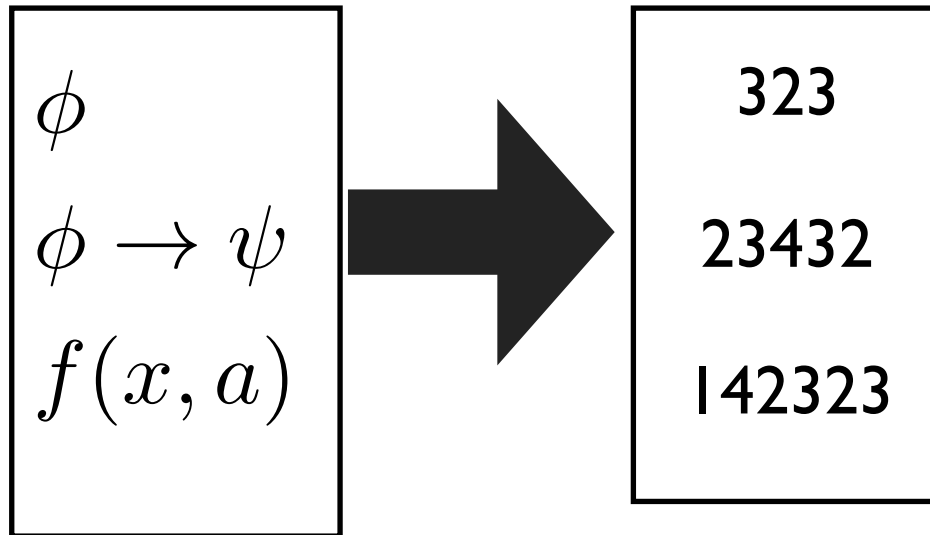


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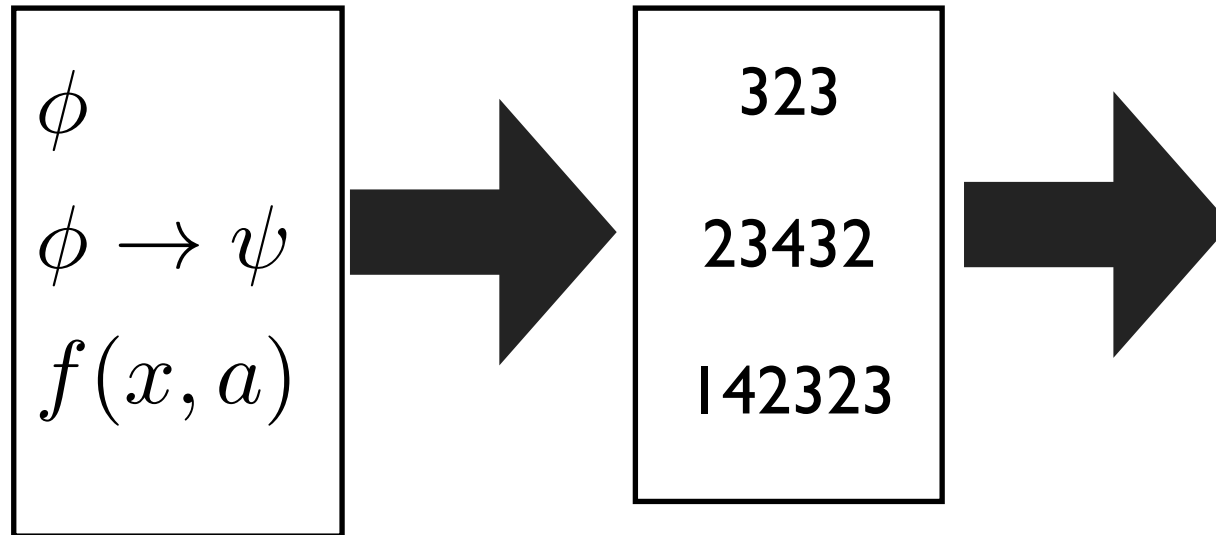
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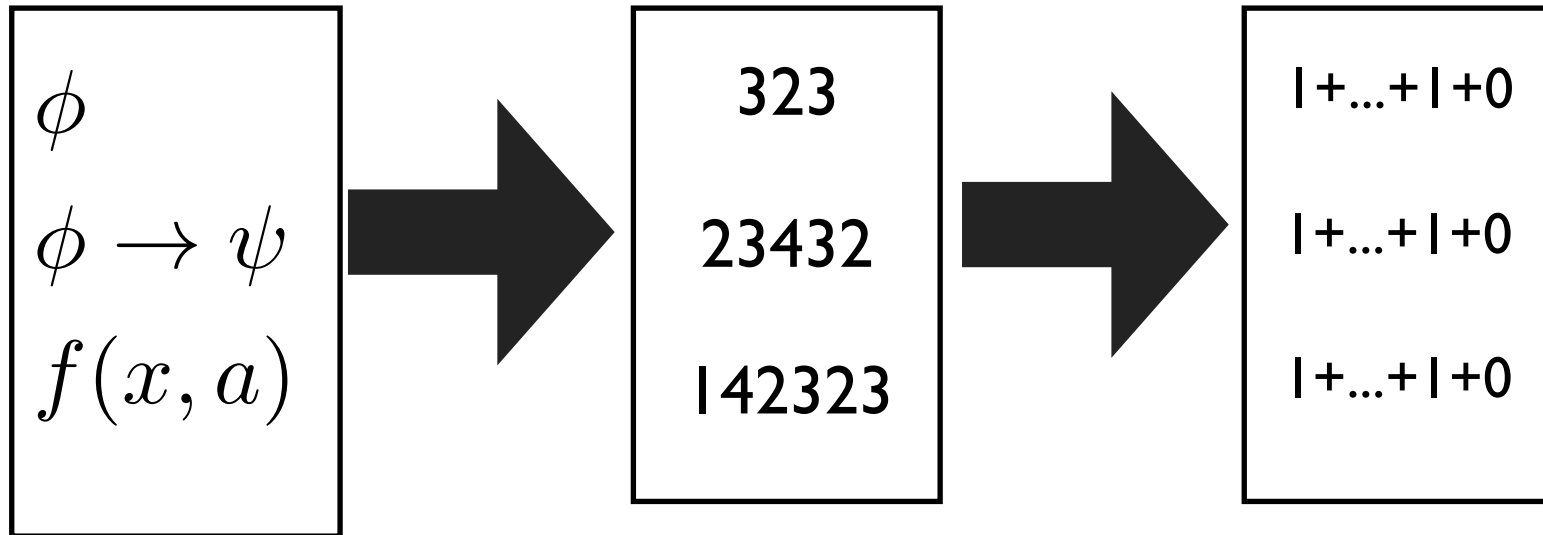
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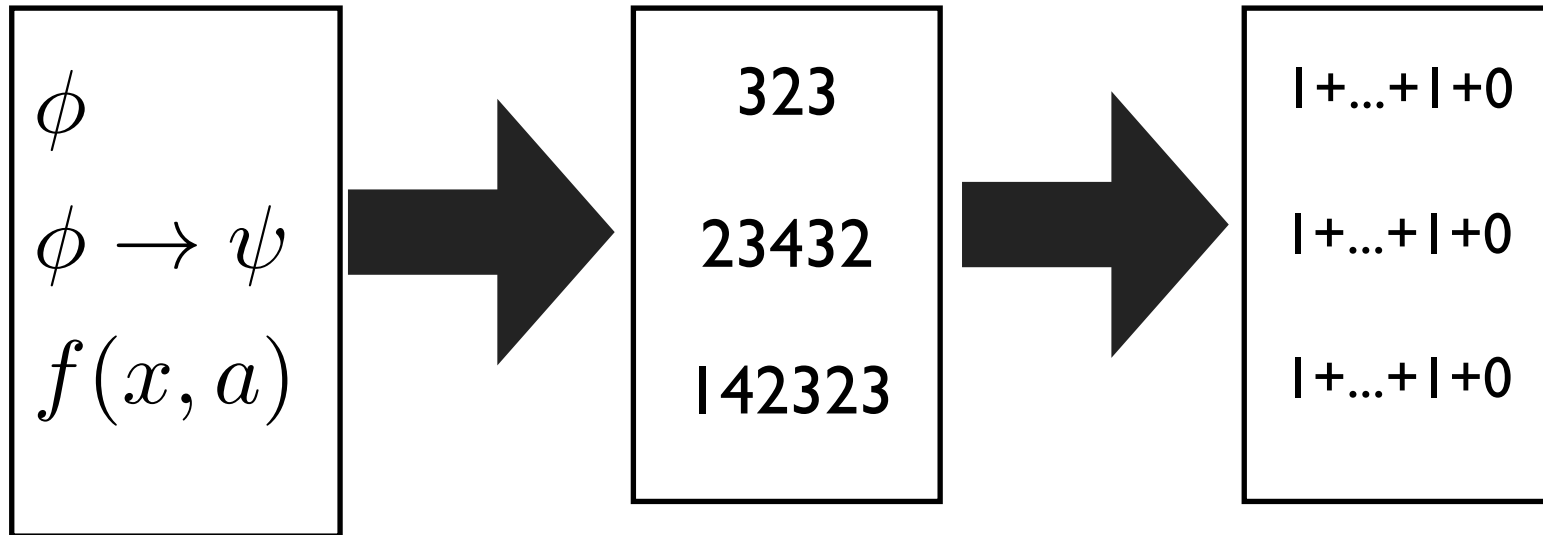
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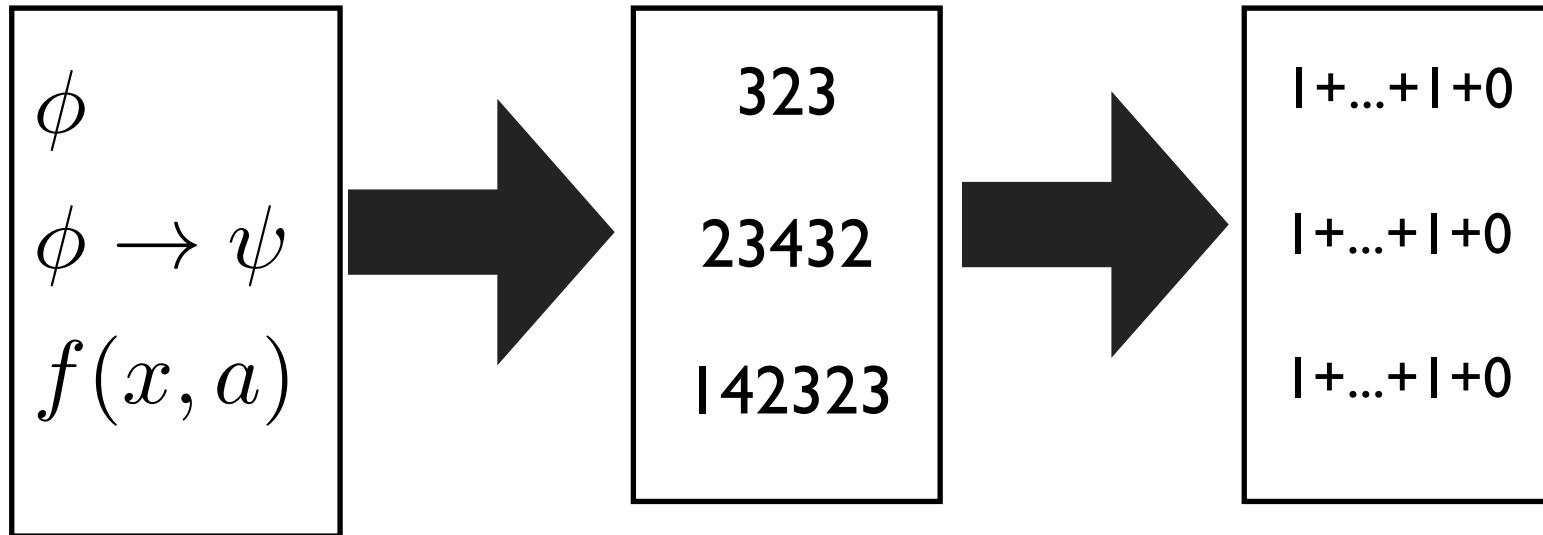
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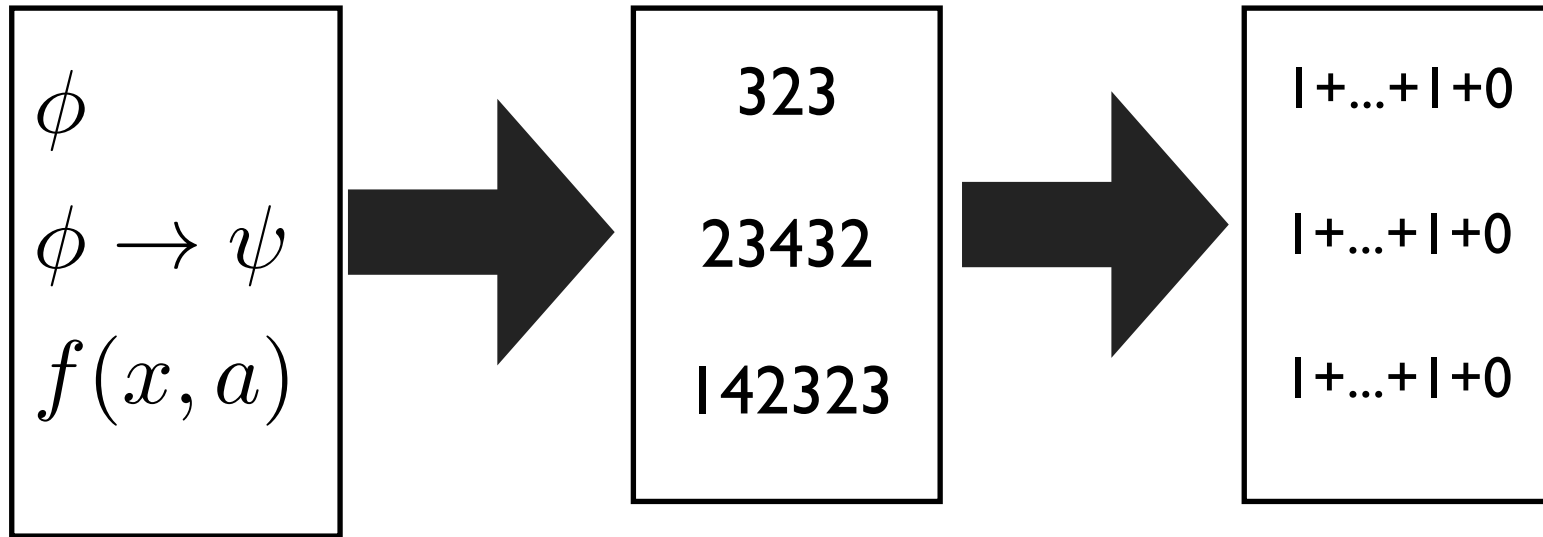
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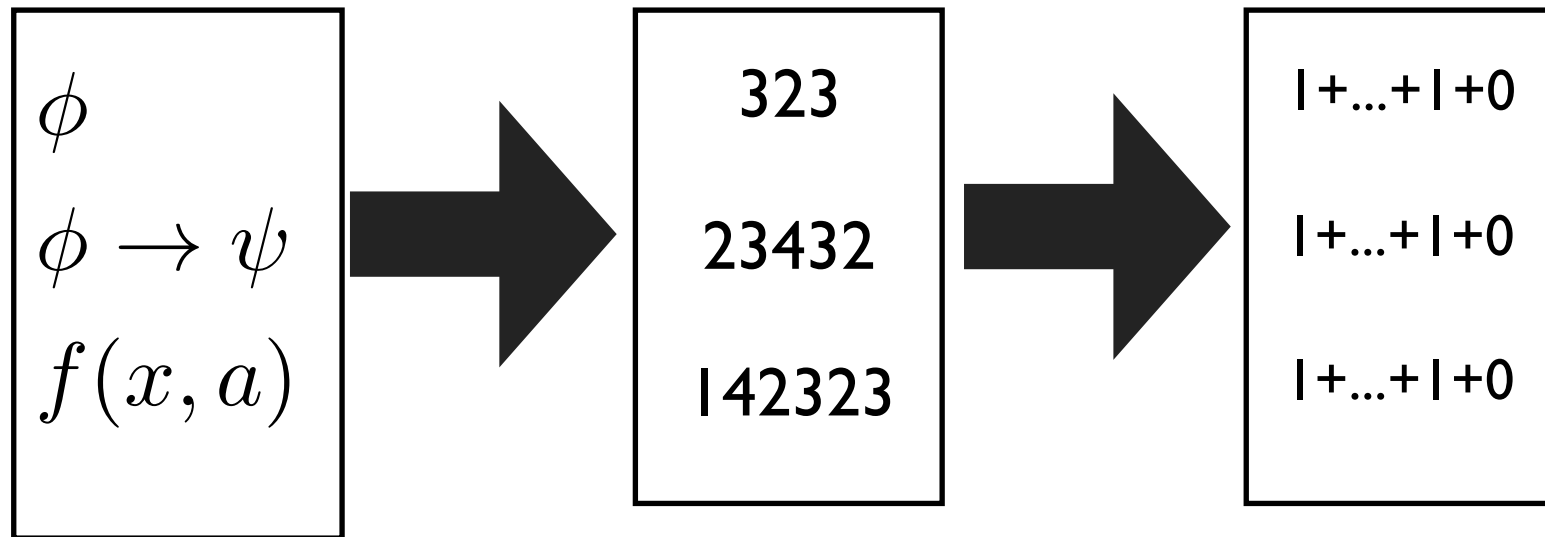
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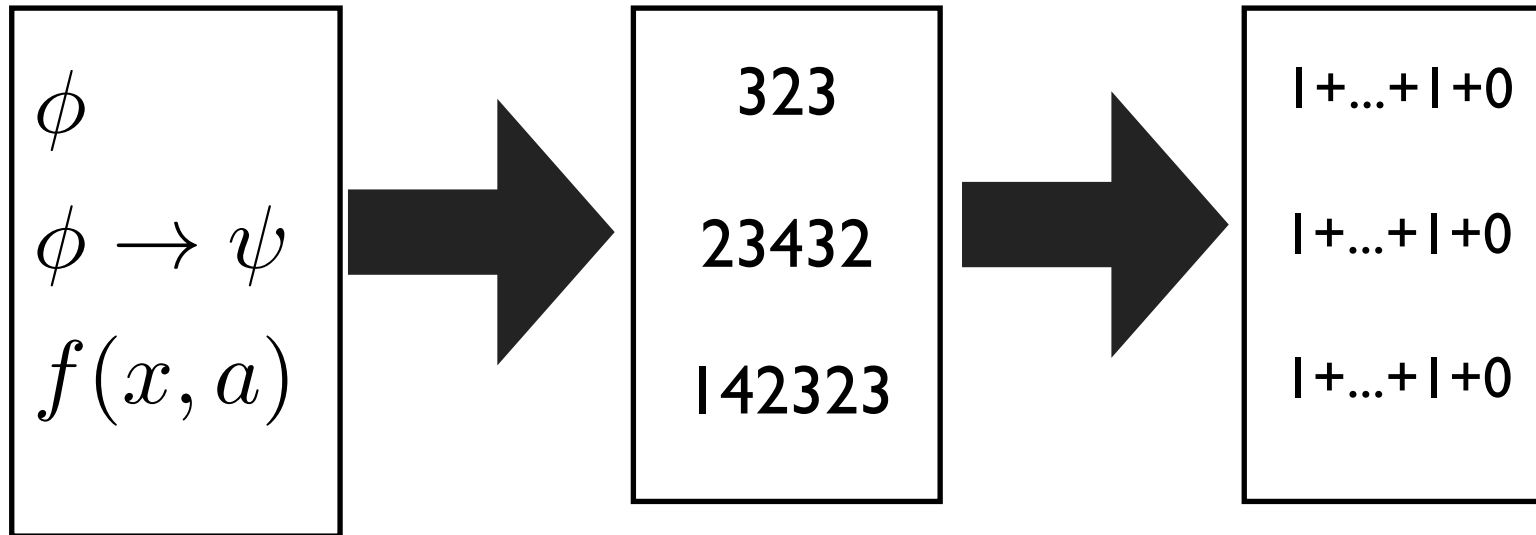
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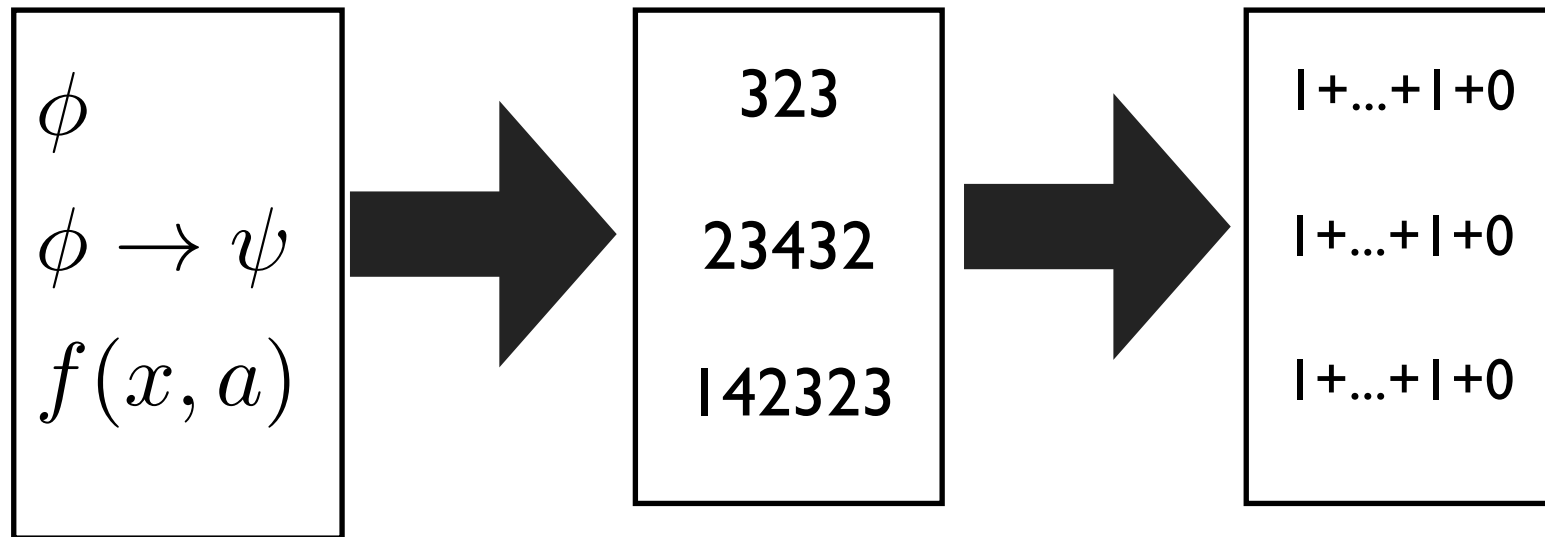
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S will often conflate.

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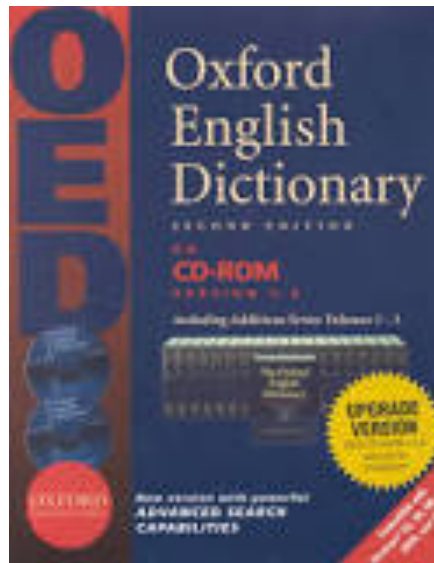
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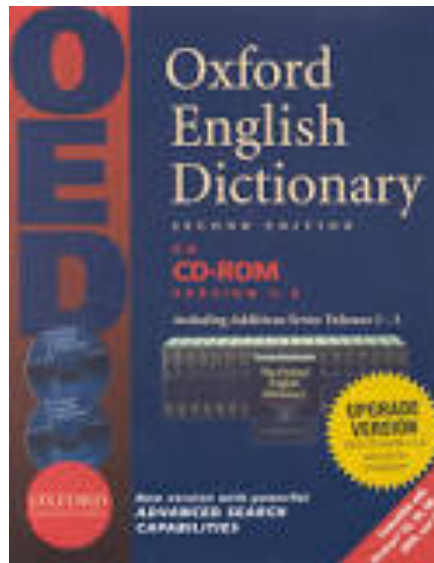
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So, gimcrack is named by some positive integer k . Hence, I can just refer to this word as “ k ” Or in the notation I prefer: k^{gimcrack} .

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Or, every syntactically valid computer program in Clojure that you will ever write can be uniquely denoted by some number m in the lexicographic ordering of all syntactically valid such programs. So your program π can just be coded as a numeral m^π in a formal language that captures arithmetic (i.e., an *arithmetic language*).

Gödel's First Incompleteness Theorem

Let Φ be a set of arithmetic sentences that is

- (i) consistent (i.e. no contradiction $\phi \wedge \neg\phi$ can be deduced from Φ);
- (ii) s.t. an algorithm is available to decide whether or not a given string u is a member of Φ ; and
- (iii) sufficiently expressive to logician all of the operations of a standard computing machine (e.g. a Turing machine, register machine, KU machine, etc.).

Then there is an “undecidable” arithmetic sentence \mathcal{G} from Gödel that can't be proved from Φ , nor can the negation of this sentence (i.e. $\neg\mathcal{G}$) be proved from Φ !

Alas, that's painfully verbose.

Gödel's First Incompleteness Theorem

**Sometimes said to be a: "Gödel sentence," or an "undecidable" sentence, or a "mysterious" sentence.

Gödel's First Incompleteness Theorem

Suppose $\Phi \supset \mathbf{PA}$ (= Φ contains \mathbf{PA}) that is

- (i) Con Φ ;
- (ii) Turing-decidable, and
- (iii) sufficiently expressive to logicize all of the operations of a Turing machine (i.e. Repr Φ).

Then there is an arithmetic X^{**} sentence \mathcal{G} s.t.
 $\Phi \not\vdash \mathcal{G}$ and $\Phi \not\vdash \neg\mathcal{G}$.

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Remember Church's Theorem!

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To prove G I, we shall
allow ourselves ...

The Fixed Point Theorem (FPT)

“The Self-Ascription Template Theorem”

Assume that Φ is a set of arithmetic sentences such that $\text{Repr } \Phi$. Then for every arithmetic formula $\psi(x)$ with one free variable x , there is an arithmetic sentence ϕ s.t.

$$\Phi \vdash \phi \leftrightarrow \psi(\hat{n}^\phi).$$

We can intuitively understand ϕ to be saying:

“I have the property ascribed to me by the formula ψ .”

“I thought there was no free lunch!”

[W]e “would hope that such a deep theorem would have an insightful proof. No such luck. I am going to write down a sentence ... and verify that it works. What I won't do is give you a satisfactory explanation for why I write down the particular formula that I do. I write down the formula because Gödel wrote down the formula, and Gödel wrote down the formula because, when he played the logic game he was able to see seven or eight moves ahead, whereas you and I are only able to see one or two moves ahead. I don't know anyone who thinks he has a fully satisfying understanding of why the Self-referential Lemma [= FPT] works. It has a rabbit-out-of-a-hat quality for everyone.”

—V. McGee, 2002; as quoted in (Salehi 2020)

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Ok; so let's do it ...

Proof: Let Φ be a set of arithmetic sentences, and suppose (for conditional intro) the antecedent of **GI** holds, i.e. (i)–(iii) hold. We must show that neither \mathcal{G} , nor the negation of this (Liar-Paradox-inspired) arithmetic sentence, can be proved from Φ . We know, respectively, that for any theorem ϕ of Φ , and from an instantiation of FPT, that:

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Here, ϕ is of course a variable in (1) for any formula; and \mathcal{P} is a logicization of Provable. Now suppose (for *reductio*) $\Phi \vdash \mathcal{G}$. By right-to-left on (1) we deduce $\text{Provable}(n^\mathcal{G})$. We can logicize this as $\neg \neg \mathcal{P}(\hat{n}^\mathcal{G})$. Then we have $\Phi \not\vdash \mathcal{G}$, since otherwise a contradiction could be deduced. Contradiction! — and our supposition for *reductio* falls.

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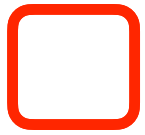
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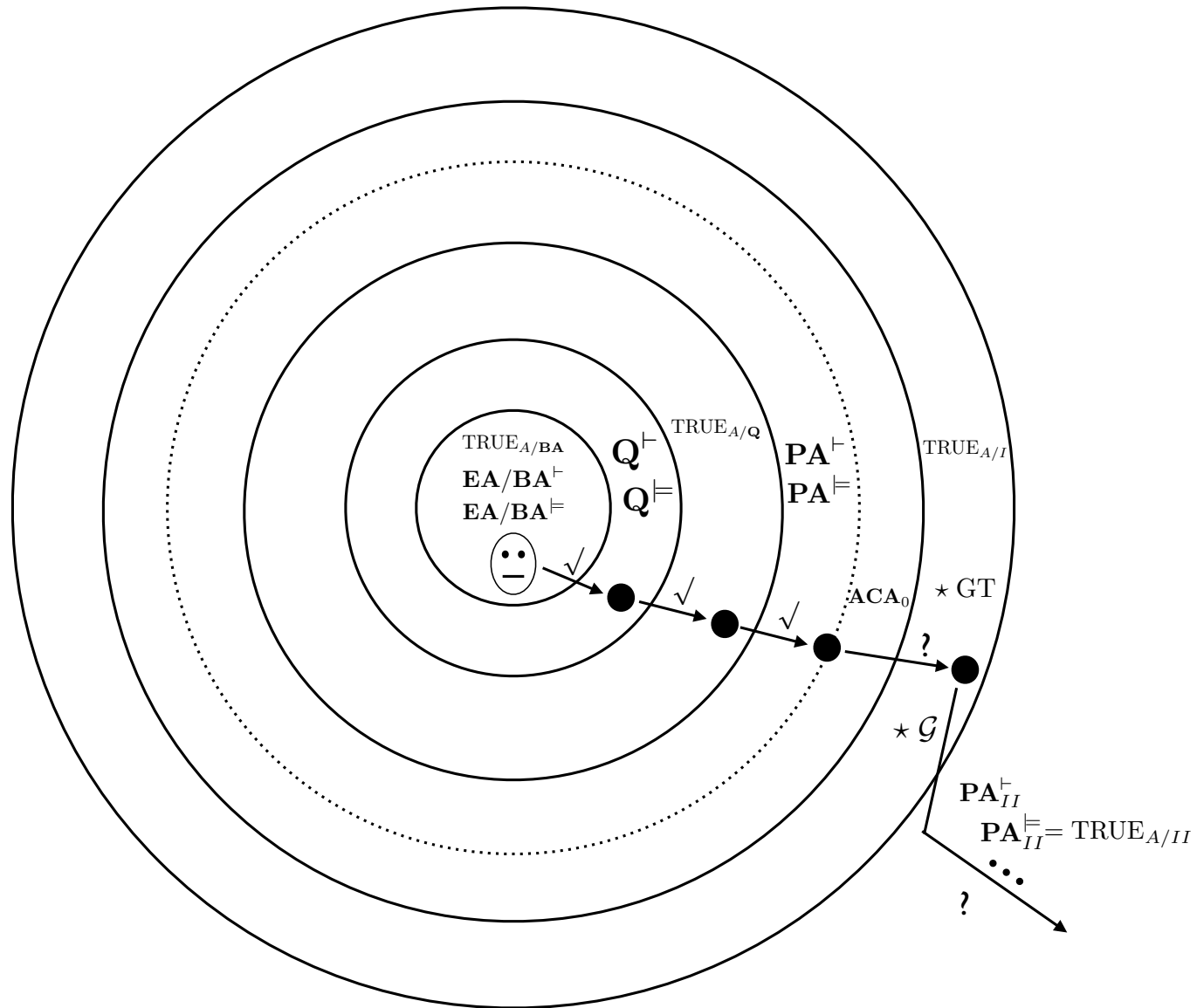
Suppose on the other hand $\Phi \vdash \neg \mathcal{G}$. And, suppose for *reductio* that $\neg \text{Provable}(n^\mathcal{G})$. We can logicize this as $\neg \mathcal{P}(\hat{n}^\mathcal{G})$, and then we can use (2) to deduce $\Phi \vdash \mathcal{G}$. But this entails $\text{Inc } \Phi = \text{not-Con } \Phi$. Yet our original assumptions (it's (i), specifically) include $\text{Con } \Phi$, so: contradiction. Therefore (by negation elim) we have $\text{Provable}(n^\mathcal{G})$. But from this, left-to-right on (1), we have $\Phi \vdash \mathcal{G}$. But then we have that \mathcal{G} is both provable and not provable from Φ , which is a contradiction with (i) = $\text{Con } \Phi$! **QED**

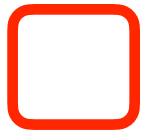
“Silly abstract nonsense! There aren’t any concrete examples of \mathcal{G} !”



Astrologic:

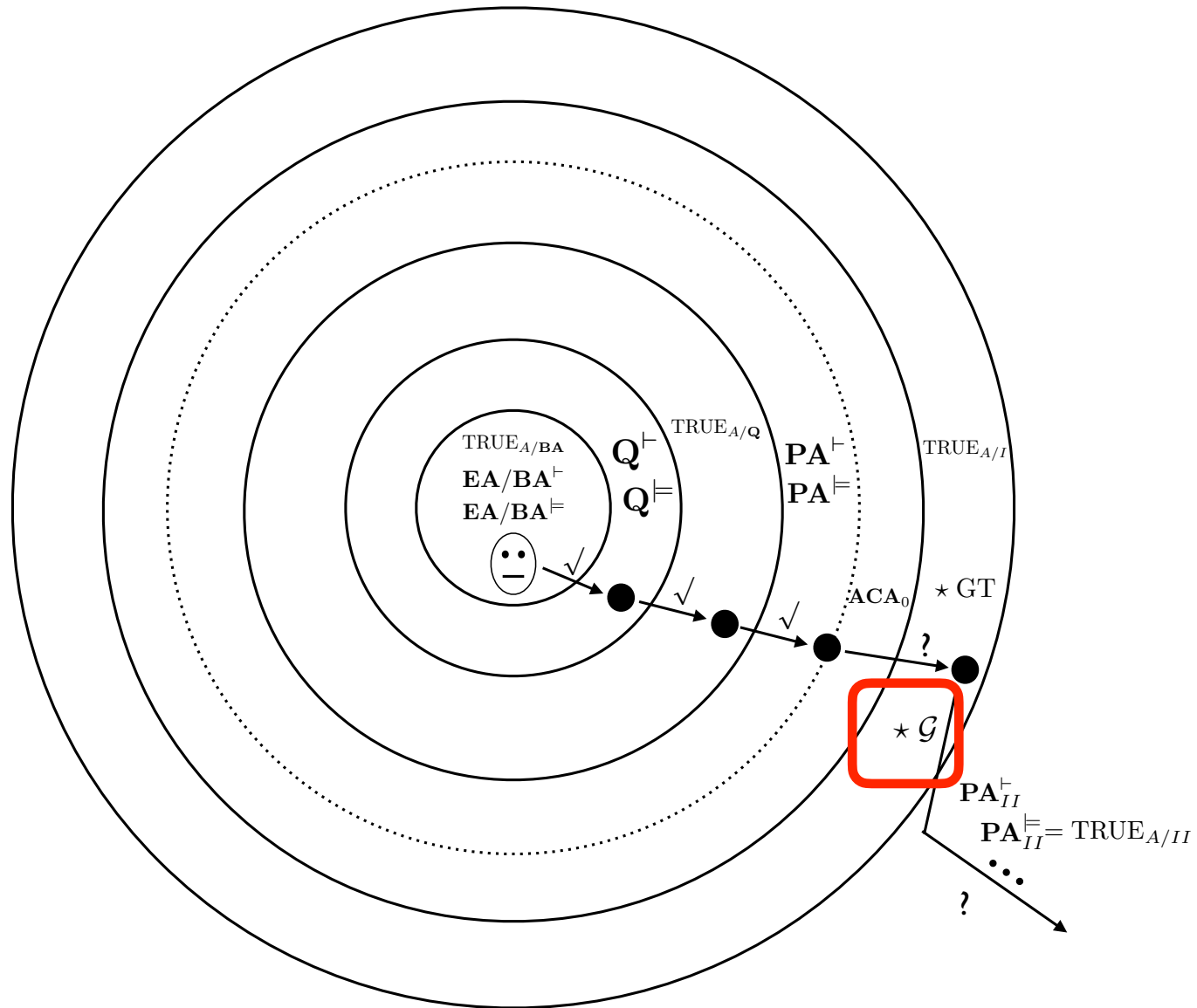
Rational Aliens Will be on the Same “Race Track”!





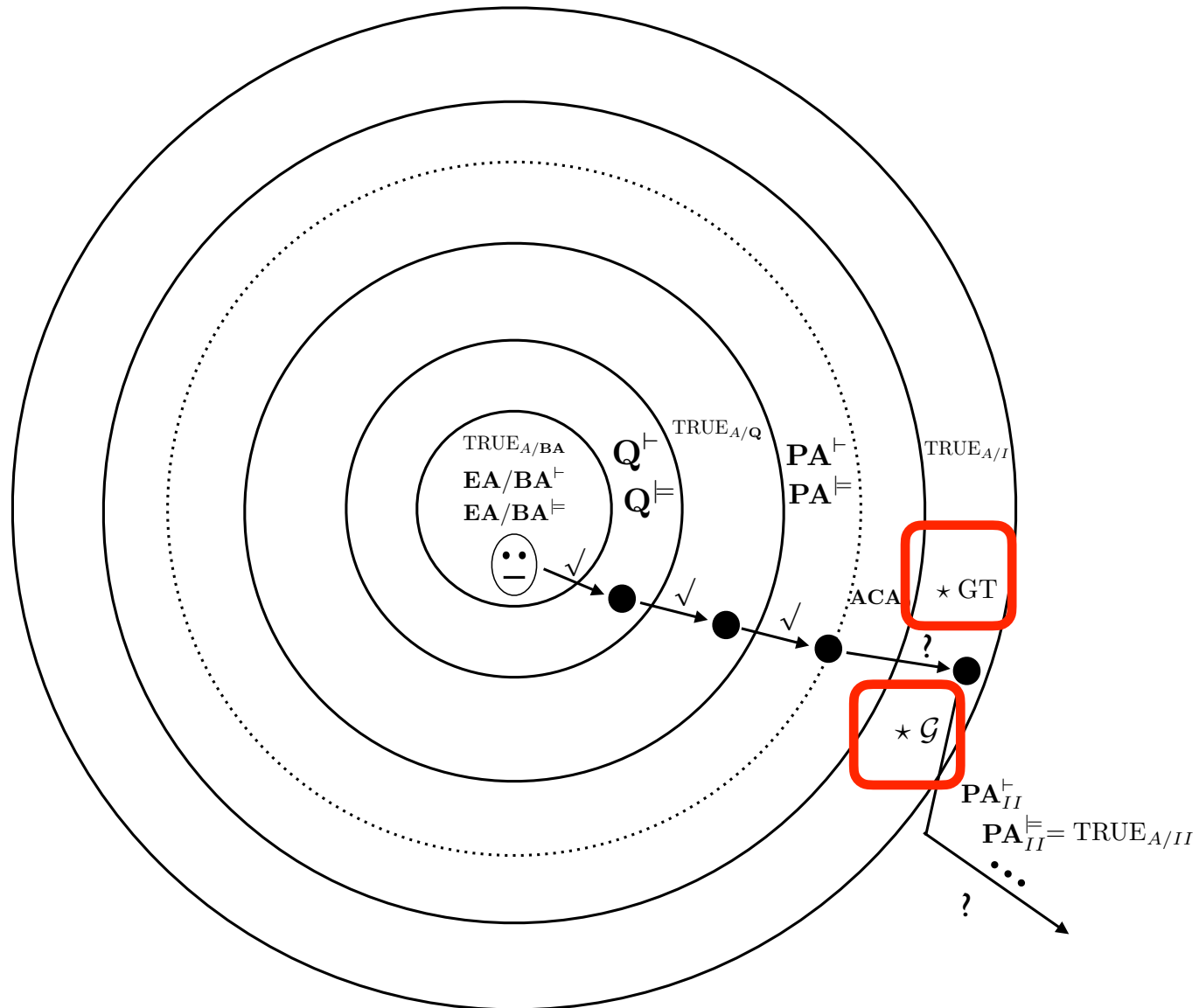
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The Goodstein Sequence goes to zero!

Pure base n representation of a number r

- Represent r as only sum of powers of n in which the exponents are also powers of n , etc.

$$266 = 2^{2^{(2^{2^0} + 2^0)}} + 2^{(2^{2^0} + 2^0)} + 2^{2^0}$$

Grow Function

$Grow_k(n)$:

1. Take the pure base k representation of n
2. Replace all k by $k + 1$. Compute the number obtained.
3. Subtract one from the number

Example of **Grow**

$Grow_2(19)$

$$19 = 2^{2^{2^{2^0}}} + 2^{2^0} + 2^0$$

$$3^{3^{3^{3^0}}} + 3^{3^0} + 3^0$$

$$3^{3^{3^{3^0}}} + 3^{3^0} + 3^0 - 1$$

7625597484990

Goodstein Sequence

- For any natural number m

m

$Grow_2(m)$

$Grow_3(Grow_2(m))$

$Grow_4(Grow_3(Grow_2(m))),$

...

Sample Values

Sample Values

m										
2	2	2	1	0						

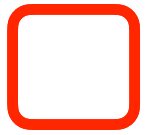
Sample Values

m										
2	2	2	1	0						
3	3	3	3	2	1	0				

Sample Values

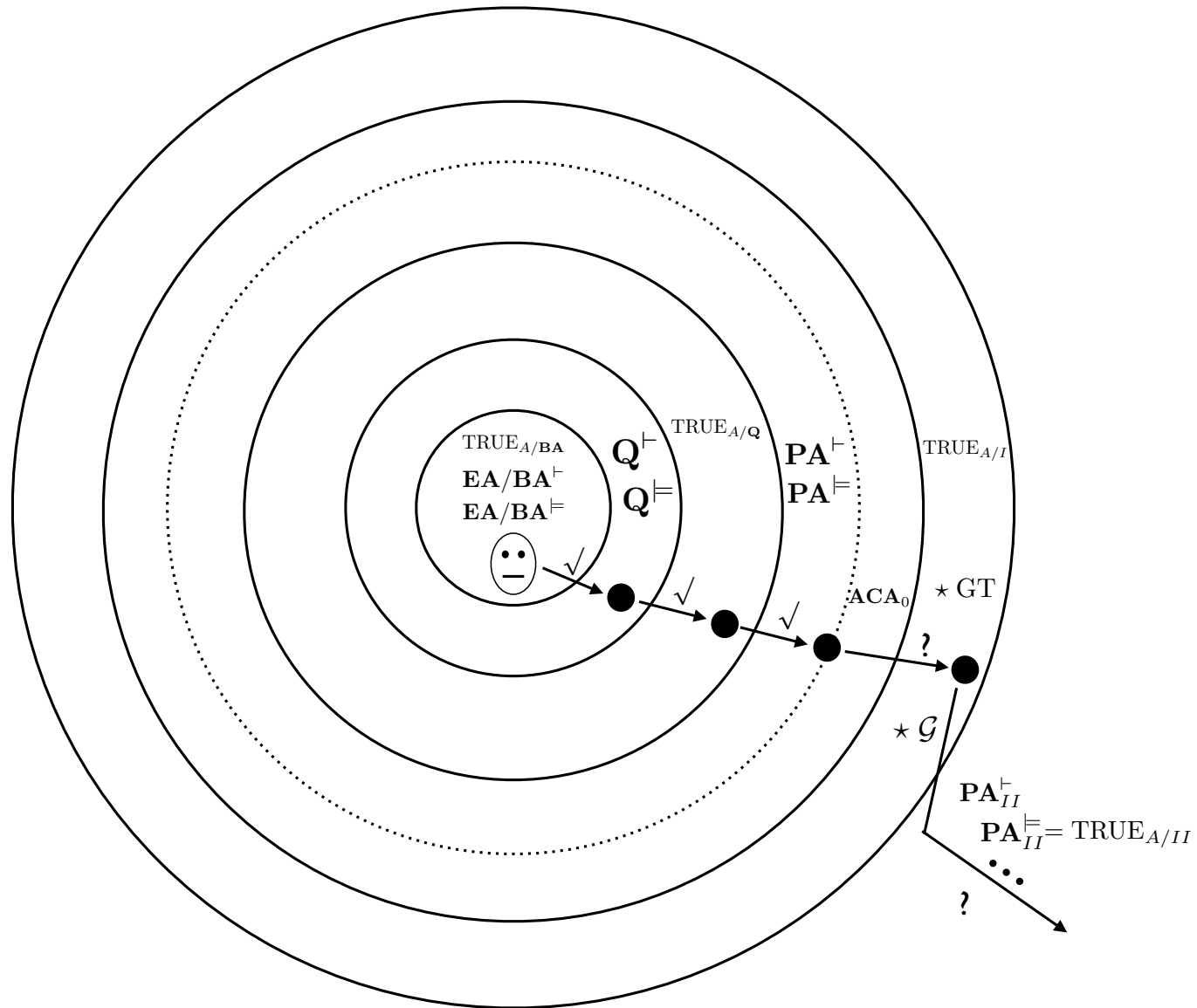
m										
2	2	2	1	0						
3	3	3	3	2	1	0				
4	4	26	41	60	83	109	139	...	11327 (96th term)	...
5	15	$\sim 10^{13}$	$\sim 10^{155}$	$\sim 10^{2185}$	$\sim 10^{36306}$	10^{695975}	$10^{15151337}$...		

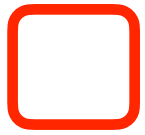
This sequence actually goes to zero!



Astrologic:

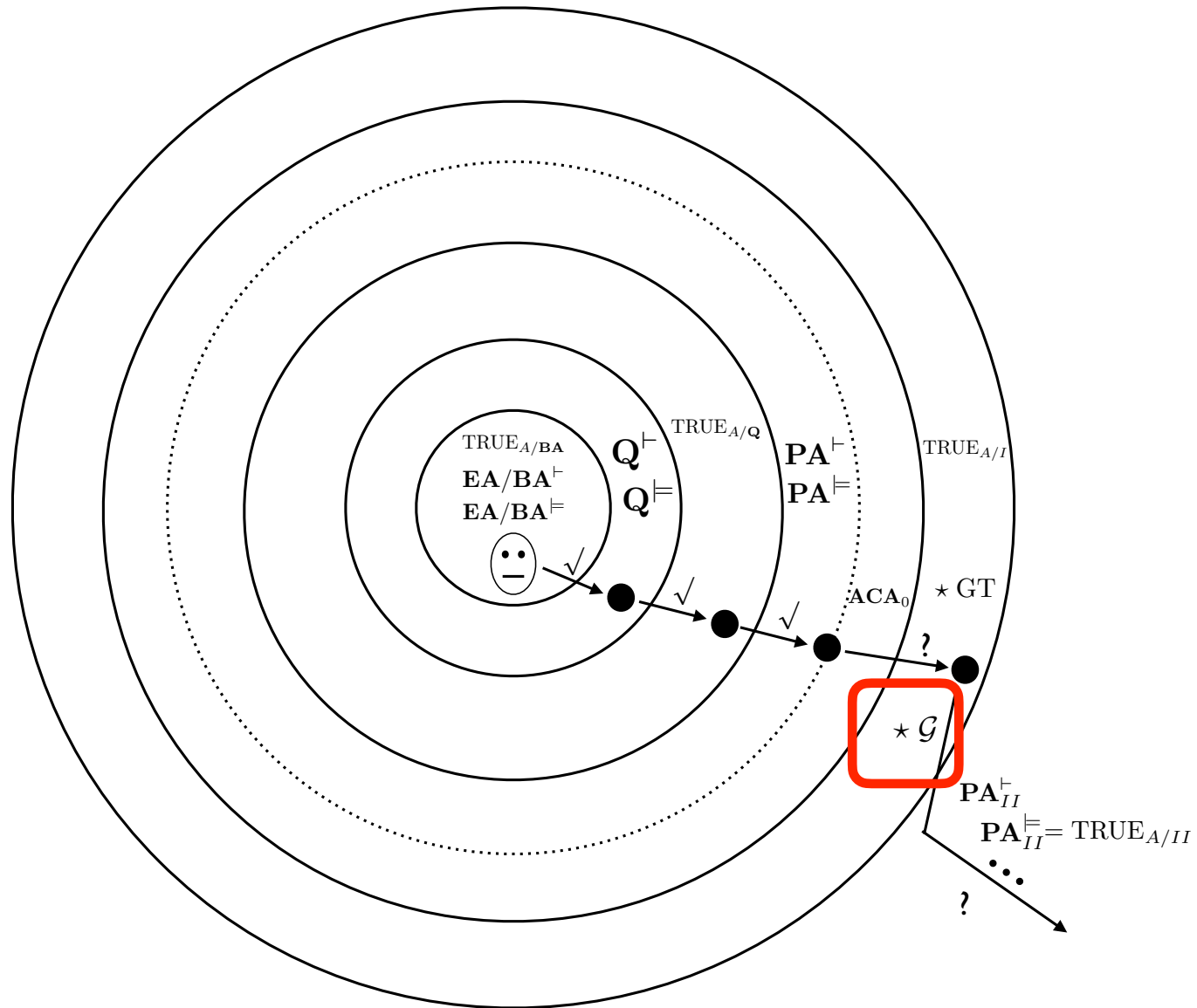
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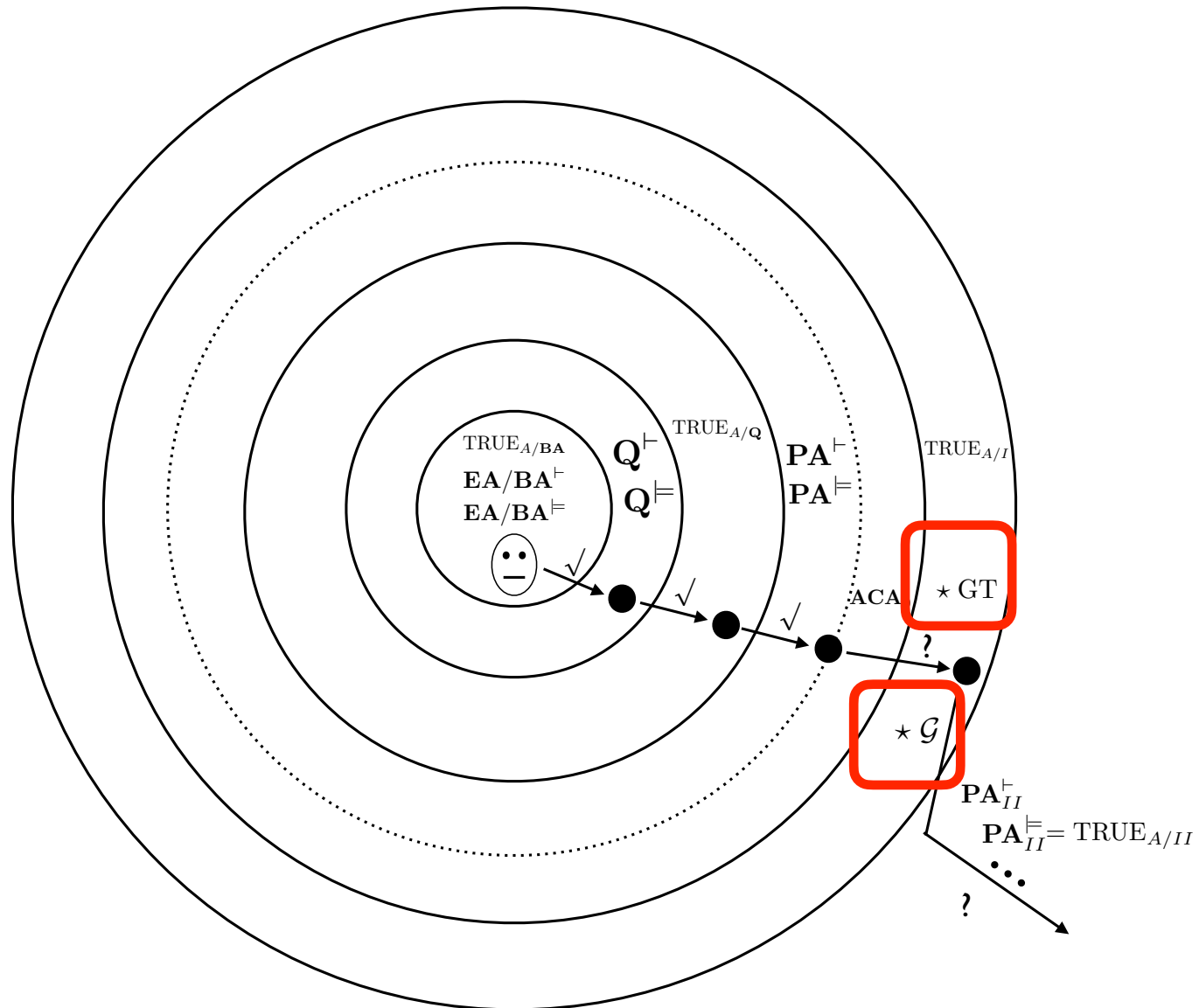
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Could an AI Ever Match Gödel's G1 & G2?

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*Med nok penger, kan
logikk løse alle problemer.*