Gödel's …
First Incompleteness Theorem
(G1)

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Department of Computer Science
Lally School of Management & Technology
Rensselaer Polytechnic Institute (RPI)
Troy, New York 12180 USA

4/3/2025

*With Naveen Sundar G's introduction to Gödel numbering, and his depiction of GT growth, etc.

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Note: This is a version designed for those who have had at least one robust, proof-intensive university-level course in formal logic to the level of $\mathscr{L}_2$.

4/3/2025

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Background Context …

Gödel's Great Theorems (OUP)

by Selmer Bringsjord

- Introduction ("The Wager")
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
- The Continuum-Hypothesis Theorem
- The Time-Travel Theorem
- Gödel's "God Theorem"
- Could a Finite Machine Match Gödel's Greatness?

STOP & REVIEW IF NEEDED!

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A corollary of the First Incompleteness Theorem: *We cannot prove (in classical mathematics) that mathematics is consistent.*

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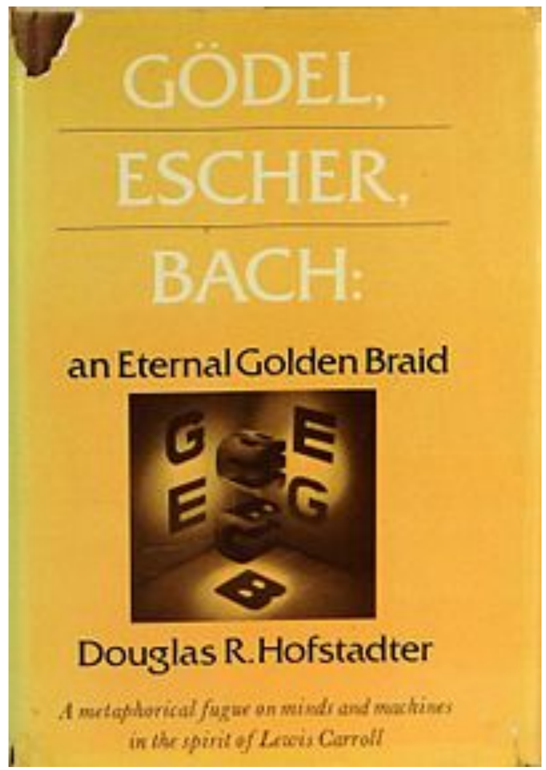
By far the greatest of GGT; Selm's analysis based Sherlock Holmes' mystery "Silver Blaze."

Deficient; Beware

Deficient; Beware



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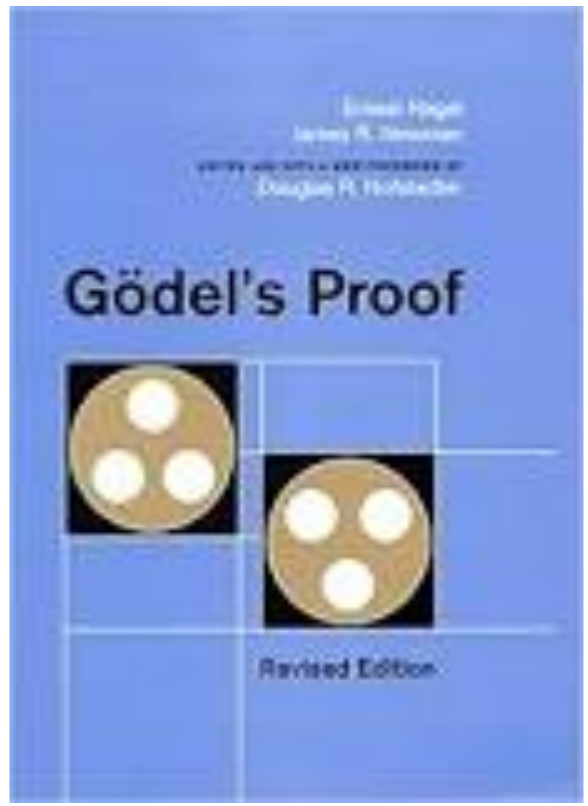

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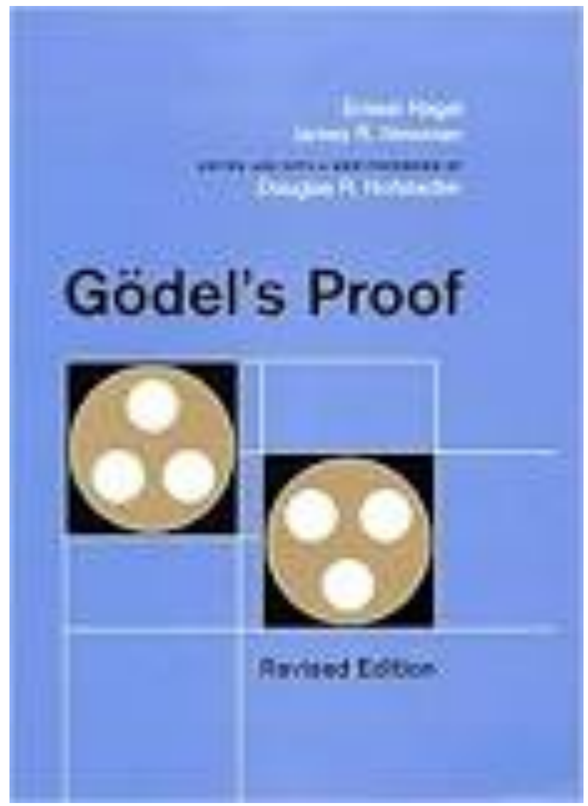

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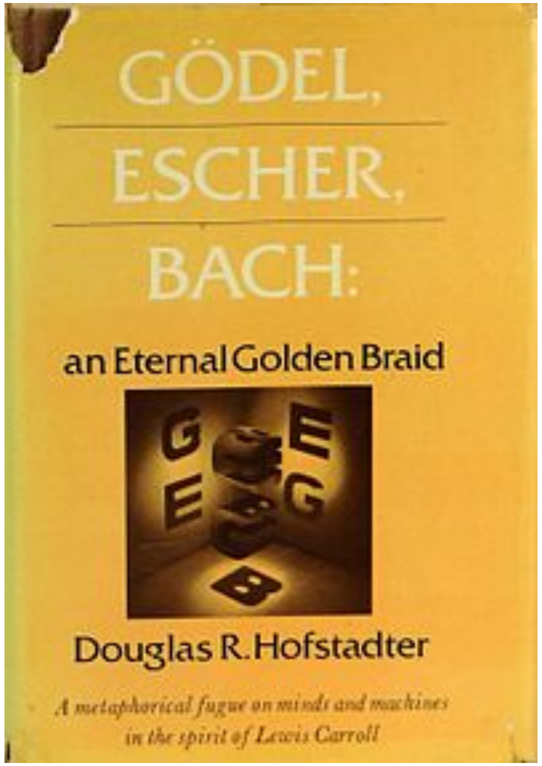

Some Timeline Points

1978 Princeton NJ USA.

1940 Back to USA, for good.

1936 Schlick murdered; Austria annexed

1933 Hitler comes to power.

1930 Announces (First) *In*completeness Theorem

1929 Doctoral Dissertation: Proof of Completeness Theorem

Undergrad in seminar by Schlick

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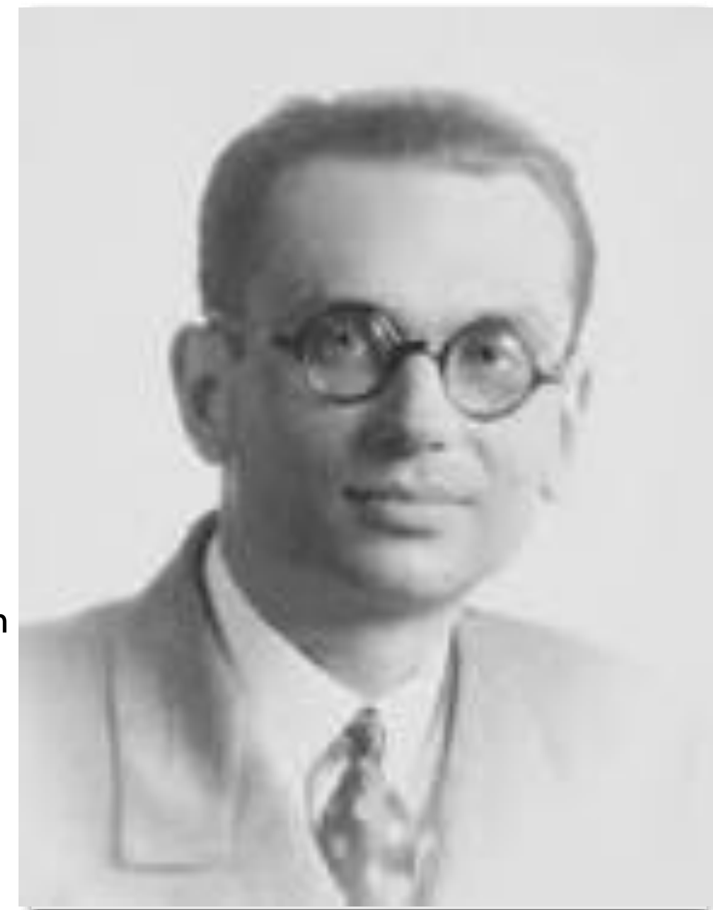
1906 Brünn, Austria-Hungary



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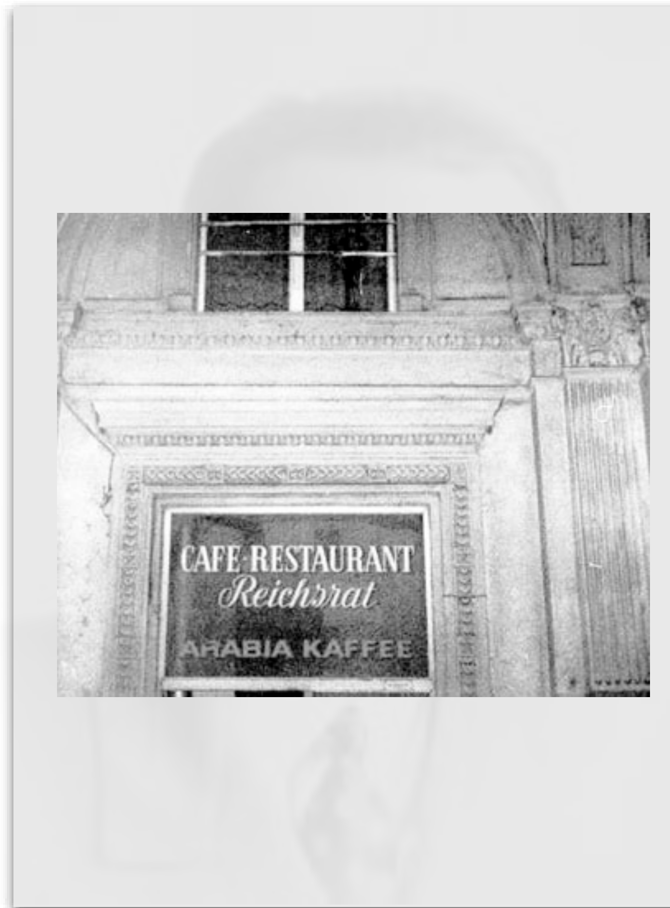

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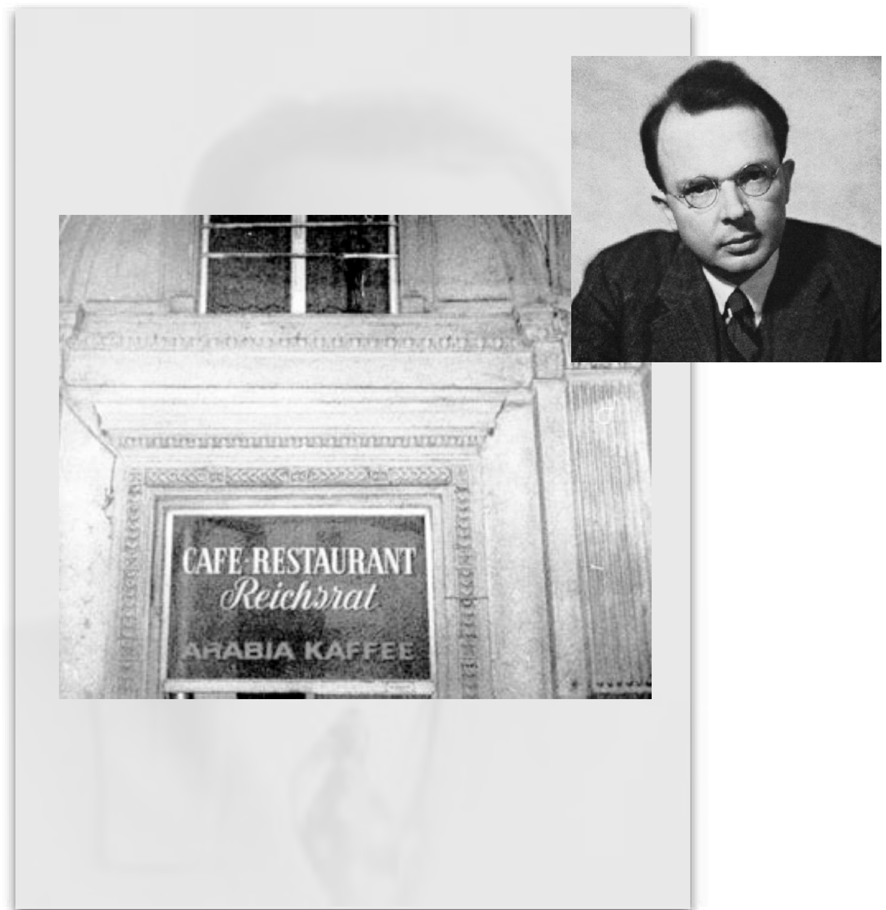

Some Timeline Points

“Well, uh, hmm, …”

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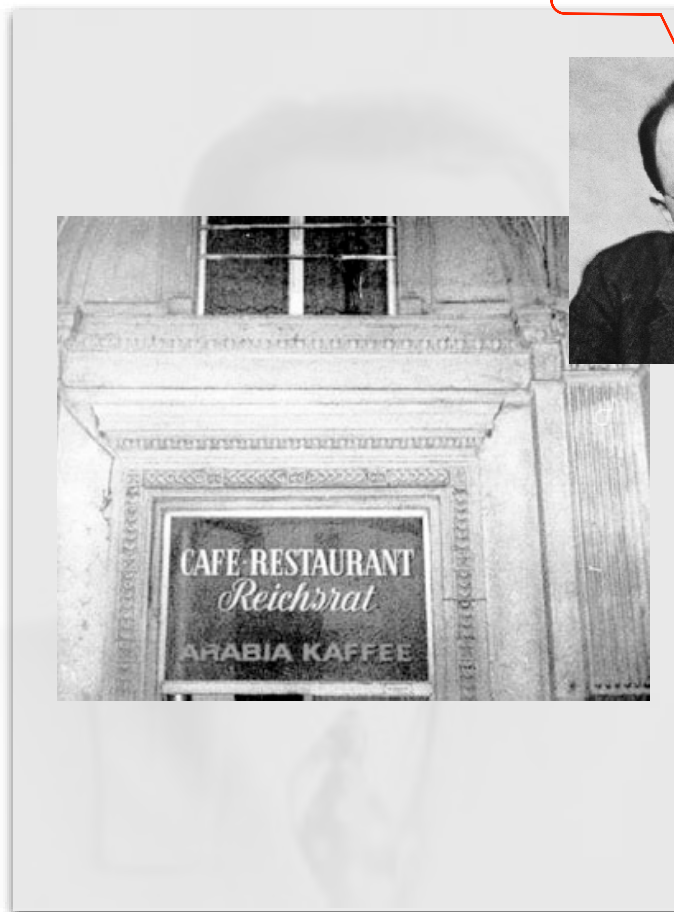

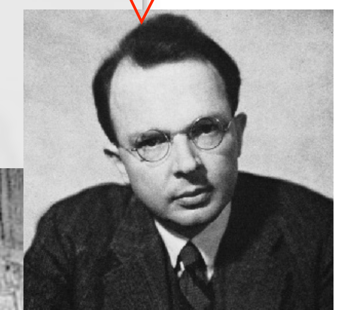

The “Liar Tree”

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The Liar Paradox

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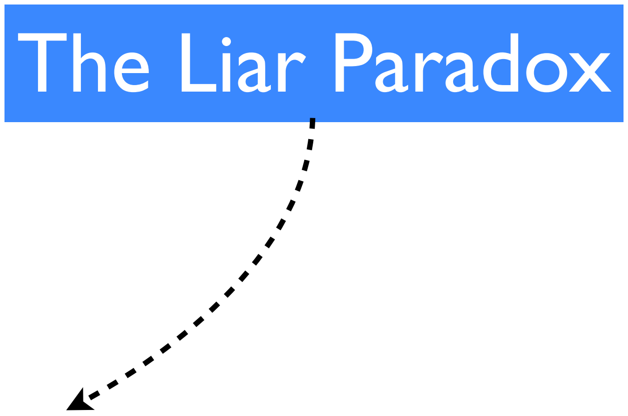

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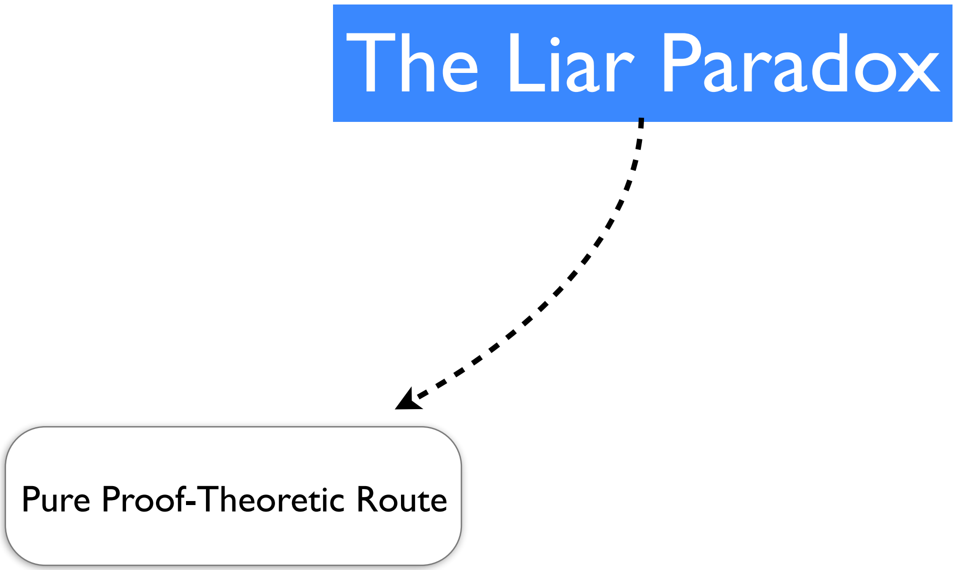

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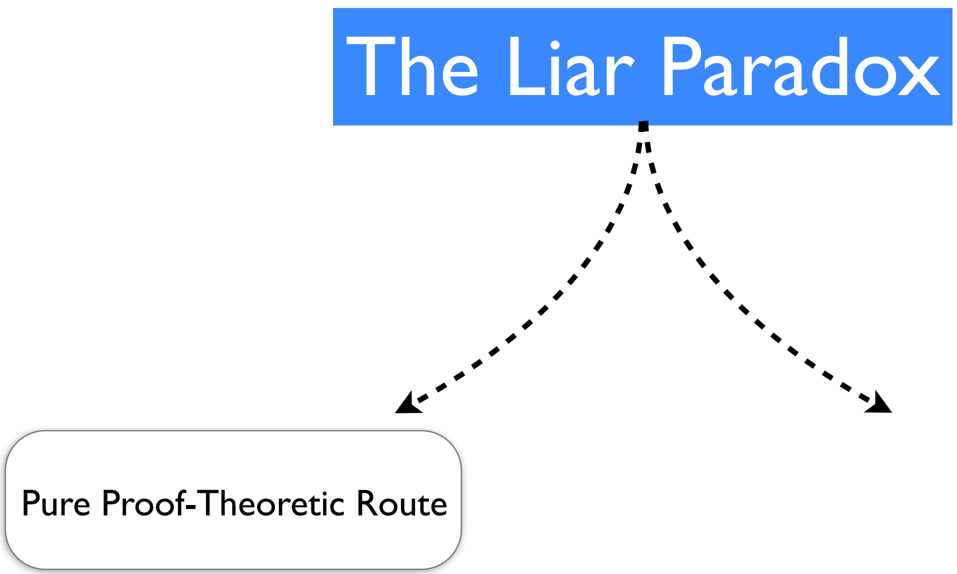

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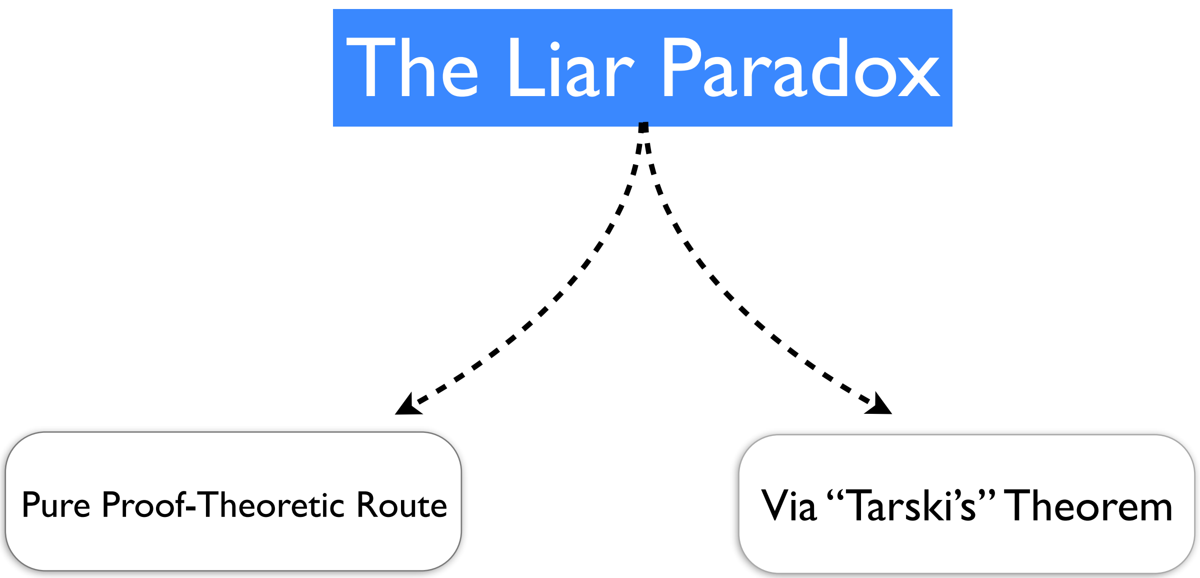

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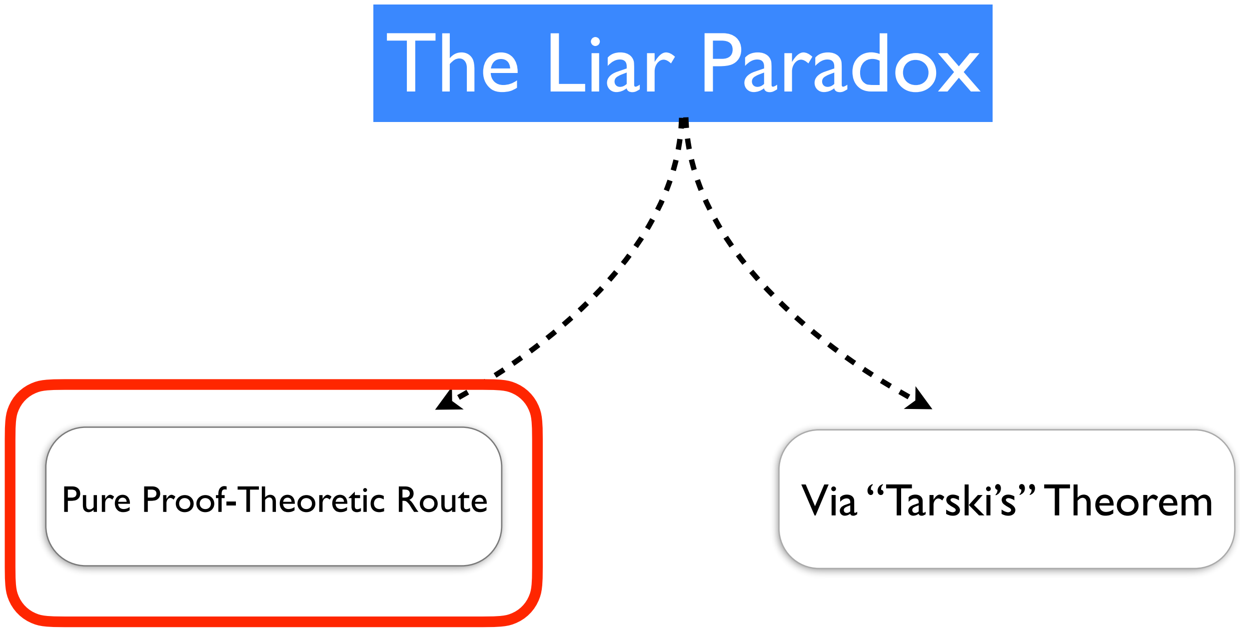

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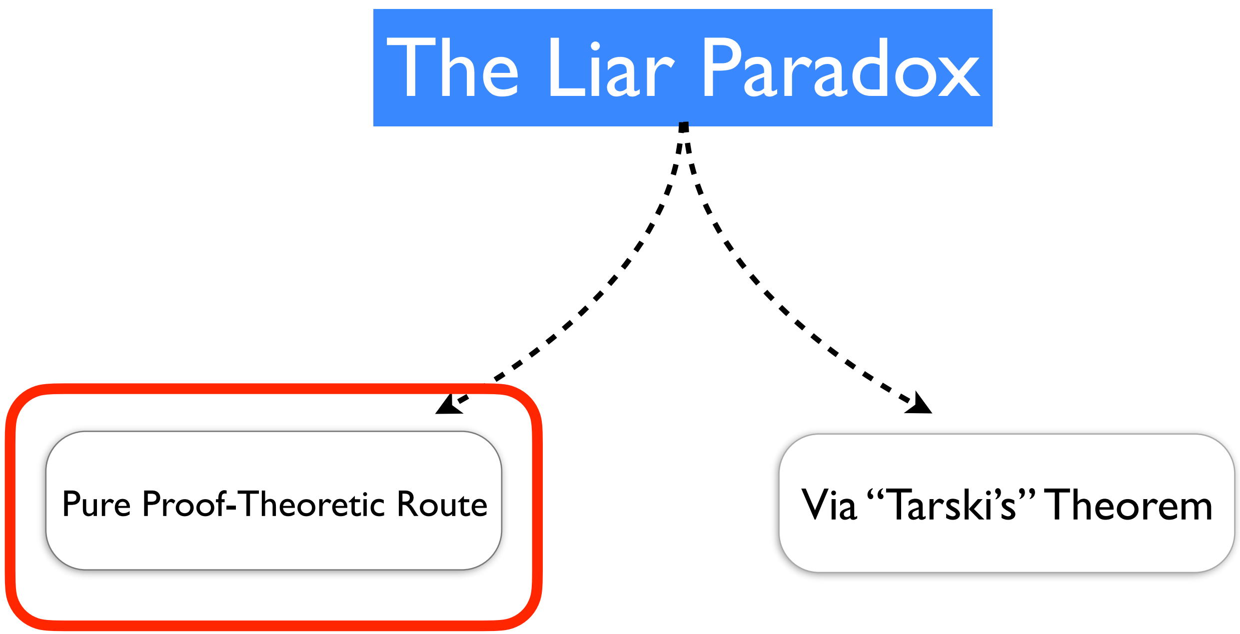

Paul Erdős

“The Book”

The "Liar Tree"

The Liar Paradox

Pure Proof-Theoretic Route

Via "Tarski's" Theorem

Paul Erdős

"The Book"

Ergo, step one: What *is* LP?

“The (Economical) Liar”

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L: This sentence is false.

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Suppose that T(**L); then** ¬T(**L**).

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Contradiction!

The “Gödelian” Liar (from me)

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$\bar{P}$: This sentence is unprovable.

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Suppose that $\bar{P}$ is true. Then we can immediately deduce that $\bar{P}$ is provable, because here is a proof: $\bar{P} \rightarrow \bar{P}$ is an easy theorem, and from it and our supposition we deduce $\bar{P}$ by *modus ponens*. But since what $\bar{P}$ says is that it's unprovable, we have deduced that $\bar{P}$ is false under our initial supposition.

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Suppose on the other hand that $\bar{P}$ is false. Then we can immediately deduce that $\bar{P}$ is unprovable: Suppose for *reductio* that $\bar{P}$ *is* provable; then $\bar{P}$ holds as a result of some proof, but what $\bar{P}$ says is that it's unprovable; and so we have contradiction. But since what $\bar{P}$ says is that it's unprovable, and we have just proved that under our supposition, we arrive at the conclusion that $\bar{P}$ is true.

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Contradiction!

All of this is fishy; but Gödel transformed it into utterly precise, impactful, indisputable reasoning …

PA (Peano Arithmetic):

$$\begin{array}{ll} \text{A1} & \forall x(0 \neq s(x)) \\ \text{A2} & \forall x \forall y(s(x) = s(y) \rightarrow x = y) \\ \text{A3} & \forall x(x \neq 0 \rightarrow \exists y(x = s(y)) \\ \text{A4} & \forall x(x + 0 = x) \\ \text{A5} & \forall x \forall y(x + s(y) = s(x + y)) \\ \text{A6} & \forall x(x \times 0 = 0) \\ \text{A7} & \forall x \forall y(x \times s(y) = (x \times y) + x) \end{array}$$

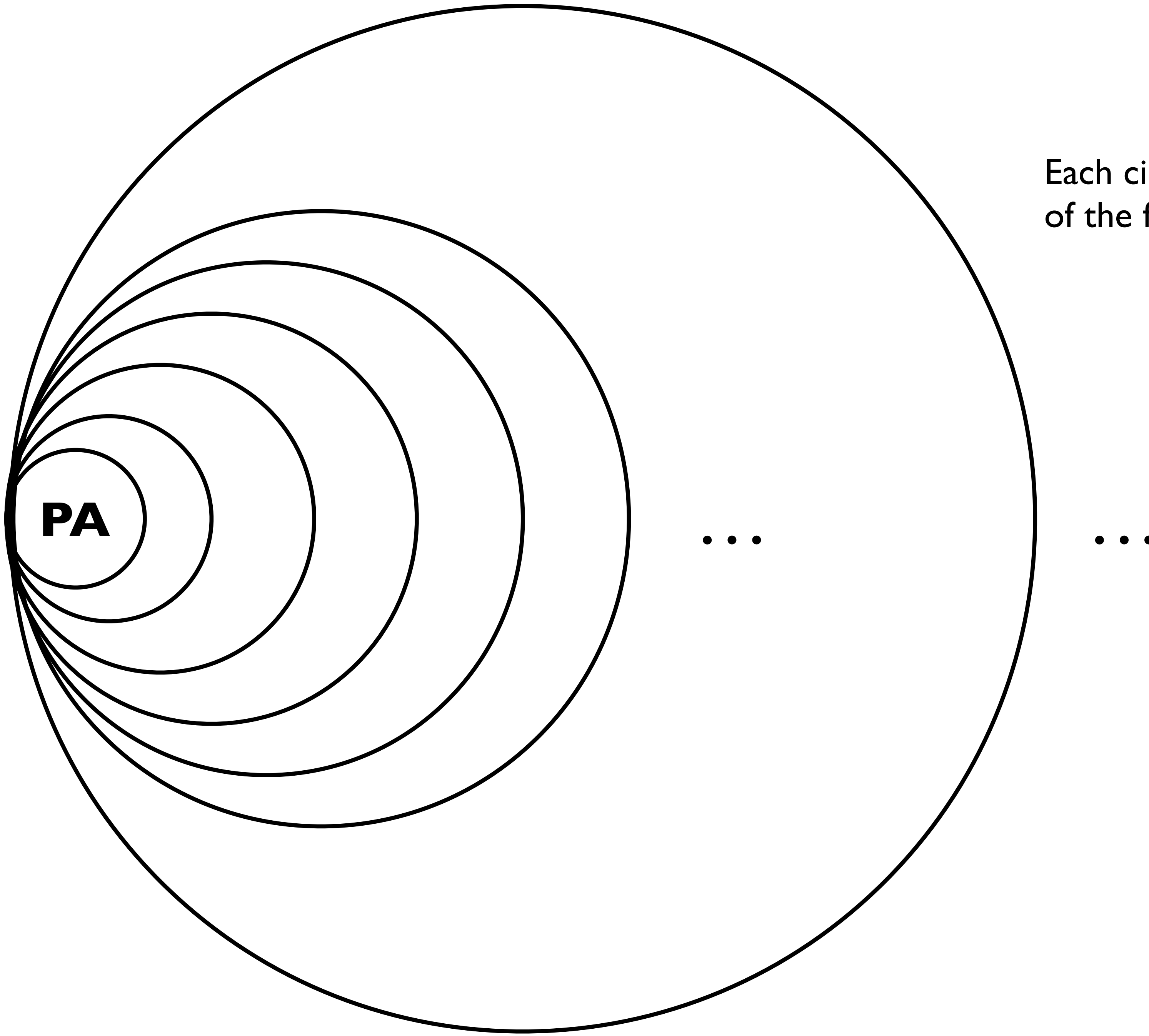
And, every sentence that is the universal closure of an instance of

$$([\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x))] \rightarrow \forall x \phi(x))$$

where $\phi(x)$ is open wff with variable x, and perhaps others, free.

Arithmetic is Part of All Things Sci/Eng/Tech!

but courtesy of Gödel: We can't even prove all truths of arithmetic!



Each circle is a larger part of the formal sciences.

Arithmetic Crucial Part of All Things Sci/Eng/Tech!

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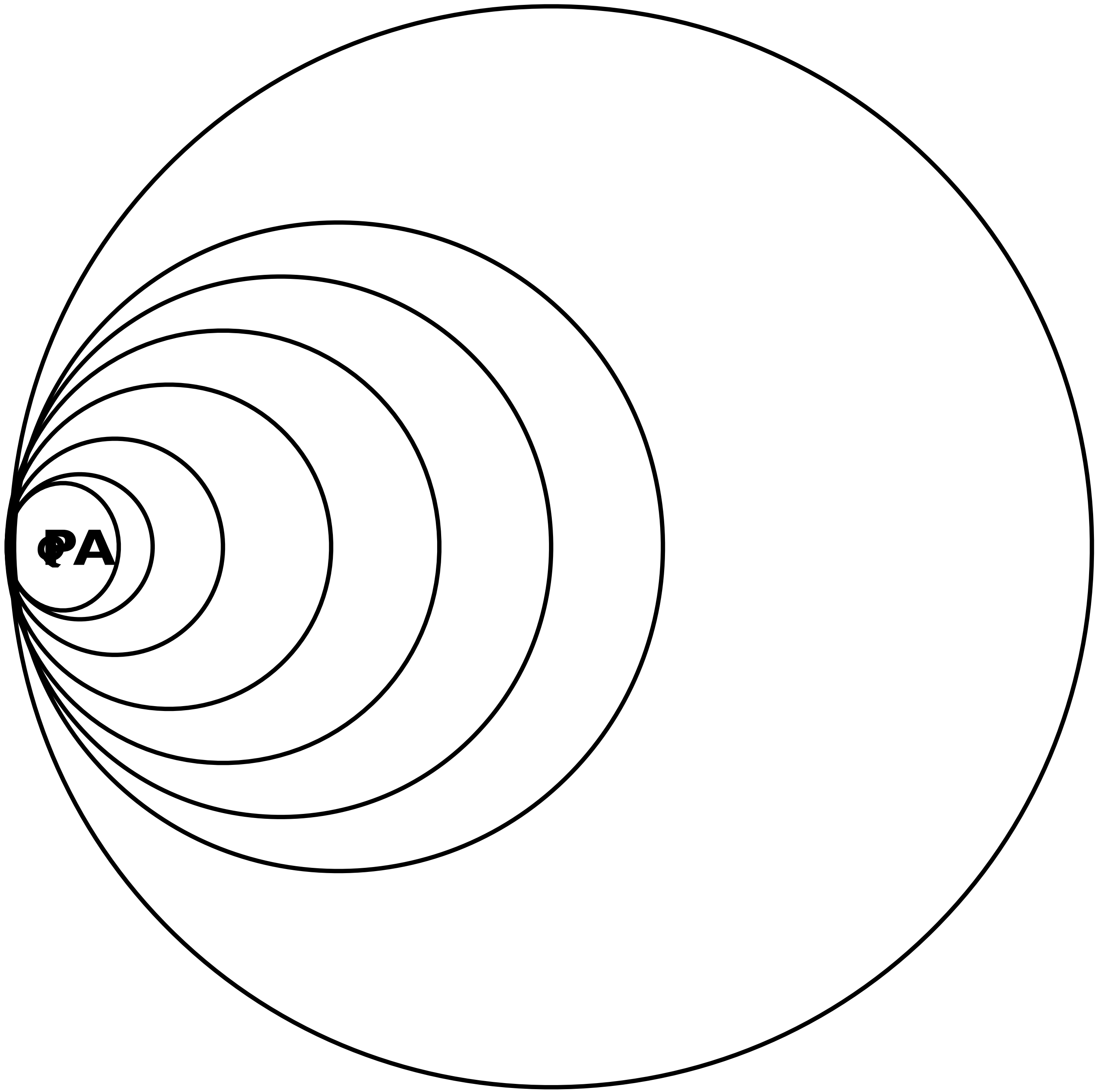

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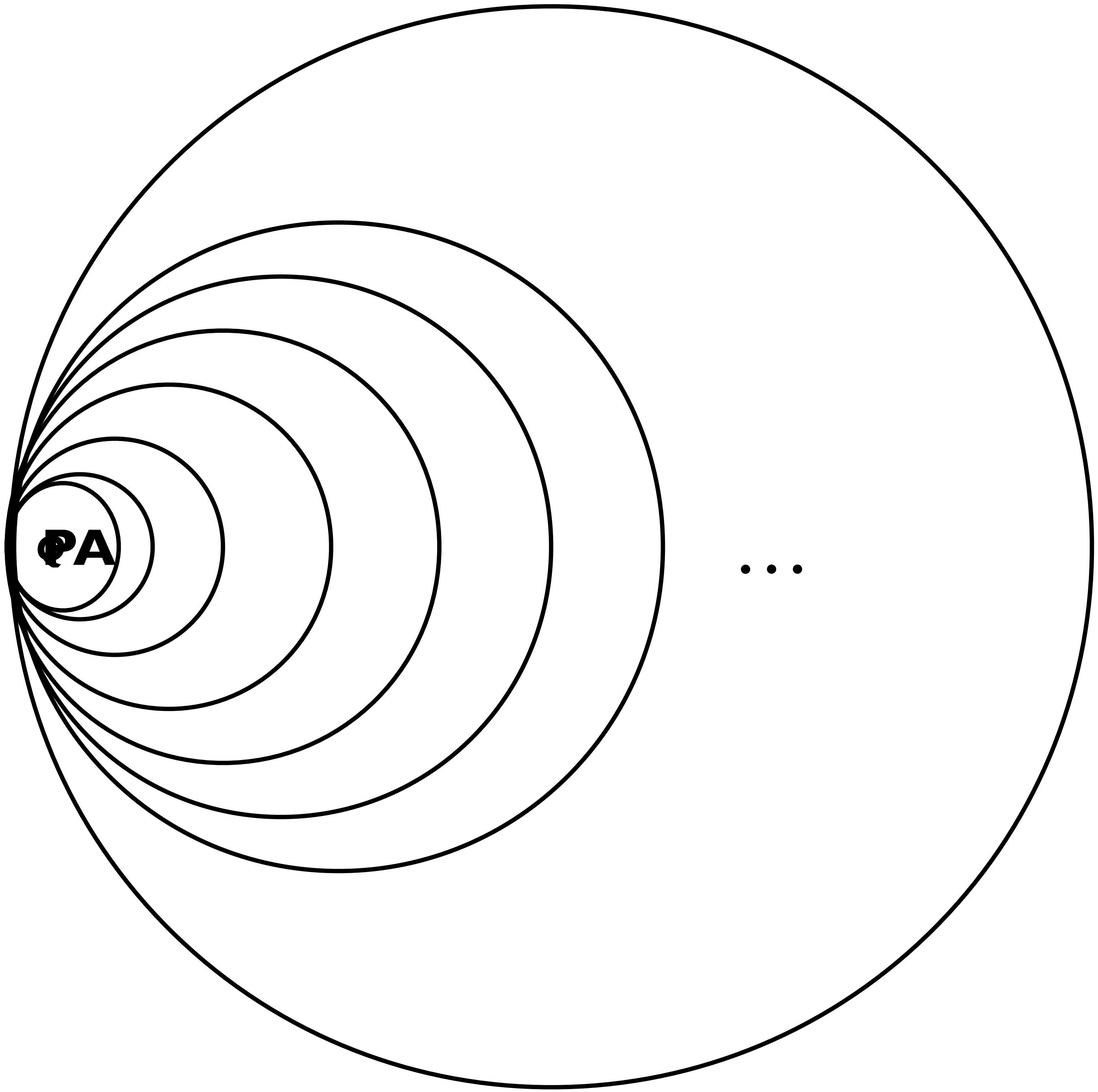

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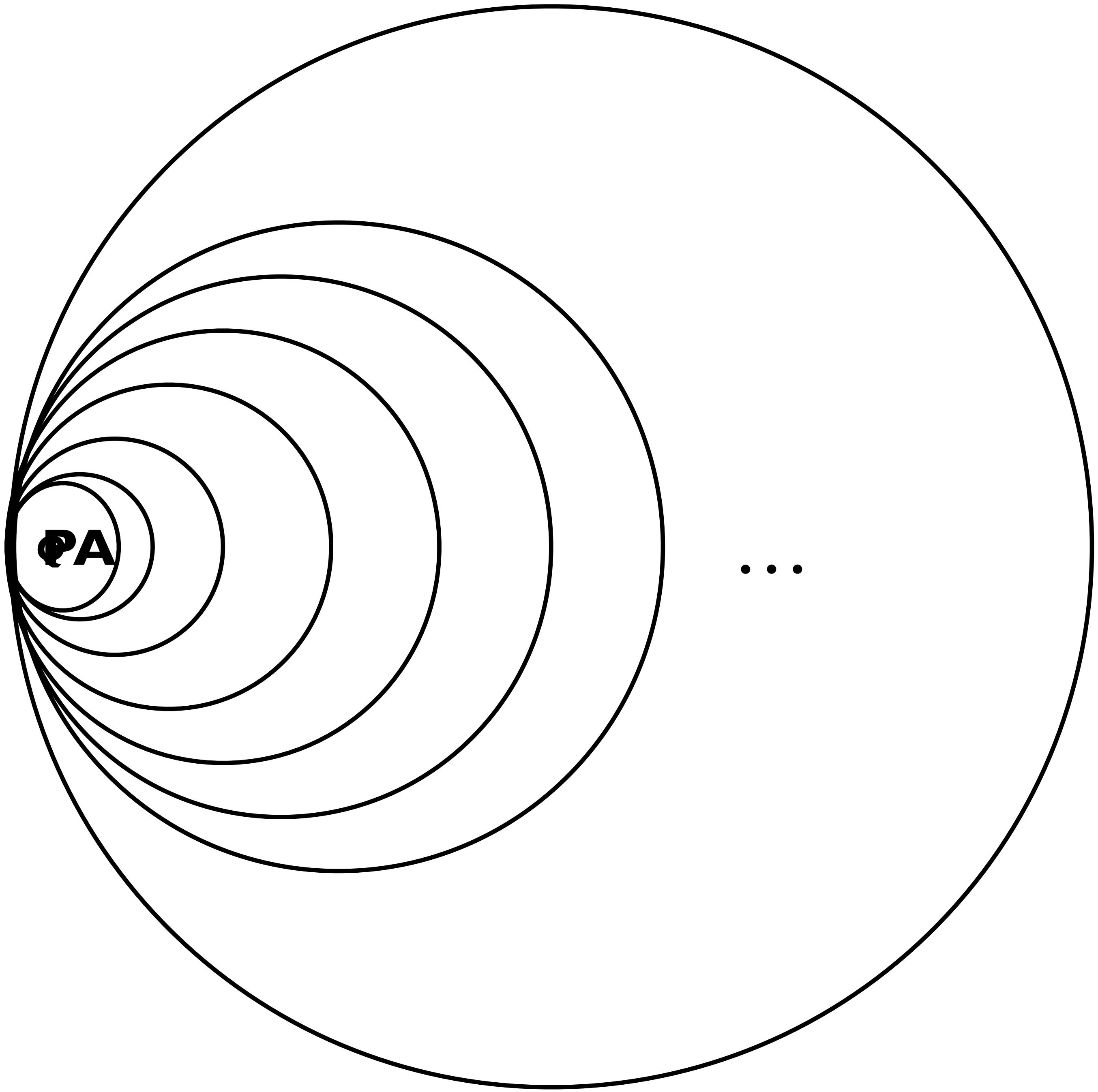

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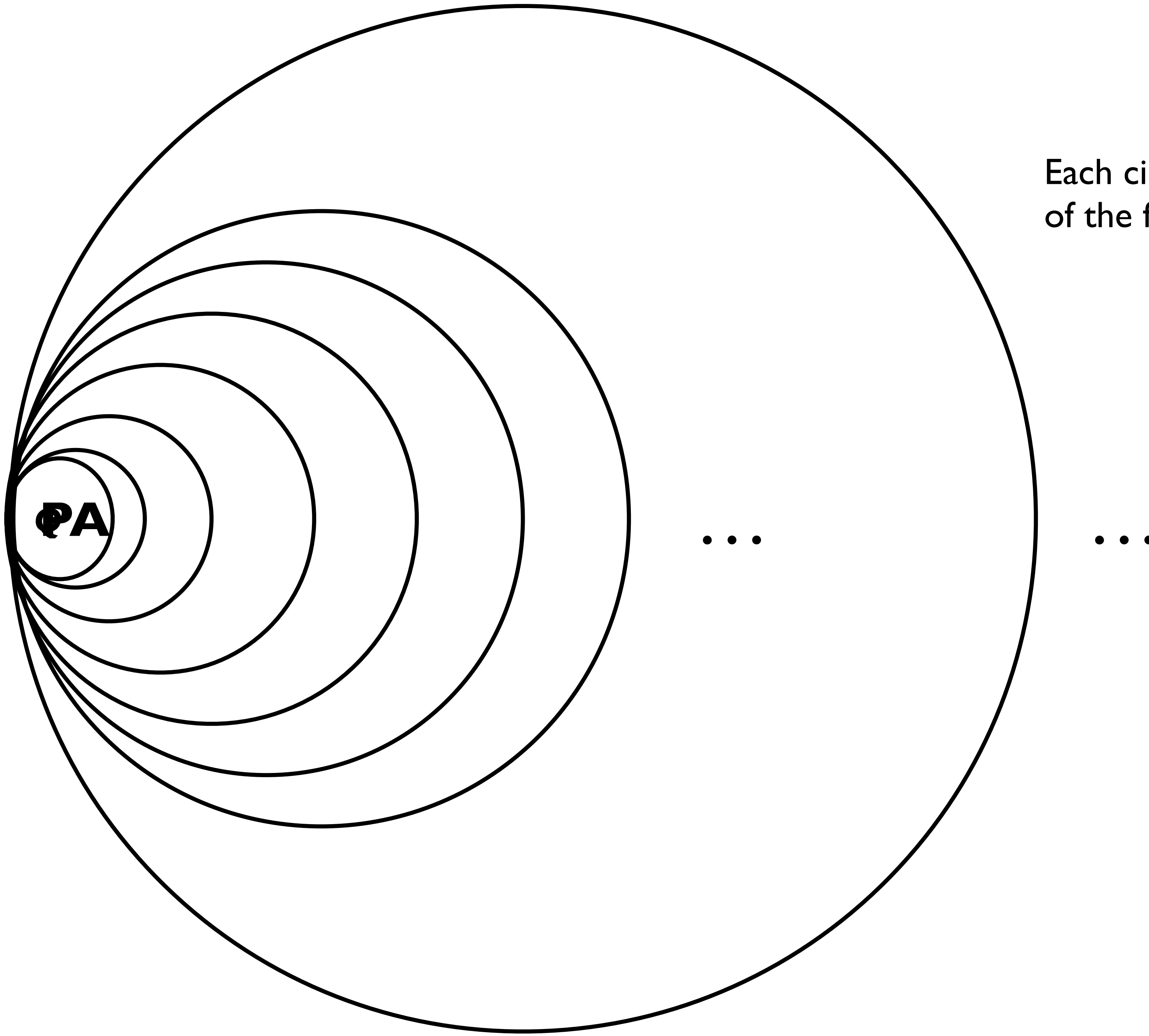

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Gödel Numbering

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Problem: How do we enable a formula to refer to other formulae and itself (and also other objects like proofs, terms etc.), in a perfectly consistent way?

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$$\phi$$

$$\phi \to \psi$$

$$f(x, a)$$

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Object-level objects in the language of $\mathscr{L}_1$

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(formulae, terms, proofs etc)

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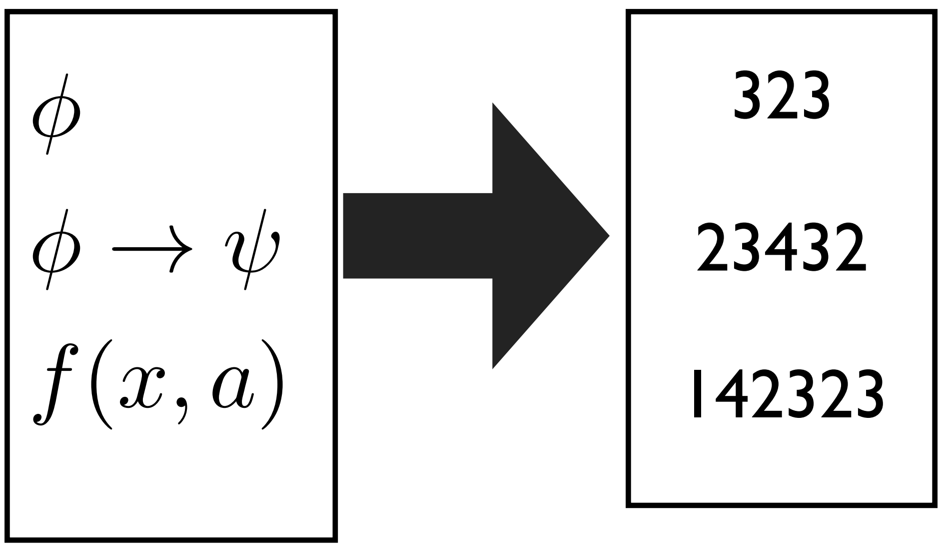

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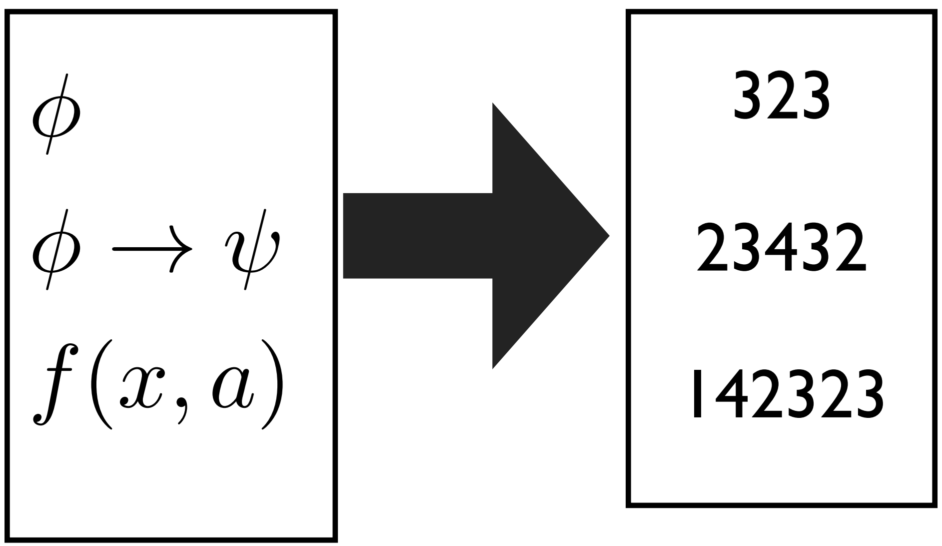

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Gödel number

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ϕ	→	323
$\phi \to \psi$	→	23432
$f(x, a)$	→	142323

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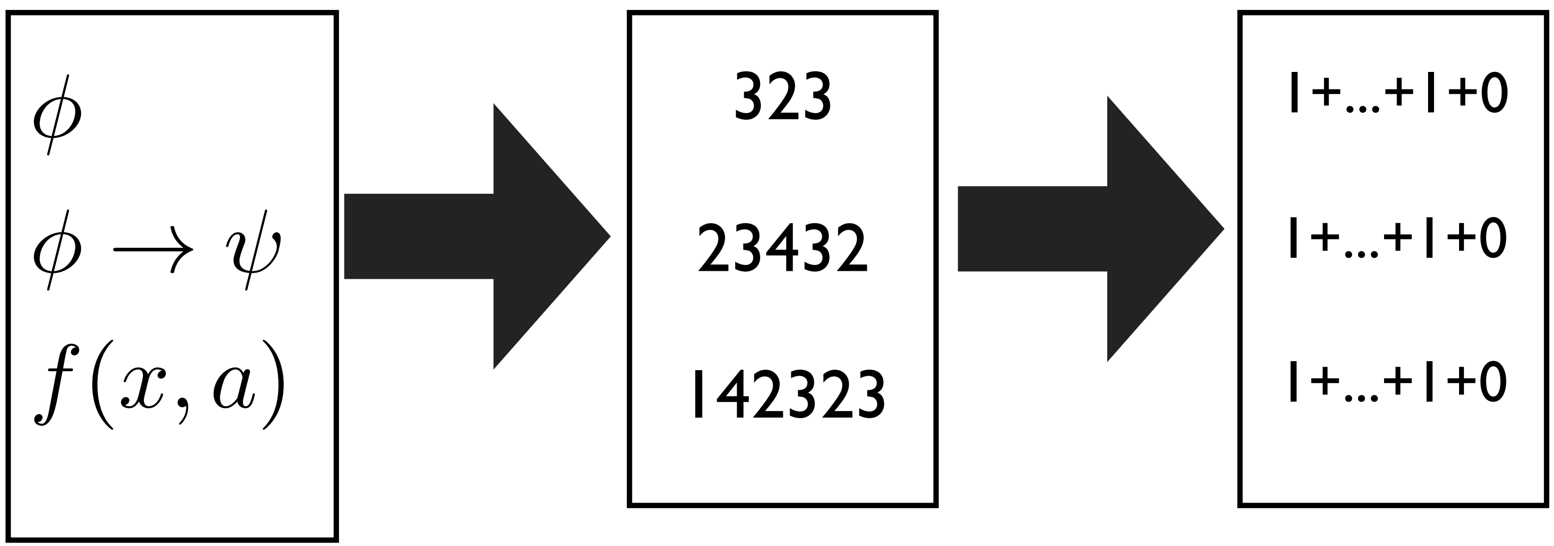

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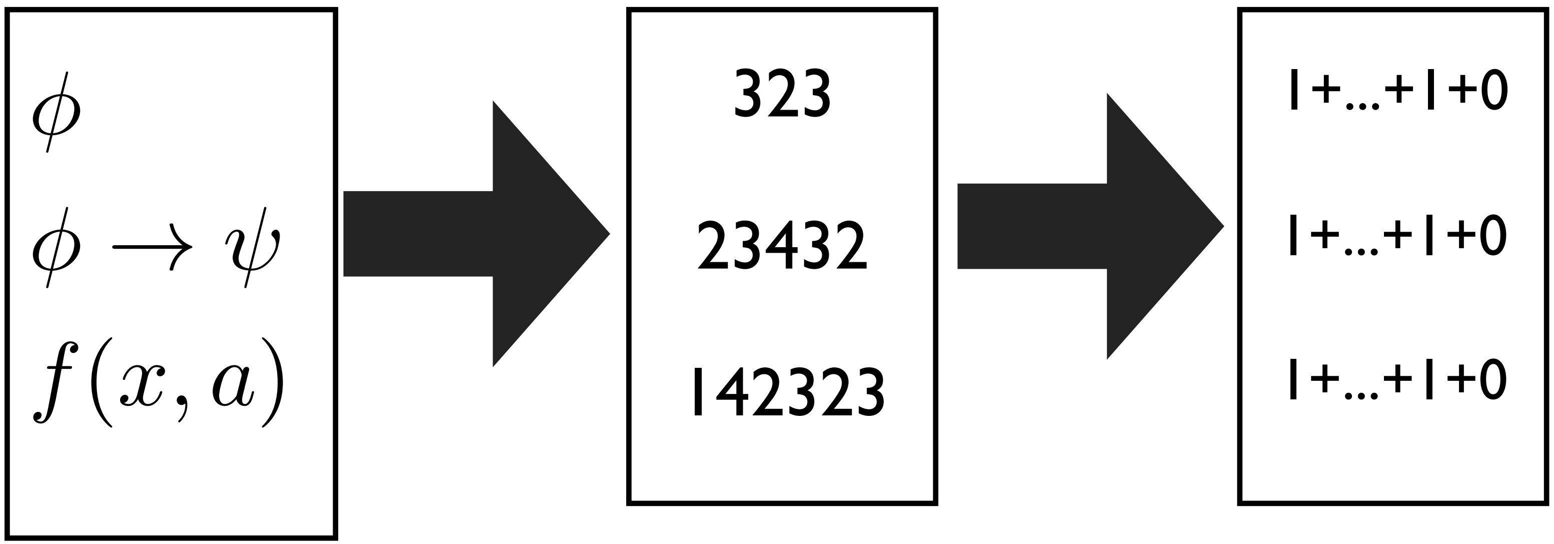

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Gödel number

Gödel numeral

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Object-level objects in the language of $\mathscr{L}_1$		Gödel number		Gödel numeral
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$\phi \rightarrow \psi$	→	23432	→	1+...+1+0
$f(x, a)$	→	142323	→	1+...+1+0

(formulae, terms, proofs etc)

$$\phi$$

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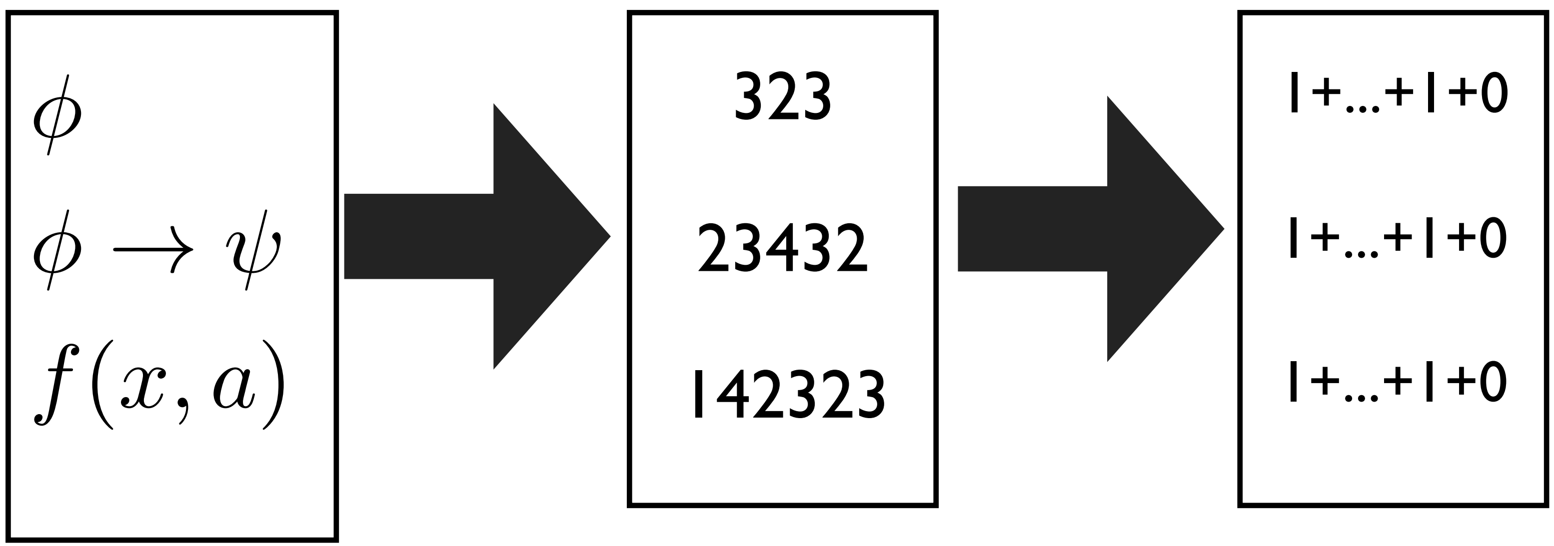

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ϕ

Gödel number

n^{ϕ}

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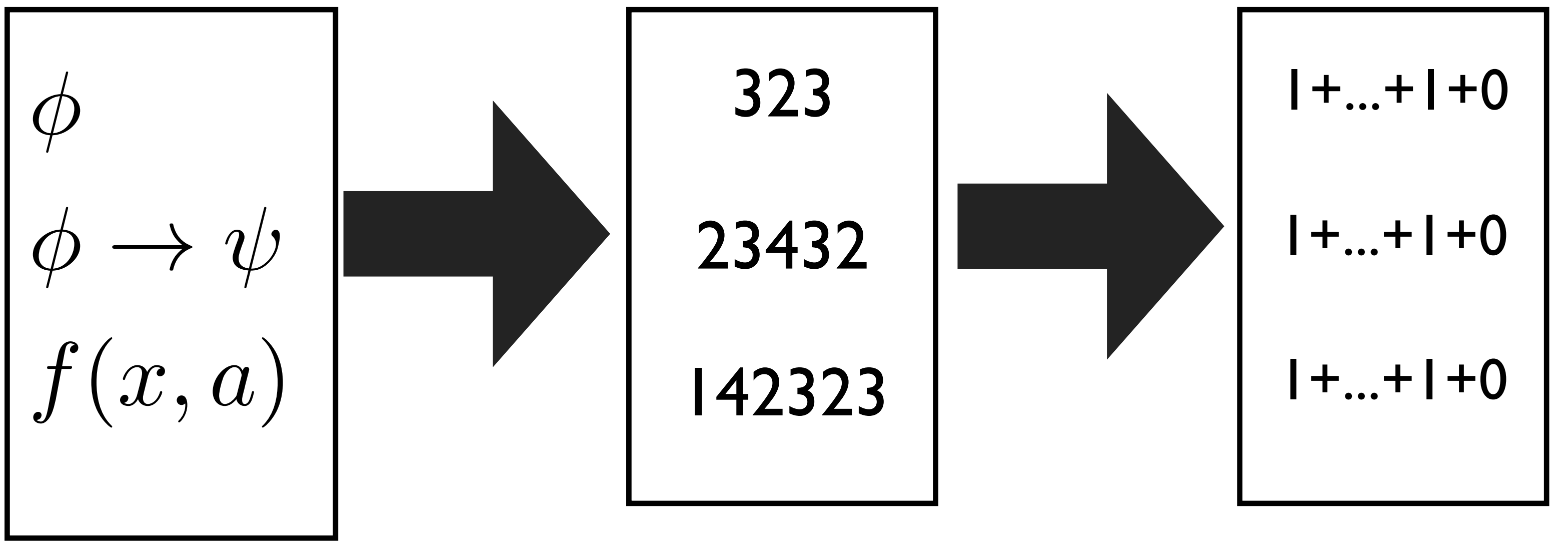

Object-level objects in the language of $\mathscr{L}_1$ (formulae, terms, proofs etc)	Gödel number	Gödel numeral
ϕ	n^{ϕ}	$\hat{n}^{\phi}$ (or just "ϕ")

Gödel Numbering

Problem: How do we enable a formula to refer to other formulae and itself (and also other objects like proofs, terms etc.), in a perfectly consistent way?

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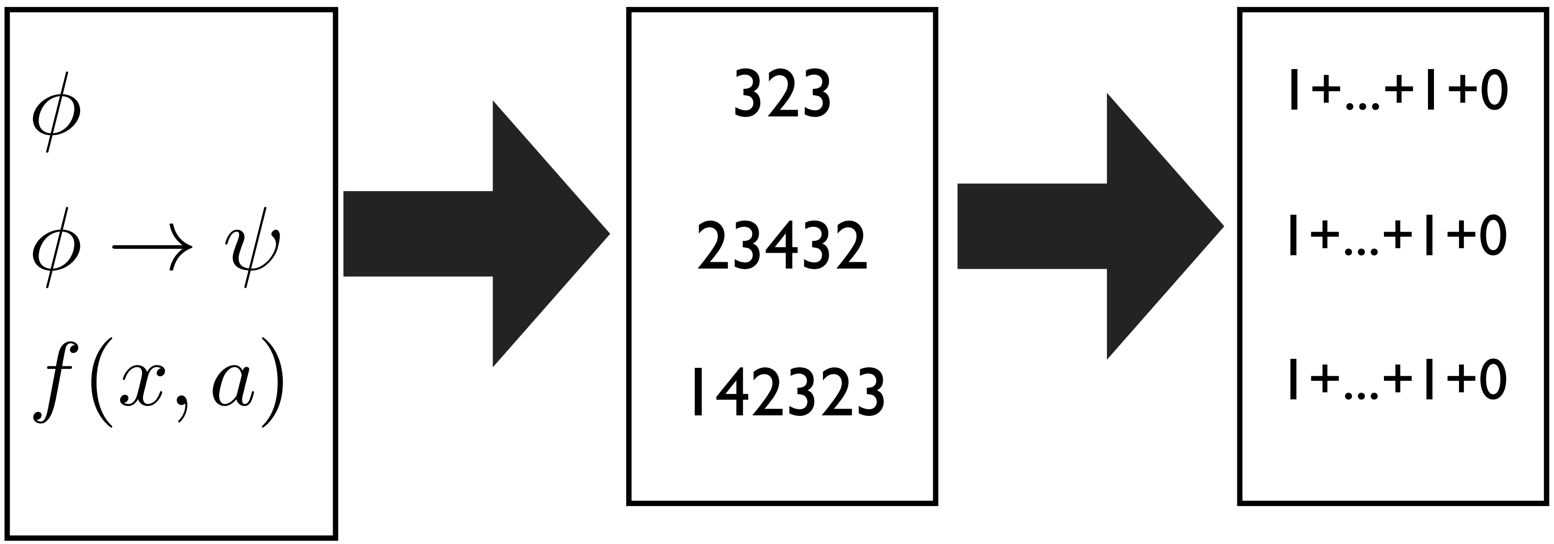

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Object-level objects in the language of $\mathscr{L}_1$		Gödel number		Gödel numeral
ϕ	→	323	→	1+...+1+0
$\phi \to \psi$	→	23432	→	1+...+1+0
$f(x, a)$	→	142323	→	1+...+1+0

(formulae, terms, proofs etc)

ϕ $\quad$ n^{ϕ} $\quad$ $\hat{n}^{\phi}$ (or just "ϕ")

S will sometimes conflate.

Gödel Numbering, the Easy Way

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Just realize that every entry in a dictionary is named by a number *n*, and by the same basic lexicographic ordering, every computer program, formula, etc. is named by a number *m* in a lexicographic ordering going from 1, to 2, to …

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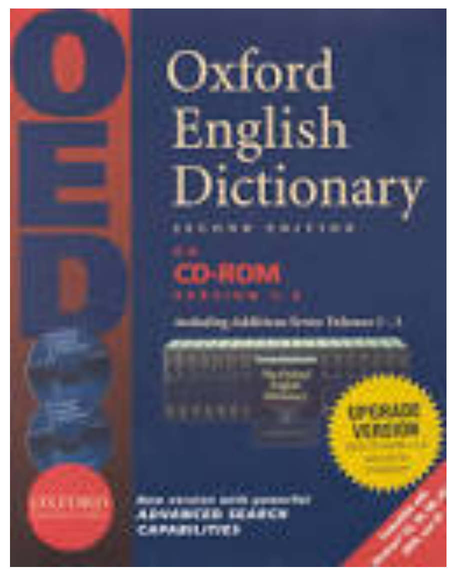

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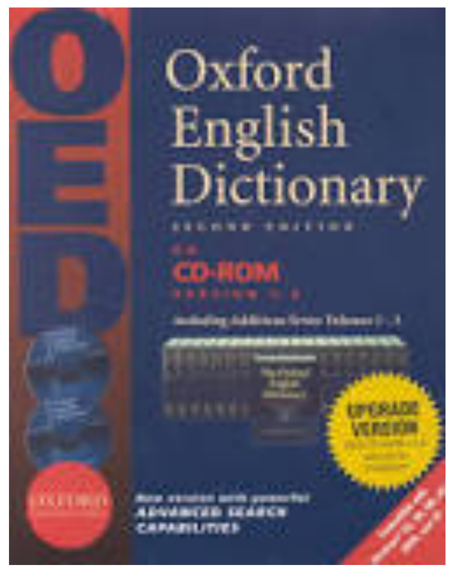

So, `gimcrack` is named by some positive integer *k*. Hence, I can just refer to this word as "*k*" Or in the notation I prefer: k^{gimcrack}.

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Or, every syntactically valid computer program in Clojure that you will ever write can be uniquely denoted by some number *m* in the lexicographic ordering of all syntactically valid such programs. So your program π can just be coded as a numeral m^{π} in a formal language that captures arithmetic (i.e., an *arithmetic language*).

Gödel's First Incompleteness Theorem

Let Φ be a set of arithmetic sentences that is

(i) consistent (i.e. no contradiction $\phi \wedge \neg\phi$ can be deduced from Φ);
(ii) s.t. an algorithm is available to decide whether or not a given string u is a member of Φ; and
(iii) sufficiently expressive to logicize all of the operations of a standard computing machine (e.g. a Turing machine, register machine, KU machine, etc.).

Then there is an "undecidable" arithmetic sentence $\mathcal{G}$ from Gödel that can't be proved from Φ, nor can the negation of this sentence (i.e. $\neg\mathcal{G}$) be proved from Φ!

Alas, that’s painfully verbose.

Gödel's First Incompleteness Theorem

**Sometimes said to be a: "Gödel sentence," or an "undecidable" sentence, or a "mysterious" sentence.

Gödel's First Incompleteness Theorem

Suppose $\Phi \supset \mathbf{PA}$ (= Φ contains **PA**) that is

(i) **Con** Φ;
(ii) Turing-decidable, and
(iii) sufficiently expressive to logicize all of the operations of a Turing machine (i.e. **Repr** Φ).

Then there is an arithmetic *X*** sentence $\mathcal{G}$ s.t. $\Phi \nvdash \mathcal{G}$ and $\Phi \nvdash \neg\mathcal{G}$.

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Gödel's First Incompleteness Theorem

Remember Church's Theorem!

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To prove G1, we shall allow ourselves …

The Fixed Point Theorem (FPT)

“The Self-Ascription Theorem” (GSAT)

Assume that Φ is a set of arithmetic sentences such that **Repr** Φ. Then for every arithmetic formula $\psi(x)$ with one free variable x, there is an arithmetic sentence ϕ s.t.

$\Phi \vdash \phi \leftrightarrow \psi(\hat{n}^{\phi})$.

We can intuitively understand ϕ to be saying:
“I have the property ascribed to me by the formula ψ.”

“I *heard* there was no free lunch!”

[W]e “would hope that such a deep theorem would have an insightful proof. No such luck. I am going to write down a sentence … and verify that it works. What I won’t do is give you a satisfactory explanation for why I write down the particular formula that I do. I write down the formula because Gödel wrote down the formula, and Gödel wrote down the formula because, when he played the logic game he was able to see seven or eight moves ahead, whereas you and I are only able to see one or two moves ahead. I don’t know anyone who thinks he has a fully satisfying understanding of why the Self-referential Lemma [= FPT] works. It has a rabbit-out-of-a-hat quality for everyone.”

—V. McGee, 2002; as quoted in (Salehi 2020)

The Fixed Point Theorem (FPT)

"The Self-Ascription Template Theorem"

Assume that Φ is a set of arithmetic sentences such that **Repr** Φ. There for every arithmetic formula $\psi(x)$ with one free variable x, there is an arithmetic sentence ϕ s.t.

$\Phi \vdash \phi \leftrightarrow \psi(\hat{n}^{\phi})$.

We can intuitively understand ϕ to be saying:
"I have the property ascribed to me by the formula ψ."

Ok; so let's do it …

Proof: Let Φ be a set of arithmetic sentences, and suppose (for conditional intro) the antecedent of **G1** holds, i.e. (i)–(iii) hold. We must show that there exits an arithmetic sentence s.t. neither it nor the negation of this (Liar-Paradox-inspired) arithmetic sentence can be proved from Φ. In homage to Gödel, we shall label this sentence '$\mathscr{G}$.'

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For *reductio*, assume the opposite; i.e. that it's *not* the case that there exists … . By quantifier shift, and propositional logic, we deduce that for very arithmetic sentence ϕ, either $\Phi \vdash \phi$ or $\Phi \vdash \neg\phi$. From this, combined with the active supposition that Φ enables logicization, we can let the key formula $\pi(v)$ in (GSAT) logicize provability in Φ. Here, then, we would instantiate the variable v to the Gödel numeral $\hat{n}^{\phi}$ of a provable formula ϕ. We thus have:

$$(1) \quad \Phi \vdash \pi(\hat{n}^{\phi}) \text{ iff } \Phi \vdash \phi\,.$$

Now let us bring on stage Gödel's self-ascribing arithmetic sentence $\mathscr{G}$, via his Self-Ascription Theorem; specifically we have:

$$(3) \quad \Phi \vdash \mathscr{G} \leftrightarrow \neg\pi(\hat{n}^{\mathscr{G}})\,.$$

We have two cases to consider, just like what we did in the original Liar Paradox. The first is that $\mathscr{G}$ is provable from Φ; suppose this holds. Then from (1) right-to-left, and *modus ponens* = conditional elim, with left-to-right (by biconditional elimination) on (3) — contradiction!

How about the second case, viz. that $\Phi \vdash \neg\mathscr{G}$? With this and (3) by *modus tollens* and simplification of a double negation we deduce $\Phi \vdash \pi(\hat{n}^{\mathscr{G}})$, which with (1) by biconditional elimination yields $\Phi \vdash \mathscr{G}$ — contradiction!, which is ruled out by the assumption that Φ is consistent. **QED**

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For *reductio*, assume the opposite; i.e. that it's *not* the case that there exists … . By quantifier shift, and propositional logic, we deduce that for very arithmetic sentence ϕ, either $\mathbf{\Phi} \vdash \phi$ or $\mathbf{\Phi} \vdash \neg\phi$. From this, combined with the active supposition that $\mathbf{\Phi}$ enables logicization, we can let the key formula $\pi(v)$ in (GSAT) logicize provability in $\mathbf{\Phi}$. Here, then, we would instantiate the variable v to the Gödel numeral $\hat{n}^{\phi}$ of a provable formula ϕ. We thus have:

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"I'm unprovable!"

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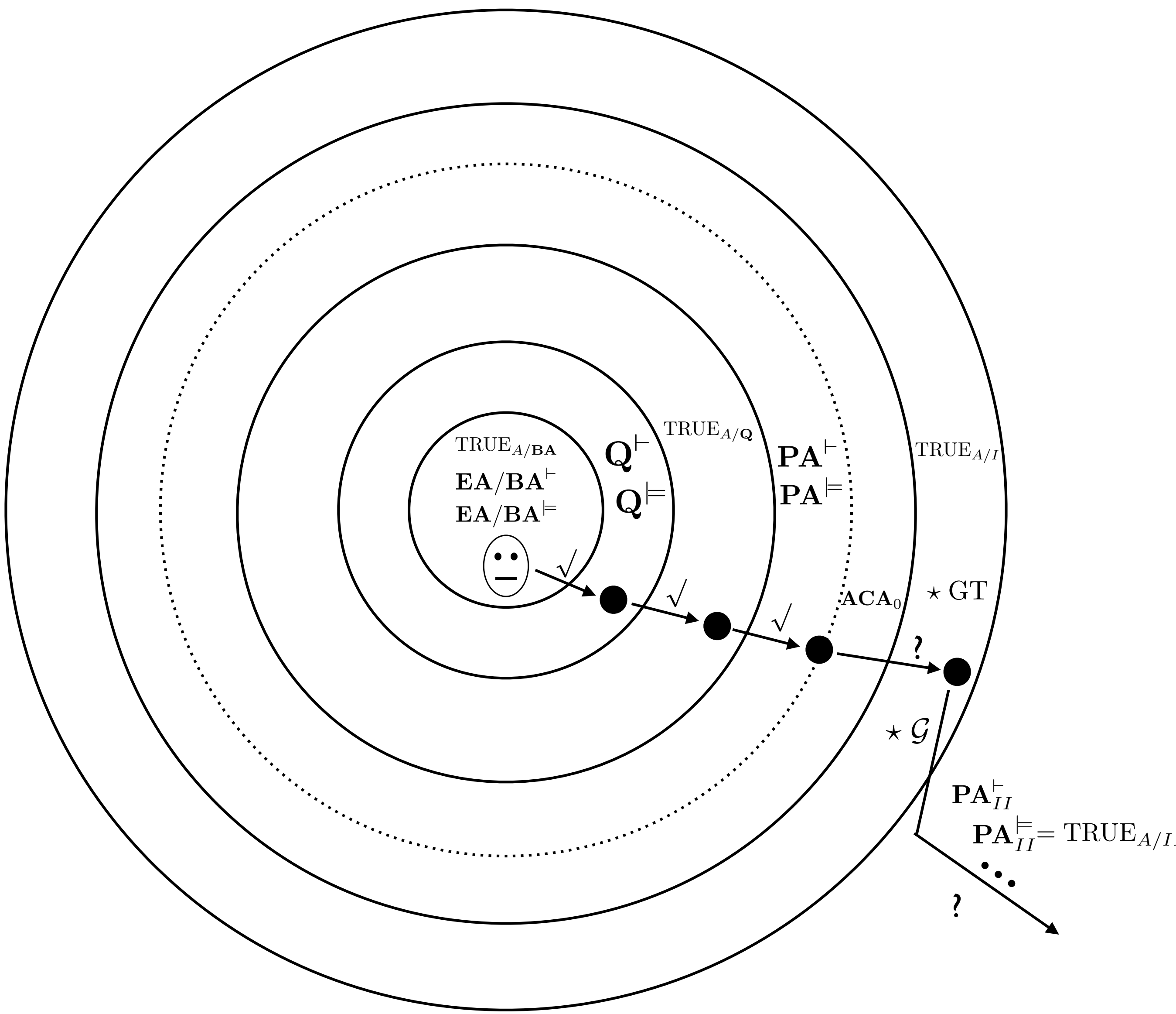
“Silly abstract nonsense! There aren’t any concrete examples of $\mathscr{G}$!”

Astrologic:
Rational Aliens Will be on the Same “Race Track”!



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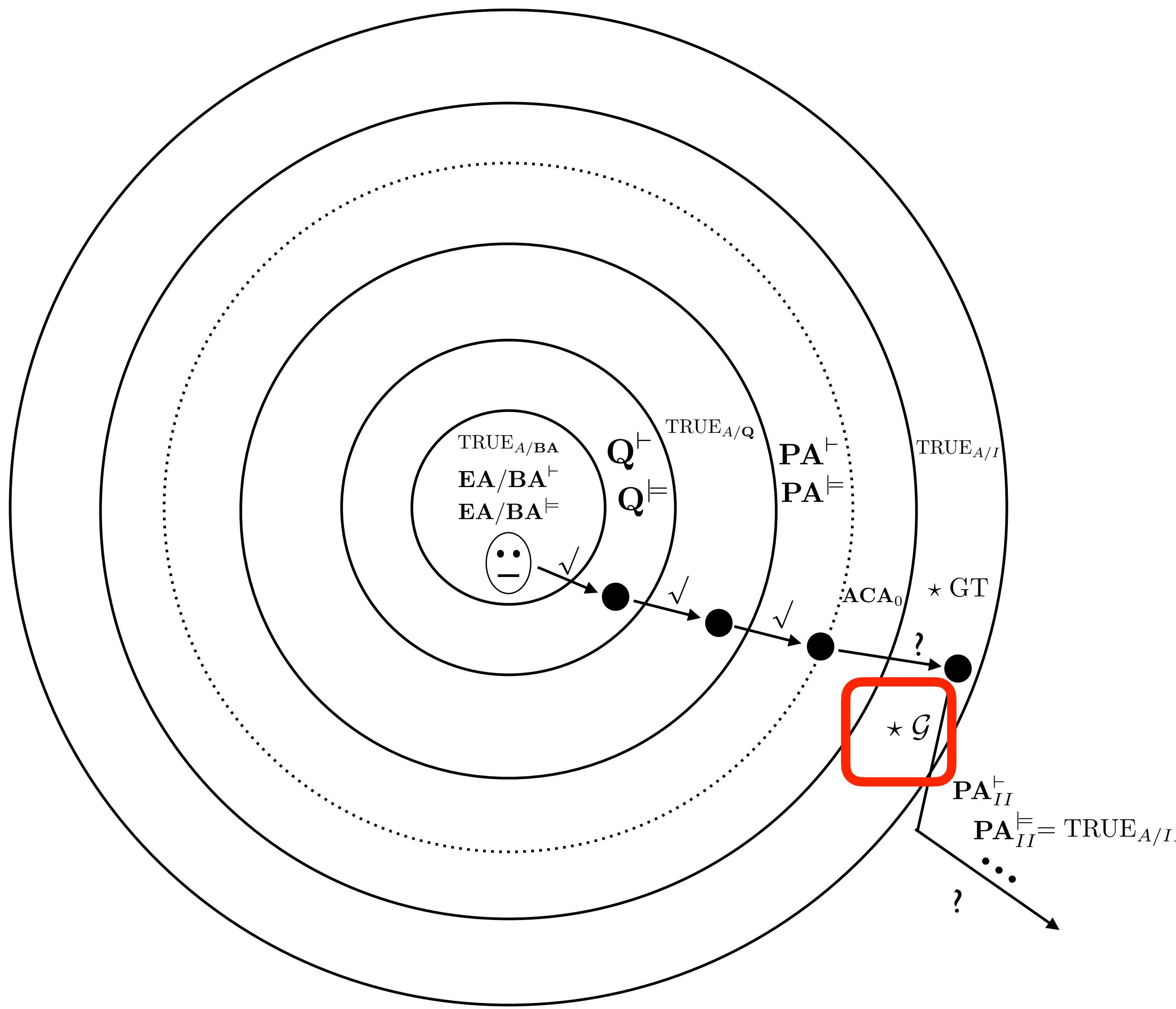

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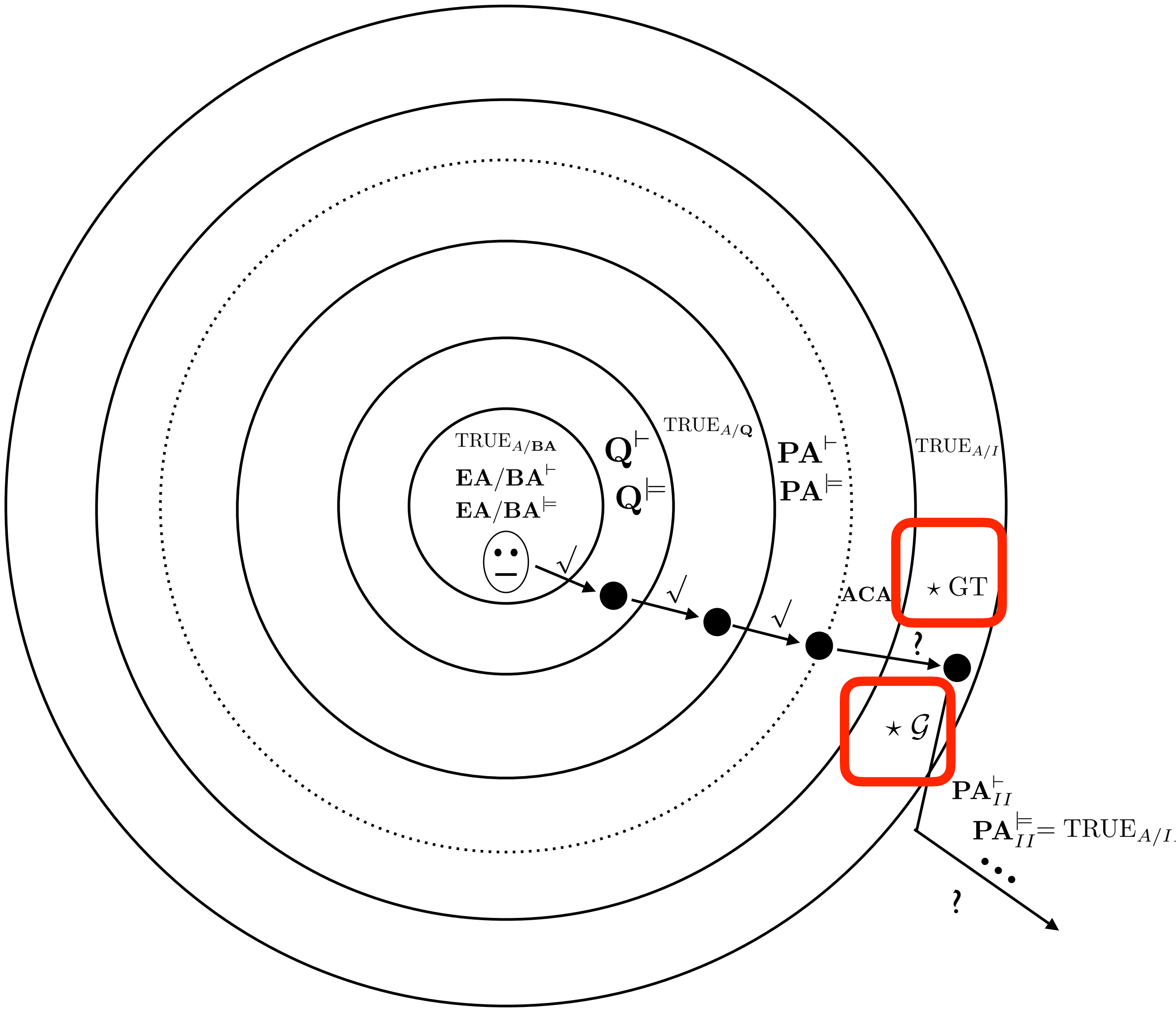

Ah, but e.g.: Goodstein's Theorem!

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The Goodstein Sequence goes to zero!

Pure base *n* representation of a number *r*

- Represent *r* as only sum of powers of *n* in which the exponents are also powers of *n*, etc.

$$266 = 2^{2^{(2^{2^0}+2^0)}} + 2^{(2^{2^0}+2^0)} + 2^{2^0}$$

Grow Function

$Grow_k(n)$:

1. Take the pure base k representation of n
2. Replace all k by $k+1$. Compute the number obtained.
3. Subtract one from the number

Example of **Grow**

$$Grow_2(19)$$

$$19 = 2^{2^{2^{2^0}}} + 2^{2^0} + 2^0$$

$$3^{3^{3^{3^0}}} + 3^{3^0} + 3^0$$

$$3^{3^{3^{3^0}}} + 3^{3^0} + 3^0 - 1$$

$$7625597484990$$

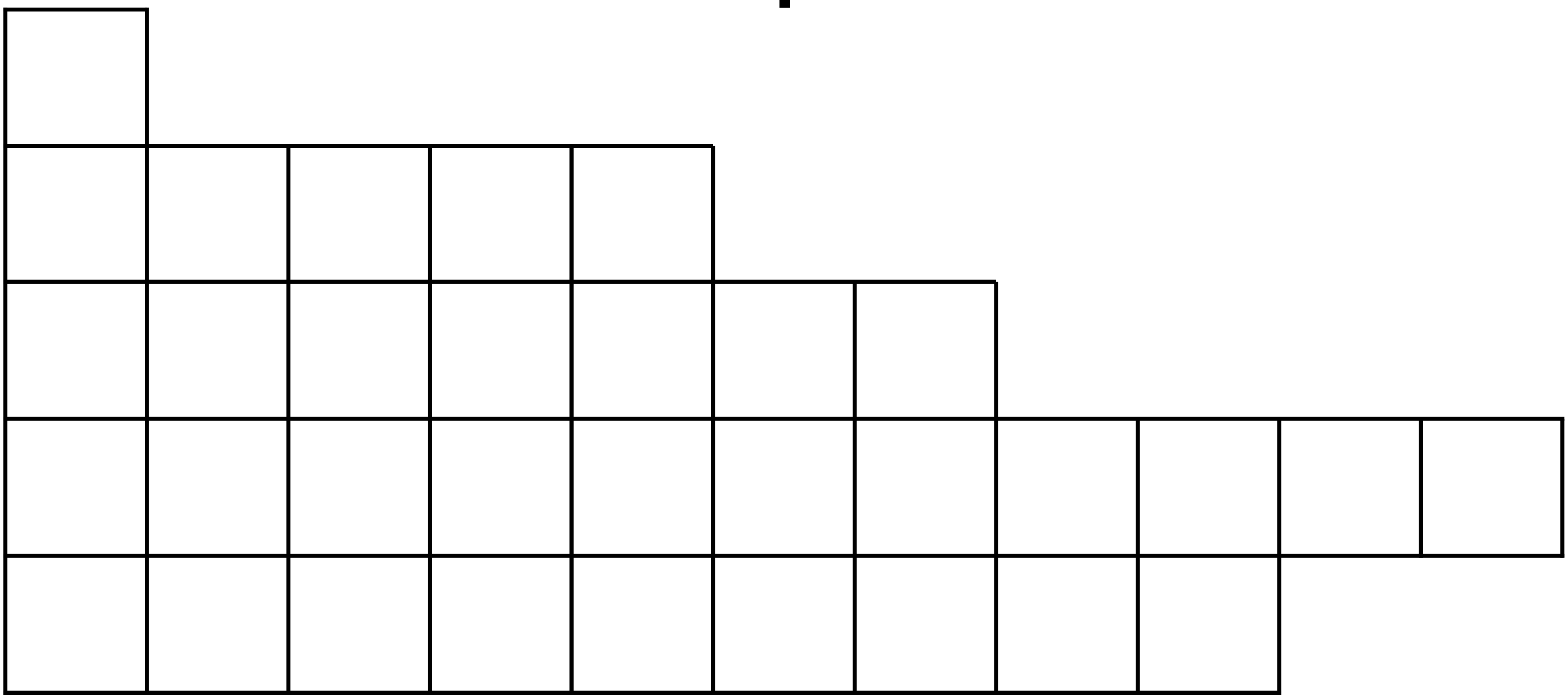
Goodstein Sequence

- For any natural number *m*

$$
\begin{aligned}
&m \\
&Grow_2(m) \\
&Grow_3(Grow_2(m)) \\
&Grow_4(Grow_3(Grow_2(m))), \\
&\ldots
\end{aligned}
$$

Sample Values

Sample Values



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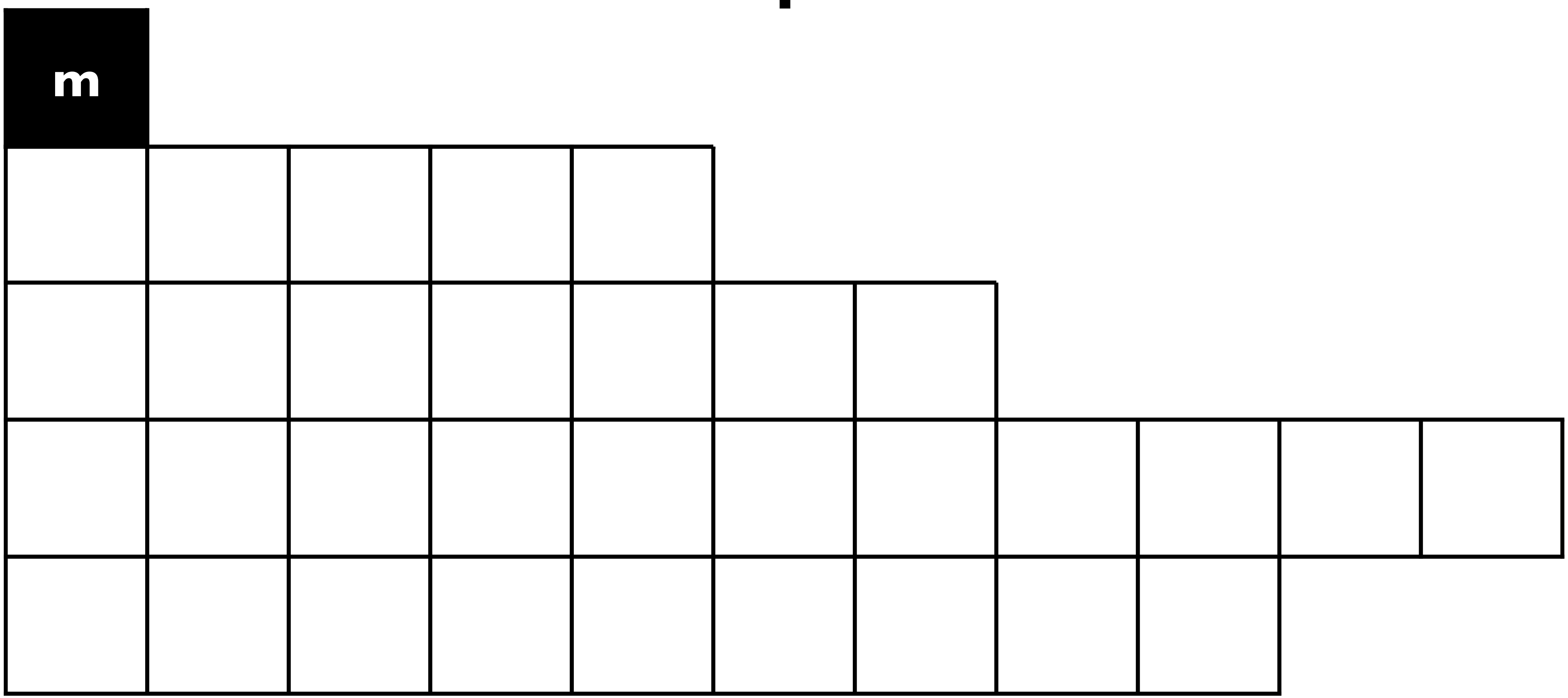

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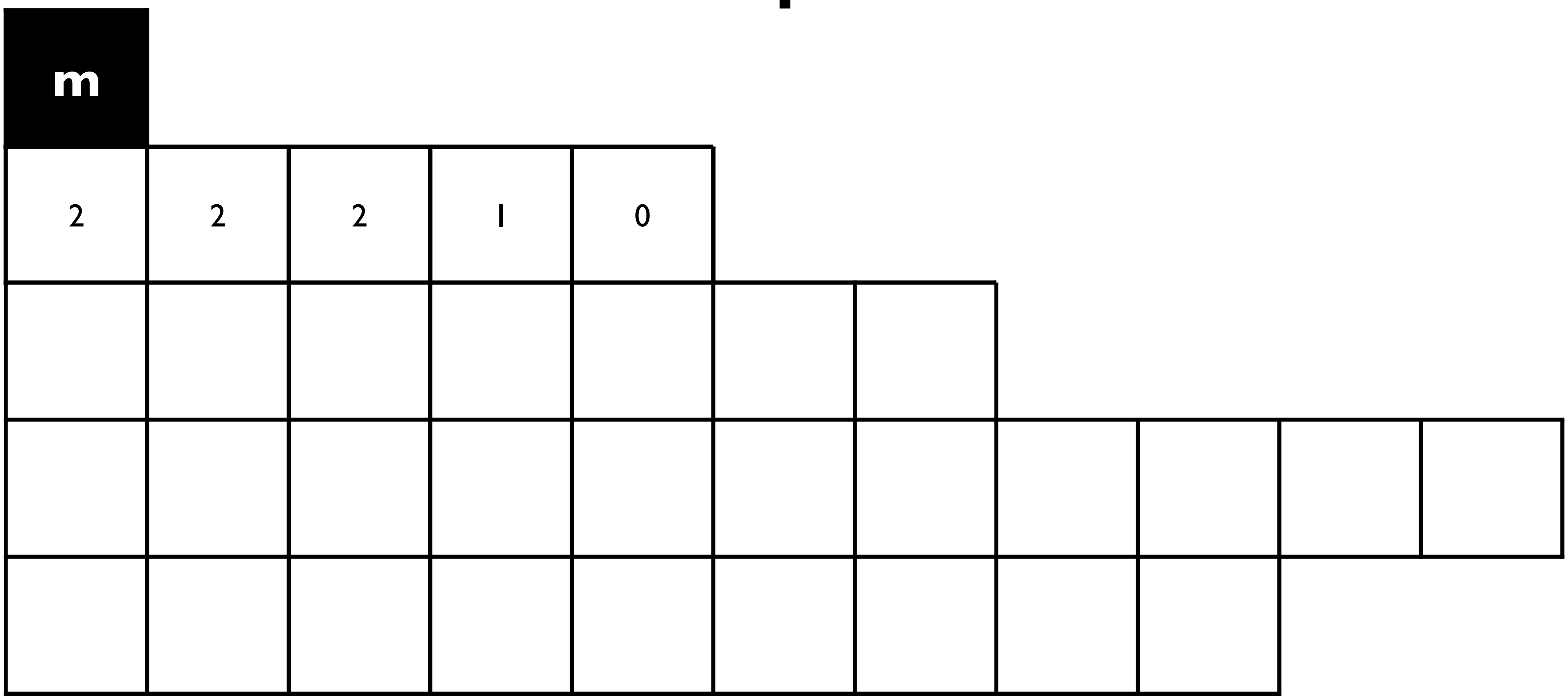

Sample Values

m										
2	2	2	1	0						
3	3	3	3	2	1	0				

Sample Values

m										
2	2	2	1	0						
3	3	3	3	2	1	0				
4	4	26	41	60	83	109	139	...	11327 (96th term)	...

Sample Values

m											
2	2	2	1	0							
3	3	3	3	2	1	0					
4	4	26	41	60	83	109	139	...	11327 (96th term)	...	
5	15	$\sim 10^{13}$	$\sim 10^{155}$	$\sim 10^{2185}$	$\sim 10^{36306}$	10^{695975}	$10^{15151337}$	...			

This sequence actually goes to zero!

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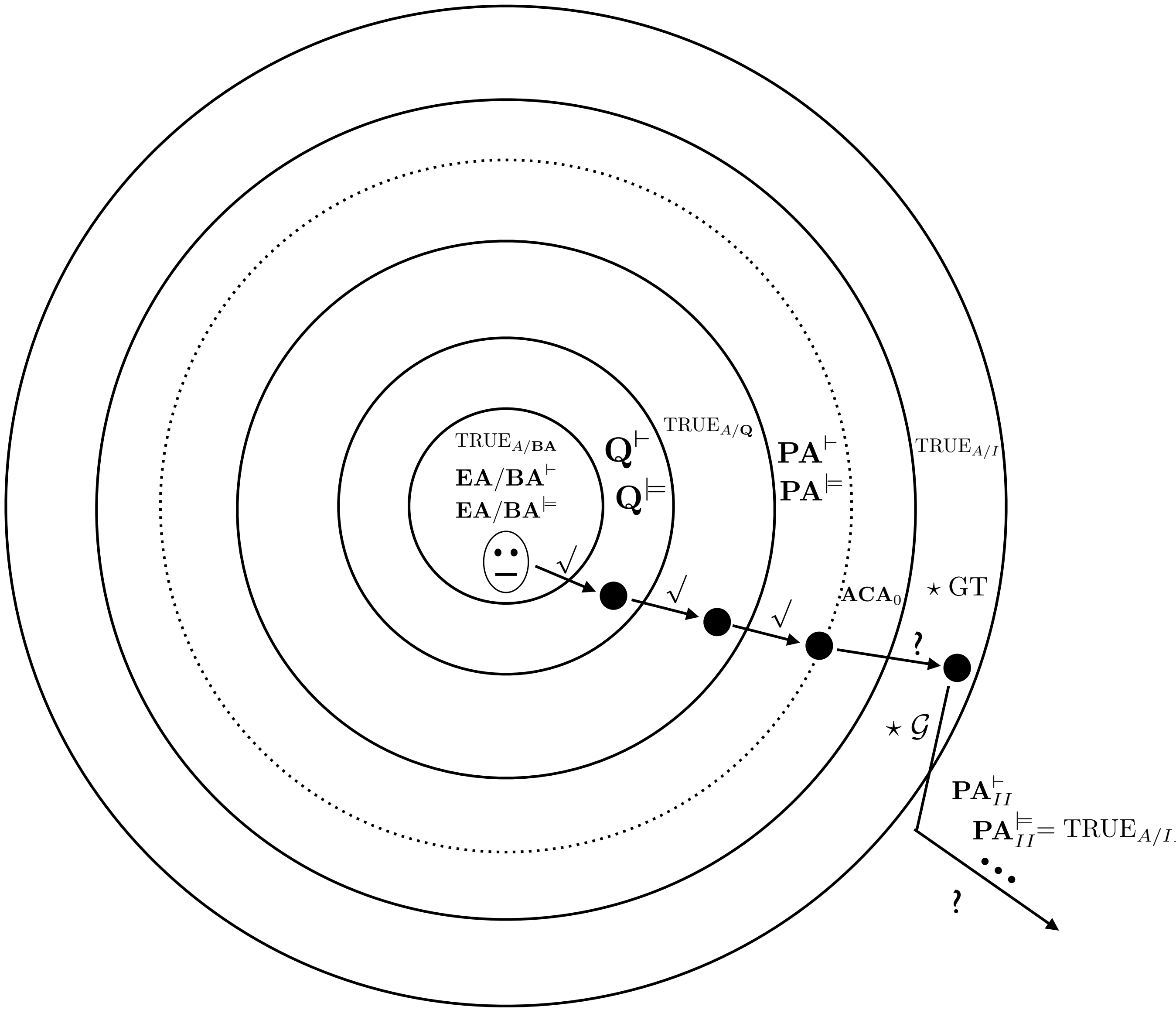

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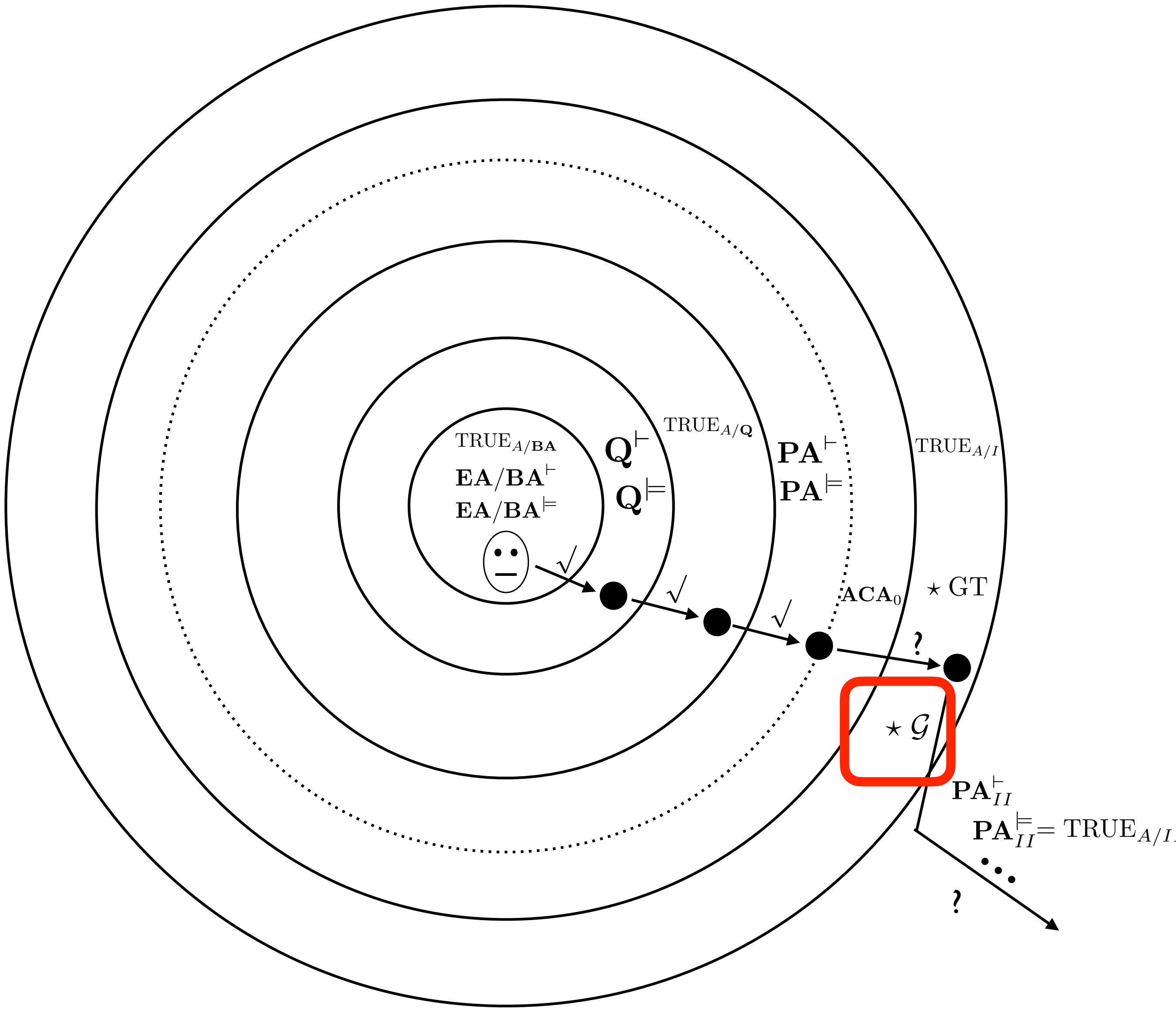

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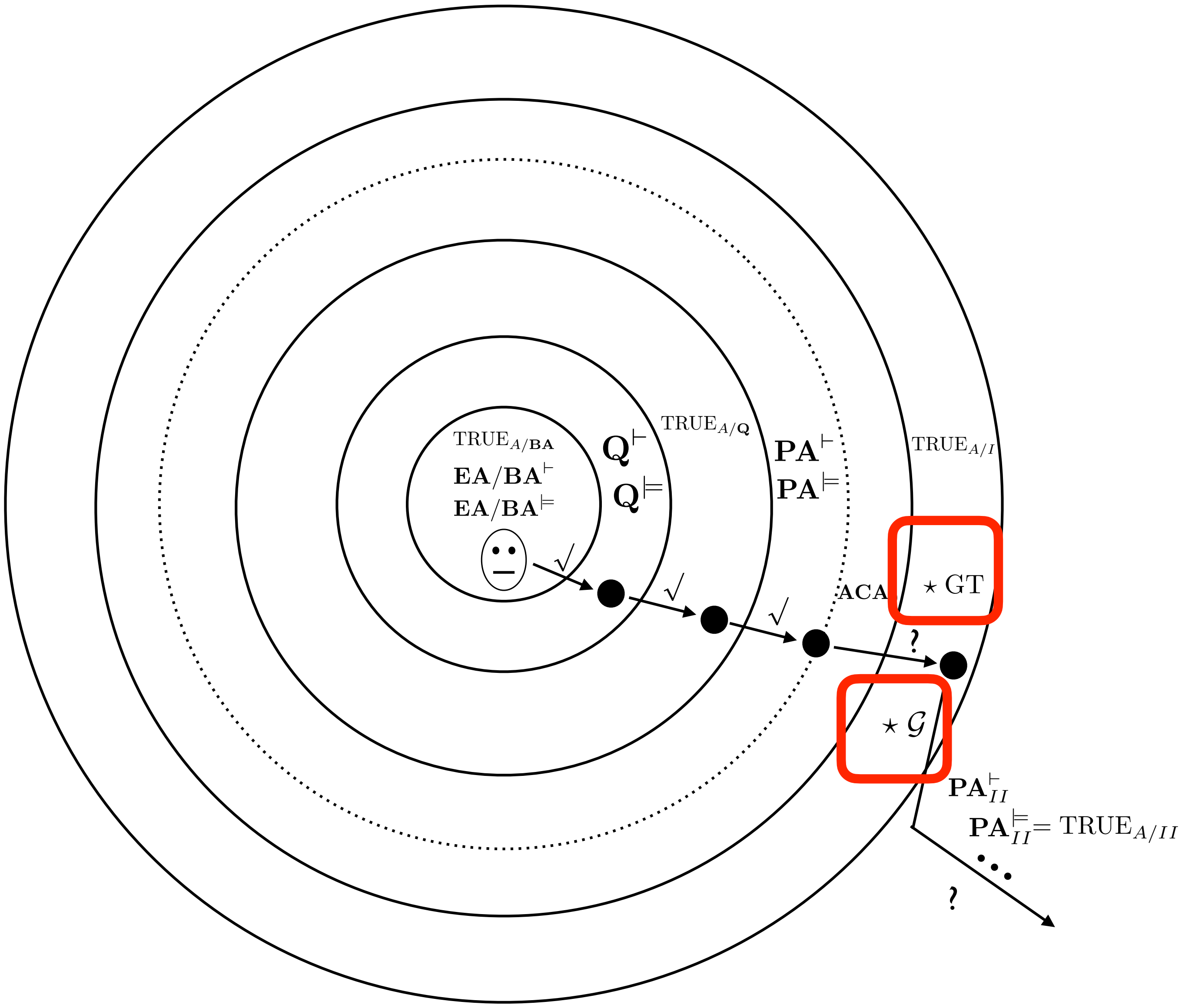

Could an AI Ever Match Gödel's G1 & G2?

Gödel's Great Theorems (OUP)

by Selmer Bringsjord

- Introduction ("The Wager")
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
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Med nok penger, kan logikk løse alle problemer.