Rebuilding the Foundations of Math via (the "Theory") <u>ZFC</u>; ZFC to Axiomatized Arithmetic (the "Theory" <u>PA</u>)

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Rensselaer AI & Reasoning (RAIR)

Intro to Logic
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R A I R
Rensselaer AI and Reasoning Lab

Reviewing the situation

…

Types of Paradoxes

- *Deductive Paradoxes* - paradoxes arrived at via deducing a contradiction from a set of assumptions. (Russell's Paradox)

- *Inductive Paradoxes* - coming (e.g. The Lottery Paradox, The Raven Paradox, The St Petersburg Paradox)

Friday's Hill, Haslemere, 16 June 1902

Dear colleague,

For a year and a half I have been acquainted with your *Grundgesetze der Arithmetik*, but it is only now that I have been able to find the time for the thorough study I intended to make of your work. I find myself in complete agreement with you in all essentials, particularly when you reject any psychological element [[Moment]] in logic and when you place a high value upon an ideography [[Begriffsschrift]] for the foundations of mathematics and of formal logic, which, incidentally, can hardly be distinguished. With regard to many particular questions, I find in your work discussions, distinctions, and definitions that one seeks in vain in the works of other logicians. Especially so far as function is concerned (§ 9 of your *Begriffsschrift*), I have been led on my own to views that are the same even in the details. There is just one point where I have encountered a difficulty. You state (p. 17 [[p. 23 above]]) that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer its opposite follows. Therefore we must conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [[Menge]] does not form a totality.

I am on the point of finishing a book on the principles of mathematics and in it I should like to discuss your work very thoroughly.[1] I already have your books or shall buy them soon, but I would be very grateful to you if you could send me reprints of your articles in various periodicals. In case this should be impossible, however, I will obtain them from a library.

The exact treatment of logic in fundamental questions, where symbols fail, has remained very much behind; in your works I find the best I know of our time, and therefore I have permitted myself to express my deep respect to you. It is very regrettable that you have not come to publish the second volume of your *Grundgesetze*; I hope that this will still be done.

Very respectfully yours,

BERTRAND RUSSELL

The above contradiction, when expressed in Peano's ideography, reads as follows:

$$w = \mathrm{cls} \cap x \ni (x \sim\varepsilon\, x) .\supset: w \,\varepsilon\, w .=. w \sim\varepsilon\, w.$$

$$\text{Axiom V} \quad \exists x \forall y [y \in x \leftrightarrow \phi(y)]$$

For formulae *Φ* with a free variable *y*, there exists a set *x* such that iff *y* is a member of *x* then formula *Φ* holds.

Russell's Theorem

$$\vdash \neg \exists x \forall y (y \in x \leftrightarrow y \notin y)$$

NO, if we take *Φ* to be the formula y ∉ y (y is not a member of itself), we are able to prove a contradiction!

You will have the honor of proving this contradiction in Hyperslate as homework…

HyperGrader™ Problems HyperSlate My Submissions Leader Board Selmer.Bringsjord@gmail.com

FregTHEN2
KnightKnave_SmullyanKKProblem1.1
AthenCfromAthenBandBthenC
BiconditionalIntroByChaining
BogusBiconditional
CheatersNeverPropser
Contrapositive_NYS_2
Disj_Syll
GreenCheeseMoon2
HypSyll
LarryIsSomehowSmart
Modus_Tollens
RussellsLetter2Frege
ThxForThePCOracle
Explosion
OnlyMediumOrLargeLlamas
GreenCheeseMoon1
Disj_Elim
kok13_28
KingAce2
kok_13_31

☑ RussellsLetter2Frege

The challenge here is to prove that from Russell's instantiation of Frege's doomed Axiom V a contradiction can be promptly derived. The letter has of course been examined in some detail by S Bringsjord (in the Mar 16 2020 lecture in the 2020 lecture lineup); it, along with an astoundingly soft-spoken reply from Frege, can be found here. Put meta-logically, your task in the present problem is to build a proof that confirms this:

$$\{\exists x \forall y((y \in x) \rightarrow (y \notin y))\} \vdash \zeta \wedge \neg\zeta.$$

Make sure you understand that the given here is an instantiation of Frege's Axiom V; i.e. it's an instantiation of

$$\exists x \forall y((y \in x) \rightarrow \phi(y)).$$

(The notation $\phi(y)$, recall, is the standard way in mathematical logic to say that y is free in ϕ.) **Note**: Your finished proof is allowed to make use the PC-provability oracle (but *only* that oracle).

(Now a brief remark on matters covered by in class by Bringsjord when second-order logic = $\mathscr{L}_2$ arrives on the scene: Longer term, and certainly constituting evidence of Frege's capacity for ingenius, intricate deduction, it has recently been realized that while Frege himself relied on Axiom V to obtain what is known as **Hume's Principle** (= HP), this reliance is avoidable. That from just HP we can deduce all of Peano Arithmetic (**PA**) (!) is a result Frege can be credited with showing; the result is known today as Frege's Theorem (= FT). Following the link just given will reward the reader with an understanding of HP, and how how to obtain **PA** from it.)

Deadline 22 Apr 2020 23:59:00 EST

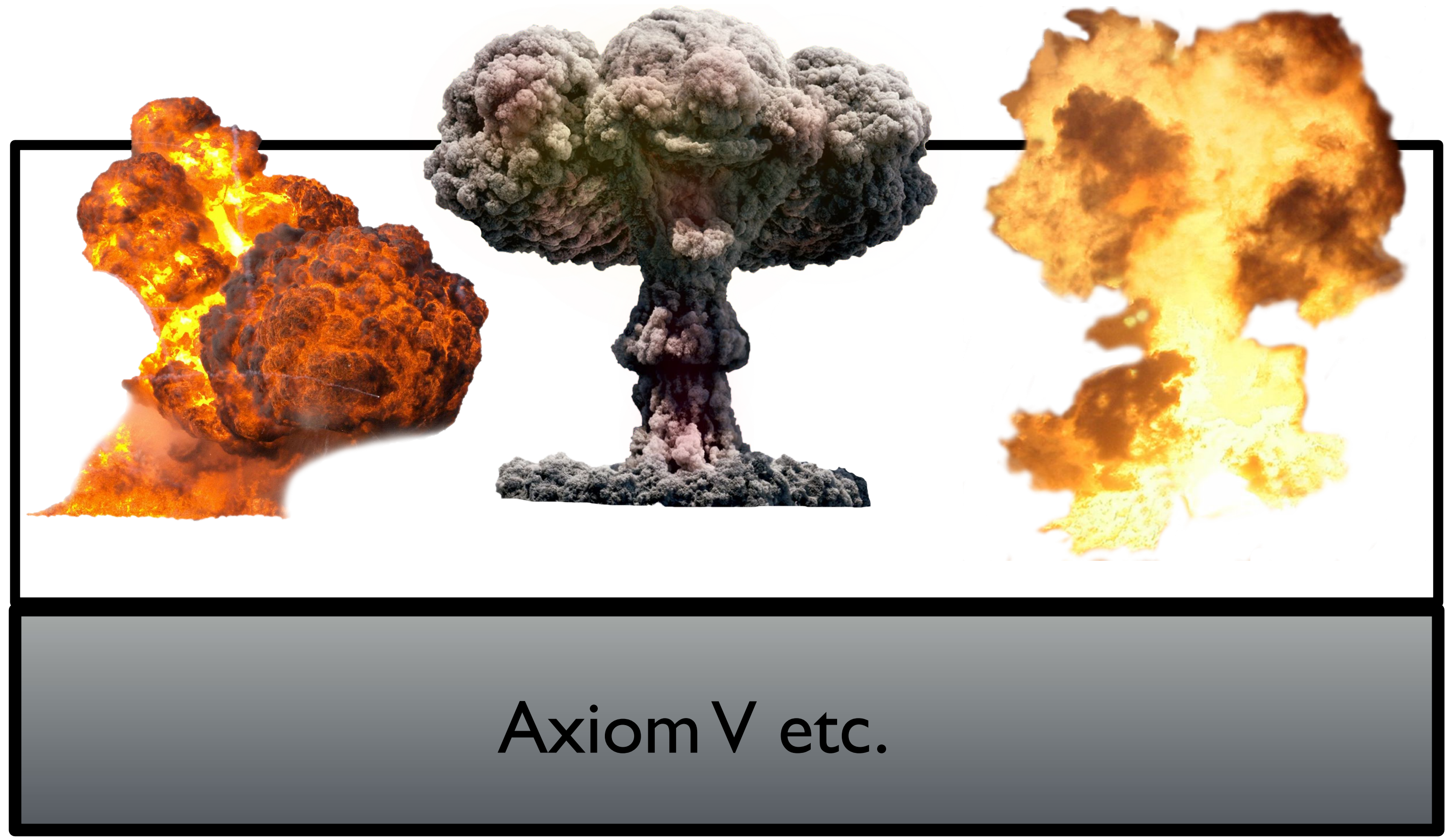
Solve

The Foundation Crumbles

Axiom V $\exists x \forall y [y \in x \leftrightarrow \phi(y)]$

a formula of arbitrary size in which the variable *y* is free; this formula ascribes a property to *y*

The Foundation Crumbles



Axiom V $\exists x \forall y [y \in x \leftrightarrow \phi(y)]$

a formula of arbitrary size in which the variable *y* is free; this formula ascribes a property to *y*

It's not just Russell's Paradox that destroys naïve set theory:

Richard's Paradox …

a

b

.

.

.

aa

ab

.

.

.

Doesn't define a real number. ~~aaa~~

.

.

.

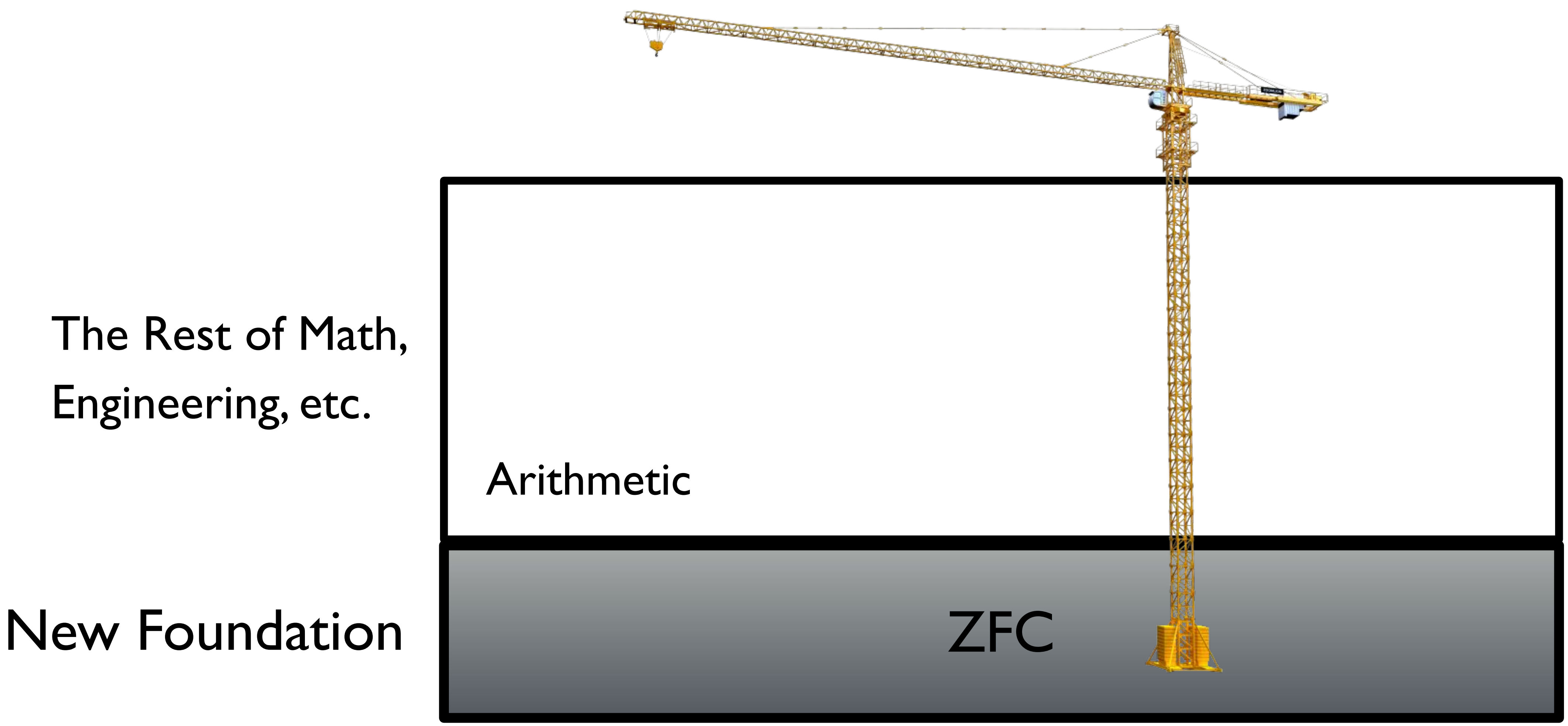
E

Definition of Richard's *N*:

"The real number whose whole part is zero, and whose *n*-th decimal is *p* plus one if the *n*-th decimal of the real number defined by the *n*-th member of *E* is *p* and *p* is neither eight nor nine, and is simply one if this *n*-th decimal is eight or nine."

Proof: *N* is defined by a finite string taken from the English alphabet, so *N* is in the sequence *E*. But on the other hand, by definition of *N*, for every *m*, *N* differs from the *m*-th element of *E* in at least one decimal place; so *N* is not any element of *E*. Contradiction! **QED**

The Foundation Rebuilt



So what are the axioms in ZFC?

Axiom *Schema* of Separation (SEP)

—

SEP

$$\forall x_1 \ldots \forall x_k \forall x \exists y \forall z [z \in y \leftrightarrow (z \in x \wedge \phi(z, x_1, \ldots, x_k))]$$

where x and y are distinct, and are both distinct from z and the x_i; and, as usual for us now, ϕ expresses a property using $\in$.

—

"Given *beforehand* some set *x* and property $\mathscr{P}$ captured by a formula ϕ that uses $\in$ for its relation, the set *y* composed of $\{z \in x : \mathscr{P}(z)\}$ exists."

How does this neutralize Russell's letter to Frege?

How does this neutralize Russell's letter to Frege?

"Given *beforehand* some set *x* and property $\mathscr{P}$ captured by a formula ϕ that uses $\in$ for its relation, the set *y* composed of $\{z \in x : \mathscr{P}(z)\}$ exists."

- This is a much stronger statement than axiom V!

Russell's paradox can be rephrased as saying the existence of the set of all sets leads to a contradiction.

Axiom V implies the existence of the set of all sets. Axiom V leads to a contradiction.

SEP only allows us to define new sets in terms of pre-existing sets, thus avoiding the existence of the set of all sets.

As an exercise: Try using $z \notin z$ for P(z)

Formal Natural-Number Arithmetic …

Q (= Robinson Arithmetic)

Define the existence of natural numbers and their relationships to each other

A1 $\forall x(0 \neq s(x))$

A2 $\forall x \forall y(s(x) = s(y) \rightarrow x = y)$

A3 $\forall x(x \neq 0 \rightarrow \exists y(x = s(y)))$

Defines addition on natural numbers

A4 $\forall x(x + 0 = x)$

A5 $\forall x \forall y(x + s(y) = s(x + y))$

Defines multiplication on natural numbers

A6 $\forall x(x \times 0 = 0)$

A7 $\forall x \forall y(x \times s(y) = (x \times y) + x)$

Notation in the Q Axioms: Quantification

The “Domain of discourse” in Q is the natural numbers.

0, 1, 2, 3, 4 …….

This means quantifiers range exclusively over them

“$\forall$x, …” reads as:
“For any number x …”
“For all numbers…”

“$\exists$x, …” reads as:
“There exists a number x such that …”
“There is a number such that …”

Notation in the Q Axioms: Successors

Robinson arithmetic defines the natural numbers in terms of their successors, denoted by the successor function *s*.

s takes a number *x* and returns the next number (*x+1*)

Thus in robinson arithmetic the natural numbers are written solely in terms of 0 and successors of 0:

0 = 0
1 = s(0)
2 = s(s(0))
3 = s(s(s(0)))
4 = s(s(s(s(0))))

This successor notion allows for a compact axiomatization of the natural numbers.

$$\text{A1} \quad \forall x(0 \neq s(x))$$

- 0 is not the successor of any natural number
- All numbers' successors are not equal to 0.

Natural Numbers Numbers start at 0!

$$\text{A2} \quad \forall x \forall y (s(x) = s(y) \rightarrow x = y)$$

- For any two numbers, if their successors are equal, than they are equal.

Simple Example: 7 = 7 thus 6 = 6 thus 5 = 5 …

Why do we care, can't we just use Equality-Intro?
No! This allows us to make more complex statements…

imagine we have a statement containing free variables "s(s(p)) = s(q)", from this we could derive "s(p) = q", Which we couldn't do with raw equality intro.

$$A3 \quad \forall x(x \neq 0 \rightarrow \exists y(x = s(y))$$

- For all numbers x, if x does not equal zero, then there exists another number y such that x is the successor of y.

In simpler terms…

- For any number that is not zero, there exists some number that comes before it.

Yes! If x = 1, y = 0; x = 2, y = 1, etc etc

We will prove a better form of A3 as an exercise from this version…

The Addition Axioms

$$\text{A4} \quad \forall x(x + 0 = x)$$

Anything plus zero is itself.

$$\text{A5} \quad \forall x \forall y(x + s(y) = s(x + y))$$

Any x plus the successor of y is the successor of x plus y

In other words…

x + (y + 1) = (x + y) + 1

Interlude: Axioms and Meaning

What do we mean when you say the axioms *define* addition?

A common confusion: "I see a + sign inside the axioms! Therefore, addition must be defined before the axioms, and you are just writing obvious properties about it!"

$$\text{A4} \quad \forall x (x + 0 = x)$$
$$\text{A5} \quad \forall x \forall y (x + s(y) = s(x + y))$$

"+" in this context is just a symbol denoting a function of two numbers, we could just as well write "x + y" as a(x, y).

By defining properties of how the symbol may be used in reasoning we are constricting the symbol + to behave as addition, imbuing it with the intuitive "meaning" of addition.

$$\text{A6} \quad \forall x(x \times 0 = 0)$$
$$\text{A7} \quad \forall x \forall y(x \times s(y) = (x \times y) + x)$$

Defining multiplication:

Anything times 0 is 0.

Any number x times the successor of y is x times y plus x

I.E

a(b + 1) = ab + a

Open Formulae?

We've already seen it in our coverage of ZFC.

$$\exists y[s(s(0)) \times y = s(s(s(s(0))))]$$

This says what?

That 2 multiplied by some number yields 4.

But this is very specific: the successor of the successor of zero is specifically 2.

Here then is the general case with an "open" wff:

$$\exists y[s(s(0)) \times y = x]$$

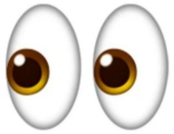
This open wff $\phi(x)$ expresses the arithmetic property 'even.'

PA (Peano Arithmetic)

$$
\begin{array}{ll}
\text{A1} & \forall x(0 \neq s(x)) \\
\text{A2} & \forall x \forall y(s(x) = s(y) \rightarrow x = y) \\
\text{A3} & \forall x(x \neq 0 \rightarrow \exists y(x = s(y))) \\
\text{A4} & \forall x(x + 0 = x) \\
\text{A5} & \forall x \forall y(x + s(y) = s(x + y)) \\
\text{A6} & \forall x(x \times 0 = 0) \\
\text{A7} & \forall x \forall y(x \times s(y) = (x \times y) + x)
\end{array}
$$

And, every sentence that is the universal closure of an instance of

$$([\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x))] \rightarrow \forall x \phi(x))$$



where $\phi(x)$ is open wff with variable x, and perhaps others, free.

What is this large scary formula?

$$([\phi(0) \land \forall x(\phi(x) \rightarrow \phi(s(x))] \rightarrow \forall x \phi(x))$$

Axiom schema of induction!

Gives us a very powerful tool for proving statements about all natural numbers.

You can say something is true about all natural number if you can first show that it is true for 0 and that it being true for an arbitrary number n implies it is true for n+1.

Domino Analogy for Induction

$$([\phi(0) \wedge \forall x(\phi(x) \to \phi(s(x))] \to \forall x \phi(x))$$

Let $\phi(x)$ be the statement: "The nth domino has been knocked over"

If the first domino is knocked over

$\phi(0)$

and for all dominos, each one will knock over its successor

$\forall x(\phi(x) \to \phi(s(x))$

then all dominos have been knocked over

$\forall x \phi(x)$

Example Proof by Induction

$$\forall n : \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

As a "forall natural numbers" statement, we can try using the induction schema to prove it!

Let $\phi(x) = \sum_{i=0}^{x} i = \frac{x(x+1)}{2}$.

We must first prove the first domino falls, this is called the base case, i.e. $\phi(0)$.

$$\sum_{i=0}^{0} i = \frac{0(0+1)}{2} \rightarrow 0 = 0$$

We must now prove that if the statement is true for one number it will be true for the next number: $\forall x : \phi(x) \rightarrow \phi(x+1)$. This is called the inductive step.

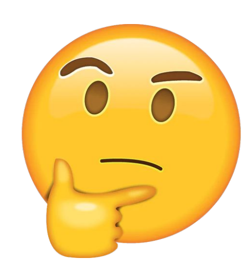
We can prove $\forall x : \phi(x) \rightarrow \phi(x+1)$ by proving $\phi(x) \rightarrow \phi(x+1)$ for this arbitrary x and using $\forall$ introduction. We will use a direct proof for $\phi(x) \rightarrow \phi(x+1)$, proving $\phi(x+1)$ using $\phi(x)$ as an assumption. $\phi(x)$ is called the "induction hypothesis".

$$\sum_{i=0}^{x+1} i = \frac{(x+1)((x+1)+1)}{2}$$

$$\sum_{i=0}^{x} i + x + 1 = \frac{(x+1)((x+1)+1)}{2}$$

$$\frac{x(x+1)}{2} + x + 1 = \frac{(x+1)((x+1)+1)}{2} \qquad \text{by } \phi(x)$$

Brush up on your highschool arithmetic
by finishing the equivalence proof…



$$([\phi(0) \wedge \forall x(\phi(x) \longrightarrow \phi(s(x))] \longrightarrow \forall x \phi(x))$$

We have $\phi(0)$ and $\forall x : \phi(x) \rightarrow \phi(x+1)$ therefore, by induction (the induction axiom) $\phi(n)$ holds for all n.

QED 🙂

Pop Quiz #1

Pull out a piece of paper and write down if the following formulae are open or closed. If they are open, list the unbound variable(s).

1) $\exists y[s(s(0)) \times y = x]$

2) s(s(0)) + s(0) = s(s(s(0)))

3) $\exists y[s(s(0)) \times y = s(s(s(s(0))))]$

4) s(a) = s(s(0))

5) $\forall x : x = 0 \vee (\exists y : x = s(y))$

6) $\sum_{i=0}^{x+1} i = \frac{(x+1)((x+1)+1)}{2}$

Pop Quiz #2

Create an FOL workspace in hyperslate named 311pop2. Using A3 we have provided as a given:

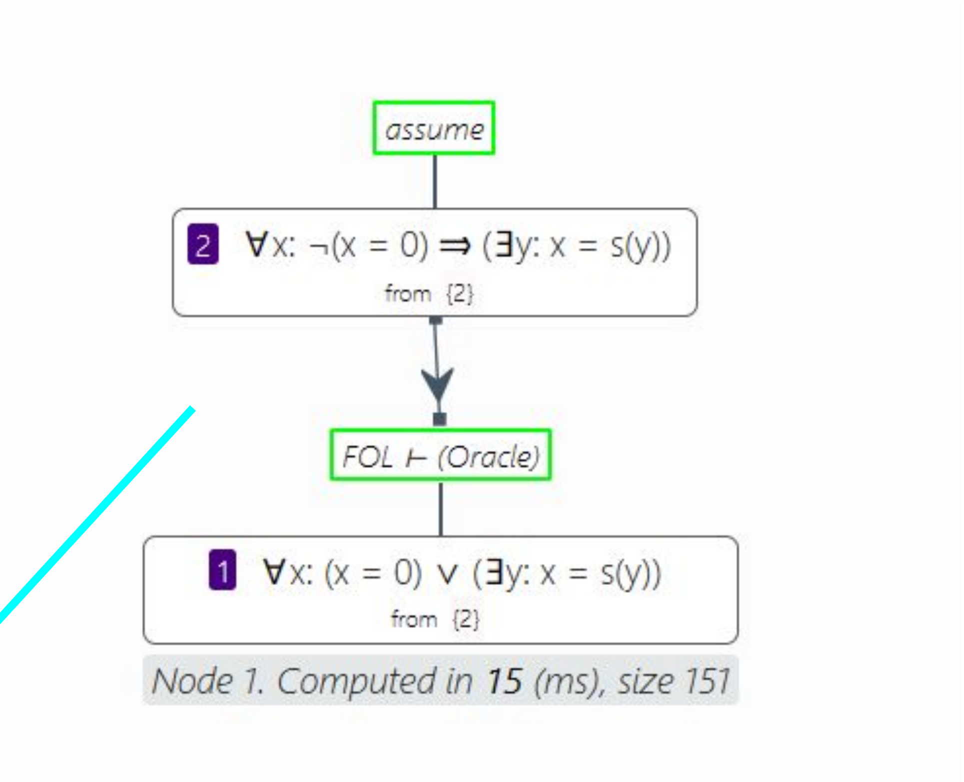
$$\text{A3} \quad \forall x(x \neq 0 \rightarrow \exists y(x = s(y)))$$

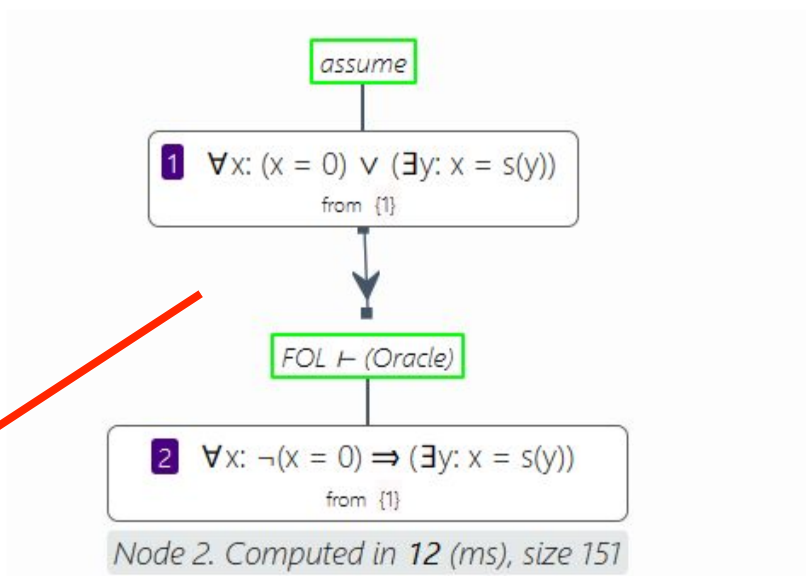
Prove that a natural number is either 0 or the successor of another natural number. No oracles.

$$\forall x : x = 0 \vee (\exists y : x = s(y))$$

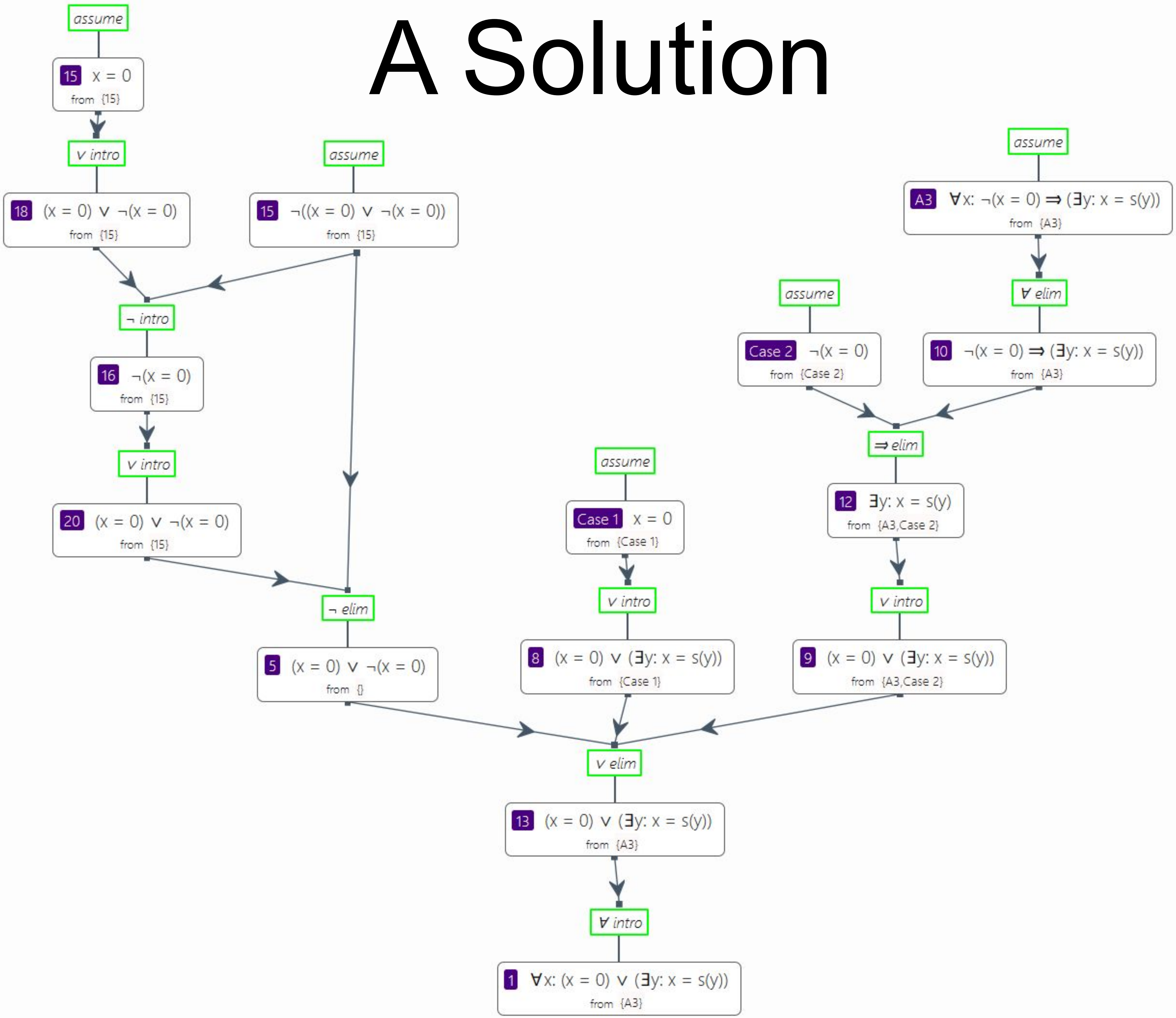
(This is an alternative formulation of axiom 3)

If you finish early, prove the other direction. Use the previous goal as an assumption and derive A3.



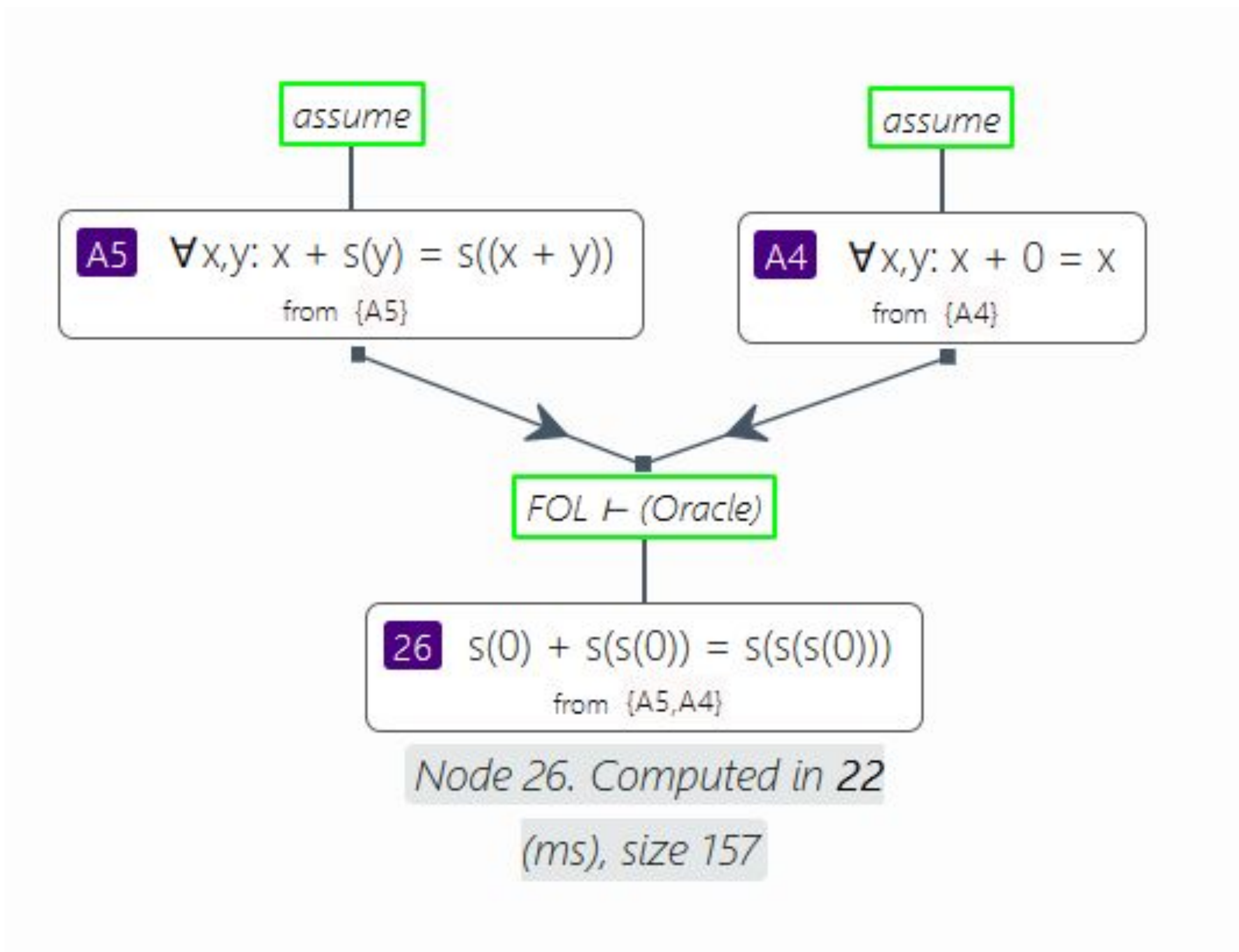


A Solution


assume
15 x = 0
from {15}
∨ intro
18 (x = 0) ∨ ¬(x = 0)
from {15}
assume
15 ¬((x = 0) ∨ ¬(x = 0))
from {15}
¬ intro
16 ¬(x = 0)
from {15}
∨ intro
20 (x = 0) ∨ ¬(x = 0)
from {15}
¬ elim
5 (x = 0) ∨ ¬(x = 0)
from {}
assume
Case 1 x = 0
from {Case 1}
∨ intro
8 (x = 0) ∨ (∃y: x = s(y))
from {Case 1}
assume
A3 ∀x: ¬(x = 0) ⇒ (∃y: x = s(y))
from {A3}
∀ elim
10 ¬(x = 0) ⇒ (∃y: x = s(y))
from {A3}
assume
Case 2 ¬(x = 0)
from {Case 2}
⇒ elim
12 ∃y: x = s(y)
from {A3,Case 2}
∨ intro
9 (x = 0) ∨ (∃y: x = s(y))
from {A3,Case 2}
∨ elim
13 (x = 0) ∨ (∃y: x = s(y))
from {A3}
∀ intro
1 ∀x: (x = 0) ∨ (∃y: x = s(y))
from {A3}

Pop Quiz #3: Basic Addition

Use A4 and A5 to prove 1+2 = 3. Do this in a new FOL workspace named 311pop3



Slutten

(Norwegian for “End”)