Natural Deduction Proof Strategies In HyperSlate®

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HyperSlate® Terms

Definition (Given): A *given* is an assumption you are allowed to use to prove a goal, it is allowed to appear on the “from:” section of your goal.

Definition (Derived): Anything formula you build from your assumptions and givens from the top down.

Definition (Theorem): A theorem is something you can prove with no givens. Its “from:” section will be empty e.g. “from:{}”

Definition (Goal): a *goal* is something we want to prove, it may be a theorem (that depends on no assumptions) or it may depend on *given assumptions.*
It will always be the bottom node in a proof.

Definition (Subgoal)*:* A *subgoal* is a smaller goal produced by breaking the goal down using (typically introduction) rules.

Common Proof Finding Heuristics and Strategies

1. Work top down and bottom up, meet in the middle
2. Try proof by cases when given a disjunction
3. Prove conjunction goals by splitting them into two subgoals
4. Prove disjunction goals by selecting one disjunct as a subgoal
5. Prove implications by assuming and the antecedent and proving the consequent
6. Introduce any assumption on any node
7. Prove negations by assuming the positive and deriving a contradiction
8. Prove biconditionals by assuming each side and proving the other side
9. Try proof by contradiction
10. See if your formulae match any common subproofs

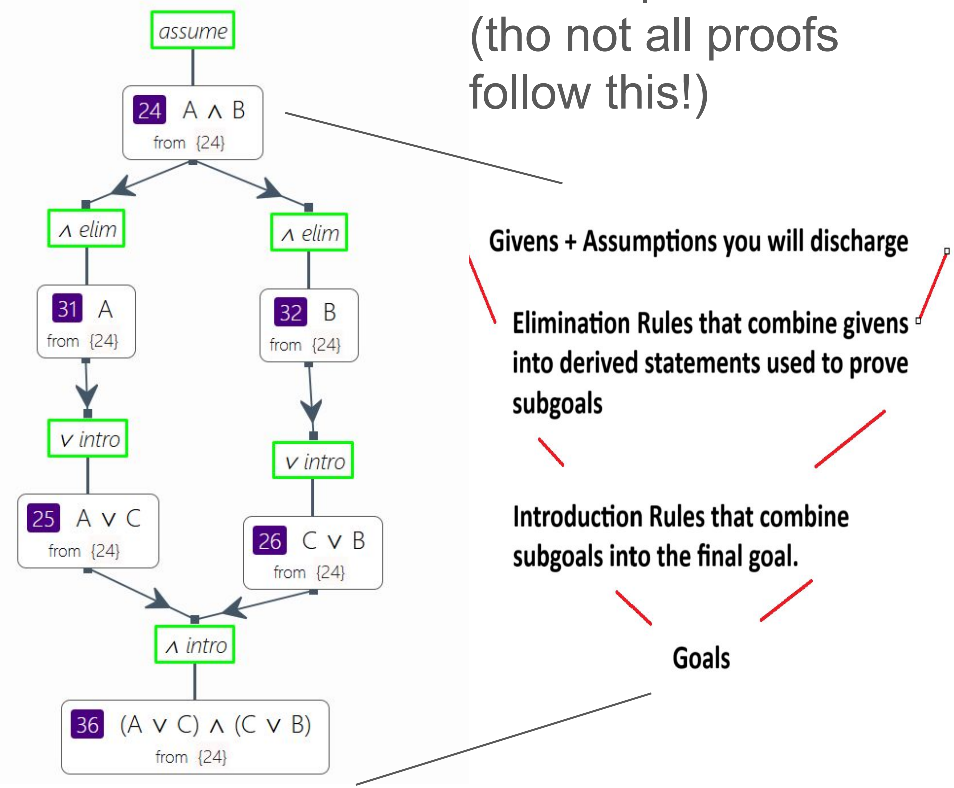
(Examples on the following slides)

General Tips

- Work both top down AND bottom up, meet in the middle!!!!
  - Use elimination rules from the top to destruct assumptions and givens.
  - Use Introduction rules from the bottom to destruct goals into multiple simpler goals.

- Goal is a theorem that can have no assumptions / You have no givens?
  - You have to work from the bottom up, you can work top down after making your own assumptions that you will later discard.

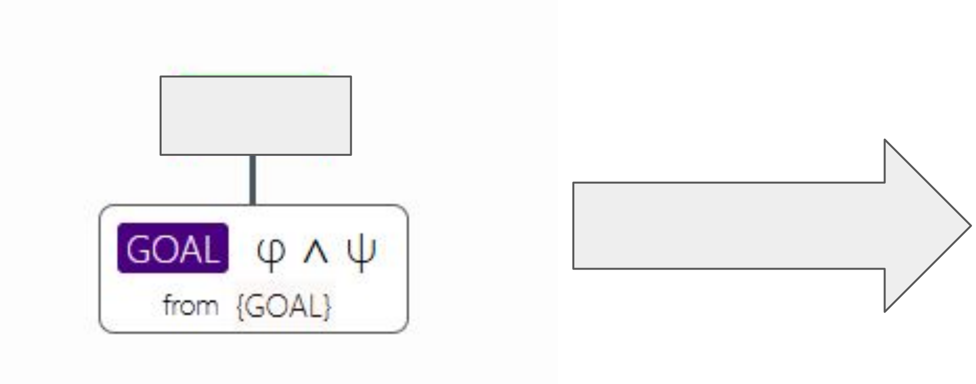
A prototypical proof and subproof structure (tho not all proofs follow this!)

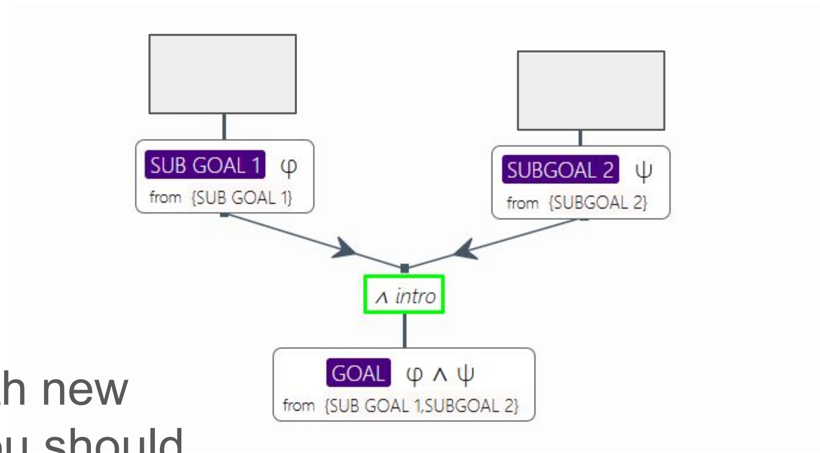


Proving Conjunction Goals (Bottom Up)

If a goal is a conjunction, use conjunction introduction to split it into two subgoals for you to prove!

You can now use the subgoals as your new goals.

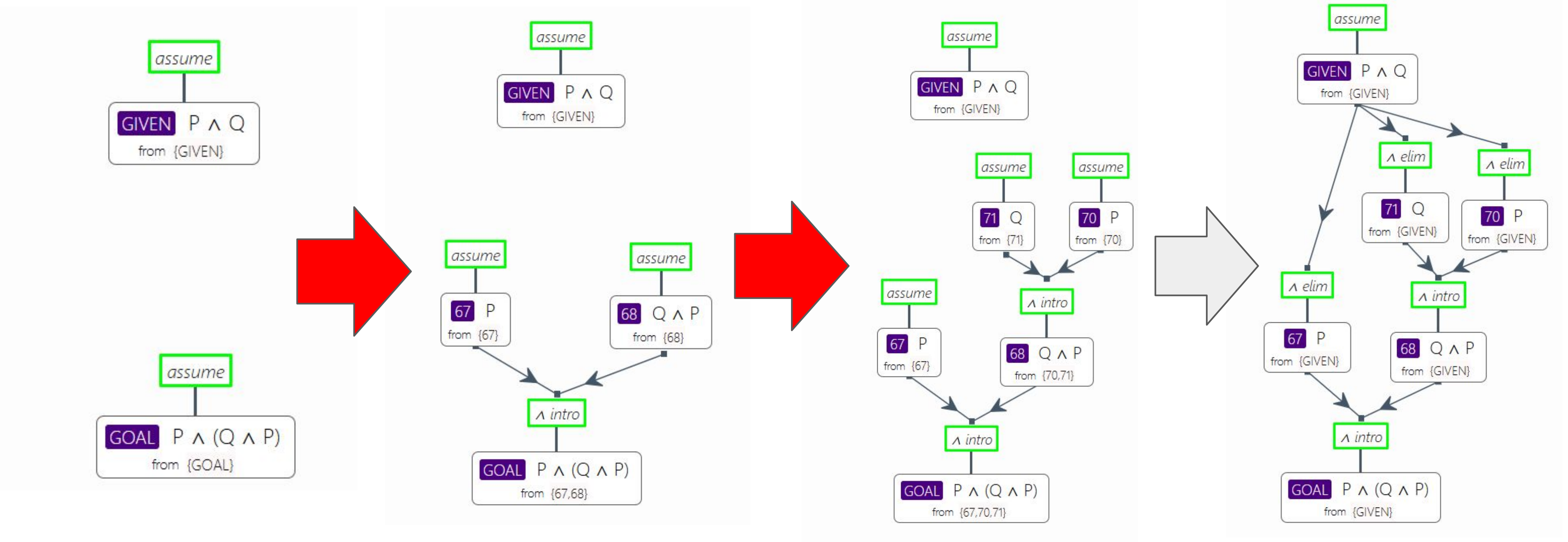




HINT: Use PC Oracle to check to make sure both new subgoals can be proven from GIVENS. If not, you should try another technique (like proof by contradiction).

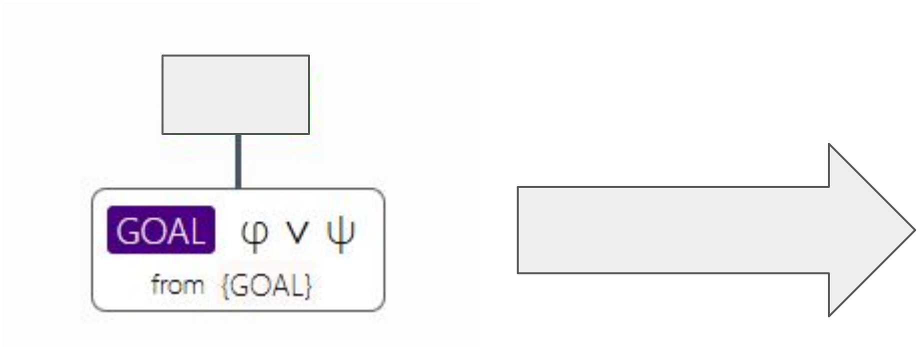
Proving Conjunction Goals Example

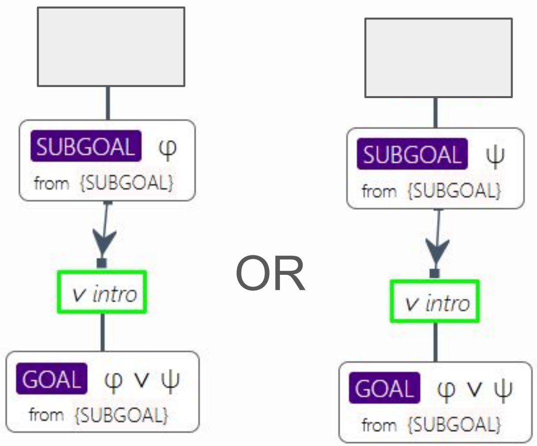
Given P and Q prove P and (Q and P)



Proving Disjunction Goals (Bottom Up)

If a goal is a disjunction, use disjunction introduction to pick one of the sides as a subgoal! You should pick the side intelligently! pick the side that looks easier to prove as your new subgoal, if you pick wrong it may be impossible to prove!





HINT: Use PC Oracle to check if you picked the correct side as a subgoal! Neither branch works? try proof by contradiction (or cases if given a disjunction).

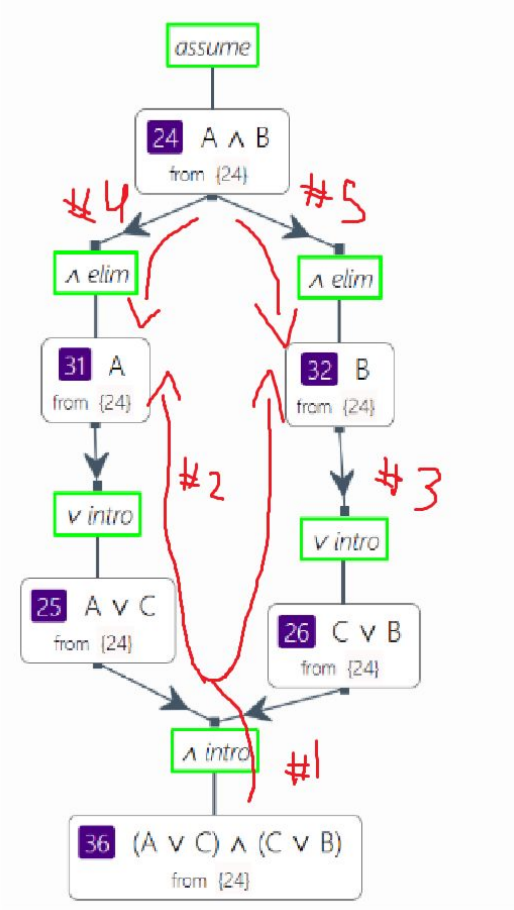
Proving Disjunction Goals Example

A and B proves (A or C) and (C or B)

Start by splitting our goal into subgoals with and intro.

Now split each subgoal into something we can prove from A and B. Choose A, B respectively (can't prove C from A and B, so don't chose it).

Now get A, B from A and B with conjunction elim.



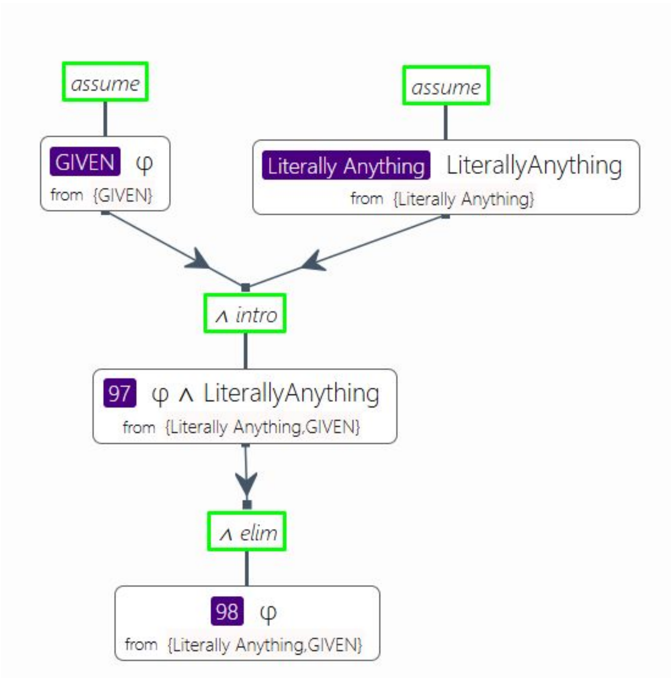
Introduce Any Assumption On Any Node

Many other rules (=> intro, ¬ intro, ¬elim, <=> intro) require a specific premise to be in the set of assumptions.

You can add ANY premice into a set of assumptions of any node using this technique.

This pattern in ND is equivalent to a derived rule called weakening:

$$\frac{\Gamma \vdash C}{\Gamma, A \vdash C}$$



NL Perspective on Inserting Assumptions

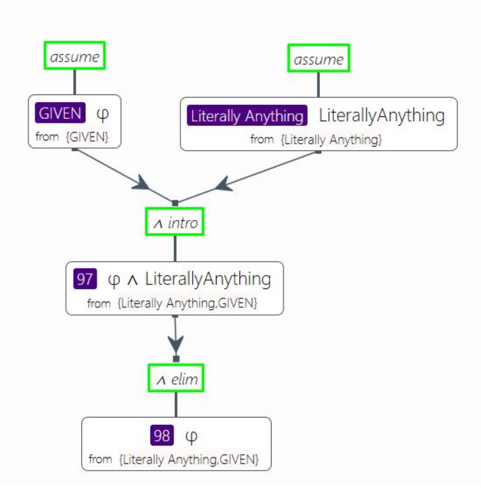
If we have

“A is provable from B” then

“A is provable from B and C”

Adding the extra assumption does not change the fact that we can use B to get A.

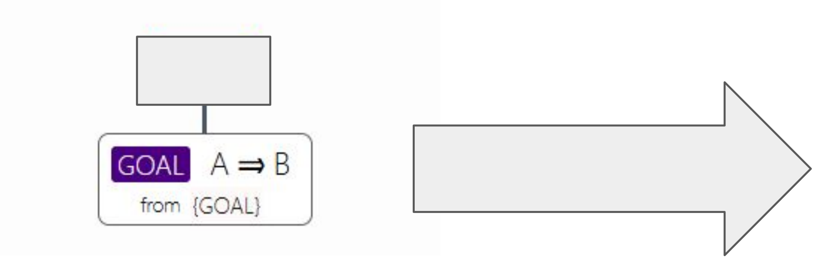
Because of this rule ND is *monotonic*. Monotonic logics are any logics where weakening is *admissible*.

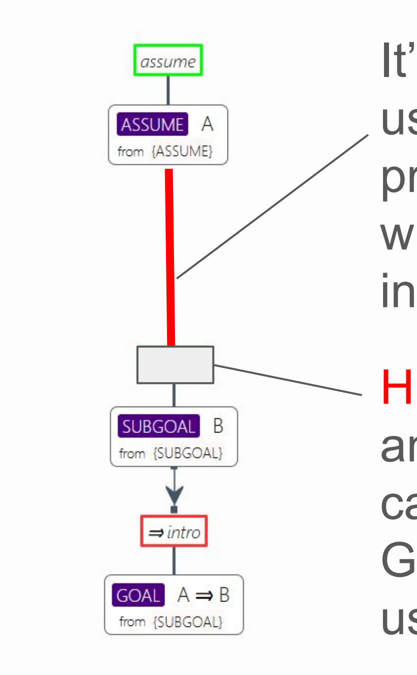


$$\frac{\Gamma \vdash C}{\Gamma, A \vdash C}$$

Proving Implication Goals (Bottom Up + Top Down)

If a goal is an implication (A => B), assume the antecedent (A) and prove the consequent (B). The assumption will be passed down your proof chain and discharged with implication (if) intro.





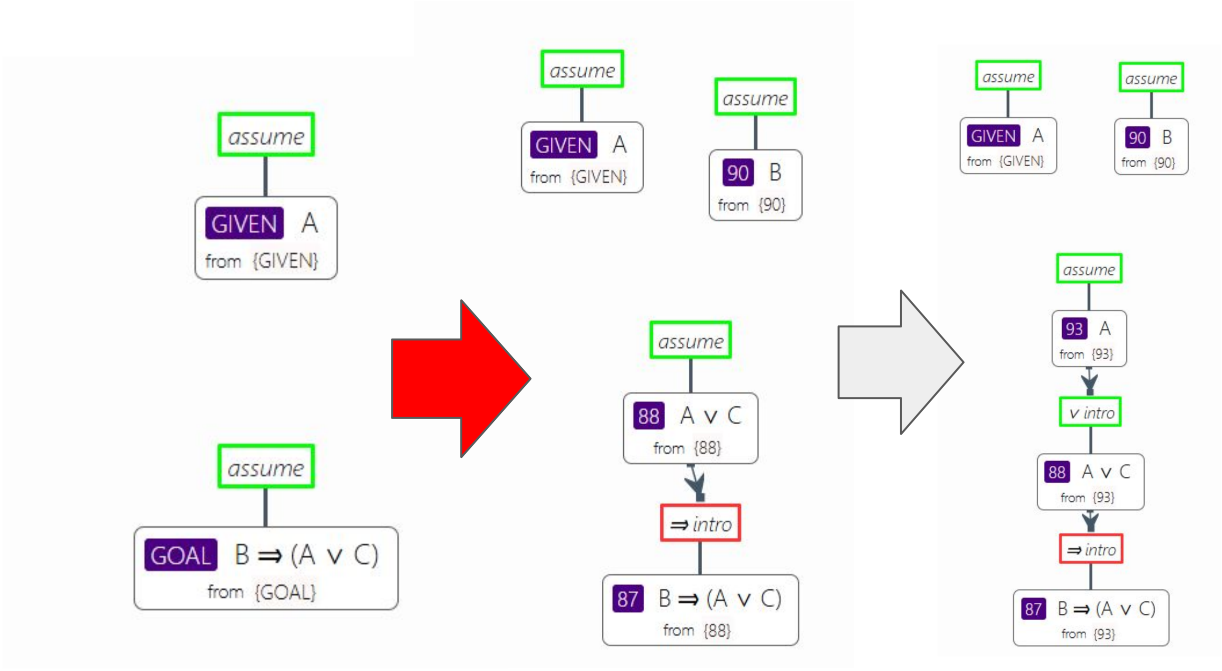
It's your job now to use A + GIVENS to prove B. Doing so will make the "=> intro" green.

HINT: Start with PC oracle and make sure subgoal can be proven from GIVENS + ASSUME, if not, use a dif technique.

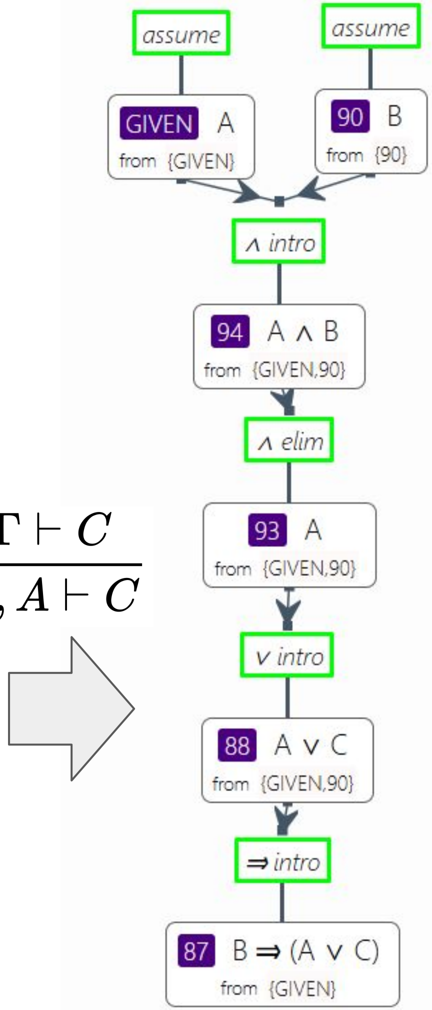
Unsure why i used A and B here,
use φ and ψ if that's easier to think about.

Proving Implication Goals Example

You will *frequently* need this weakening technique where you add an arbitrary assumption to a node.

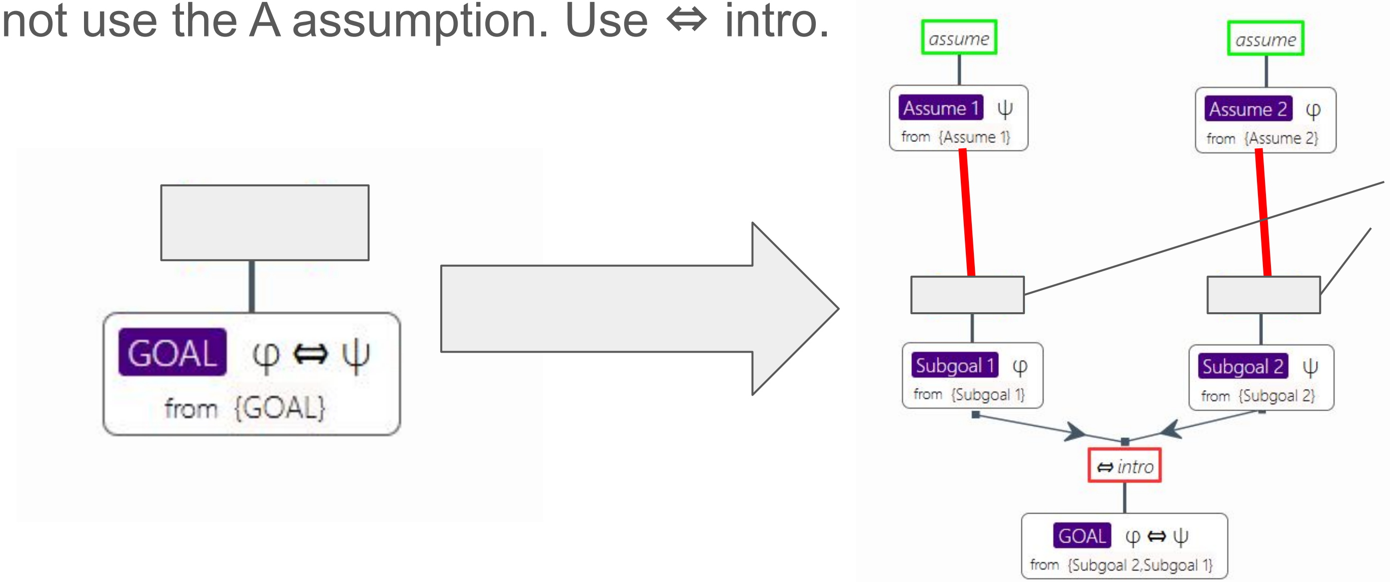


$$\frac{\Gamma \vdash C}{\Gamma, A \vdash C}$$



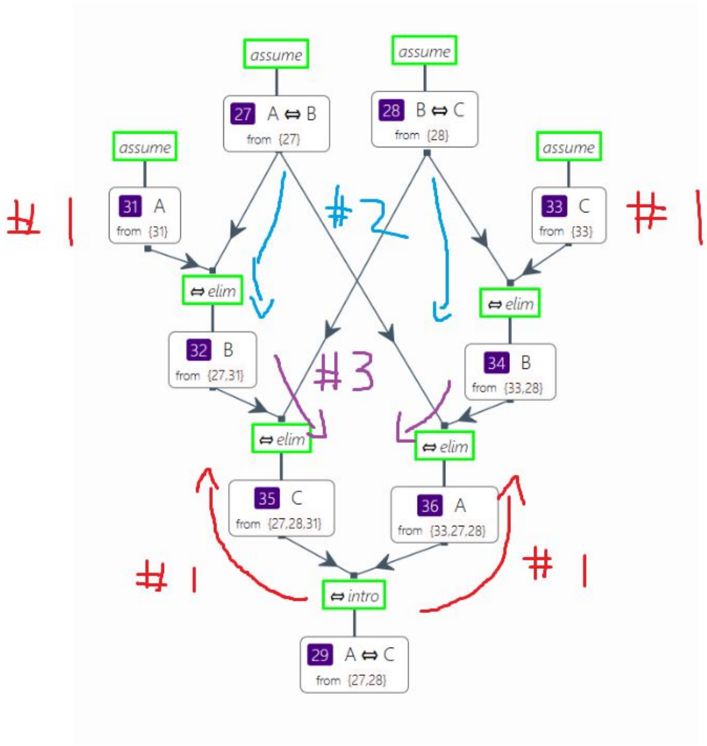
Proving Iff Goals (Bottom Up + Top Down)

If a goal is a biconditional (A ⇔ B), assume A and prove B as one proof, then assume B and prove A as a separate proofs. The first proof (proving B from A) may not use the B assumption and the second proof (proving A from the B) may not use the A assumption. Use ⇔ intro.



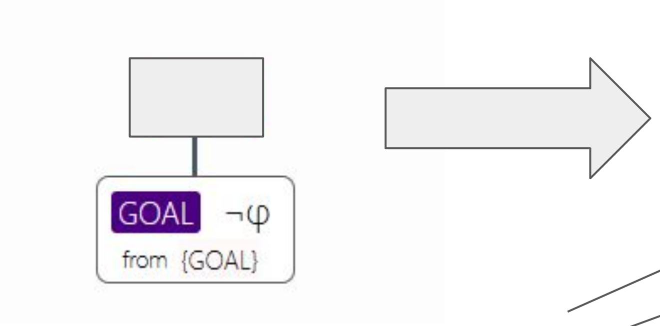
HINT: Start with PC oracle and make sure subgoals can be proven from GIVENS + respective assume, if not, use a dif technique.

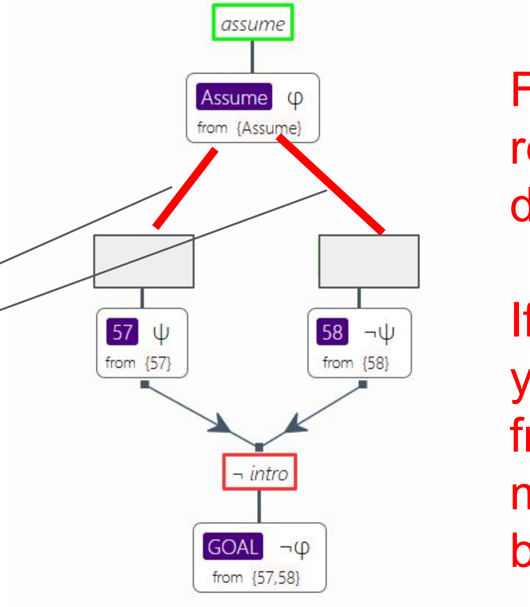
Proving Iff Goals Example



Proving Negated Goals (Bottom Up + Top Down)

If a goal is negated (¬φ), assume its non-negated form (φ). Using other givens and φ (or sometimes by directly using φ if there are no other implications) prove a contradiction and apply not intro.



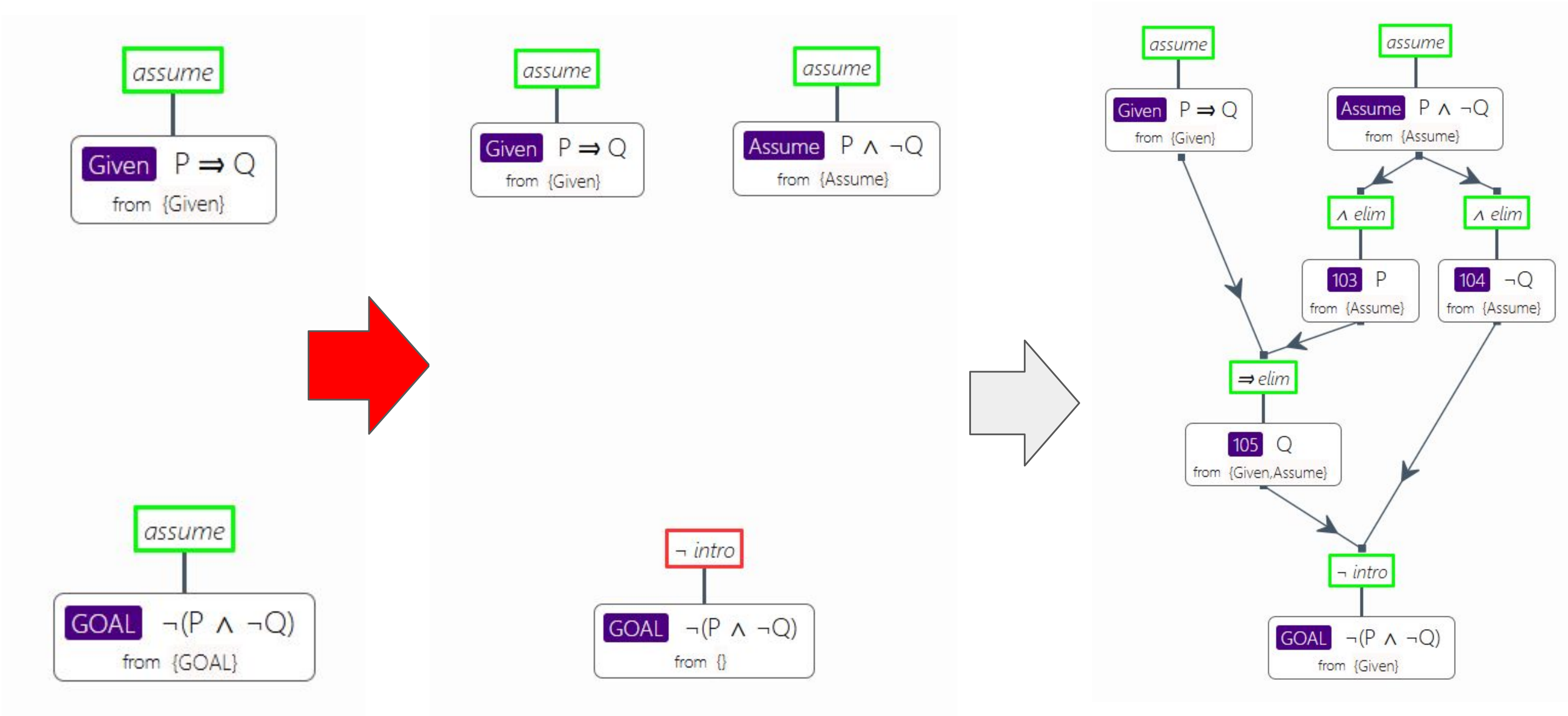


Use φ and other givens to prove two formulae: ψ and ¬ψ, which you can use to apply not introduction.

Finding a ψ may require deep reasoning about what you can derive from φ and other givens.

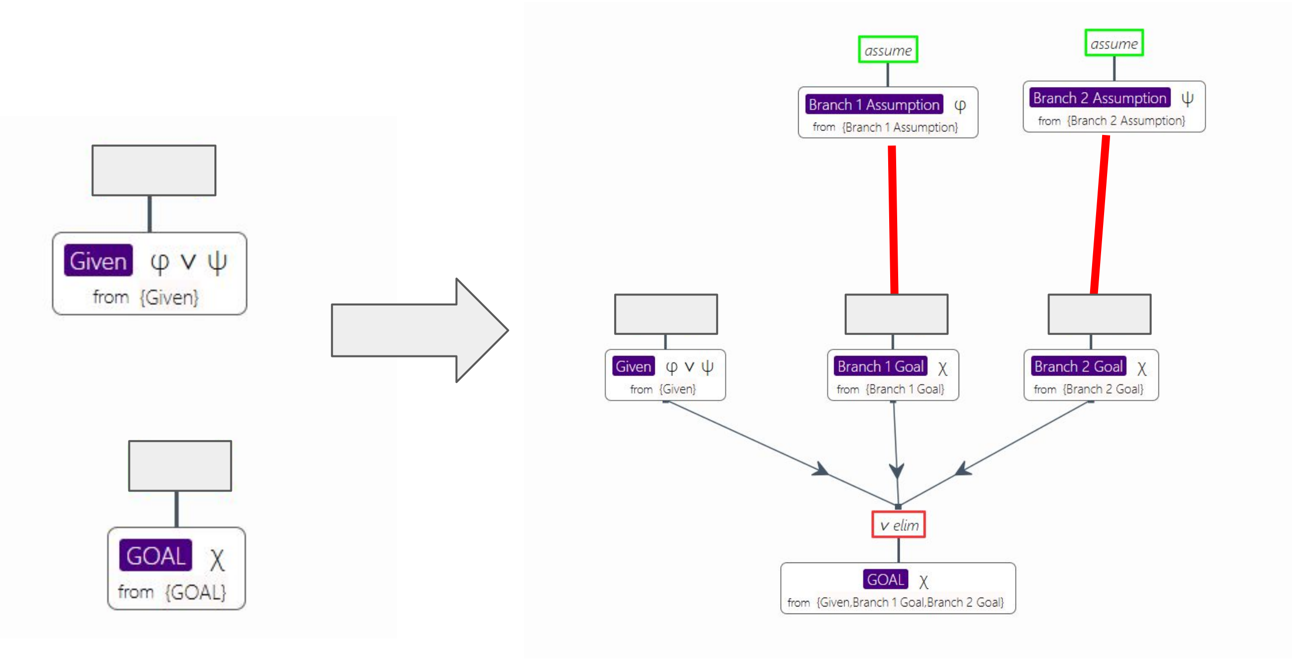
If you are lucky ψ will be one of your givens or easily derivable from it. (this is what happens most of the time, but should not be counted on)

Proving Negated Goals Example



Proof By Cases (Given Disjunction)

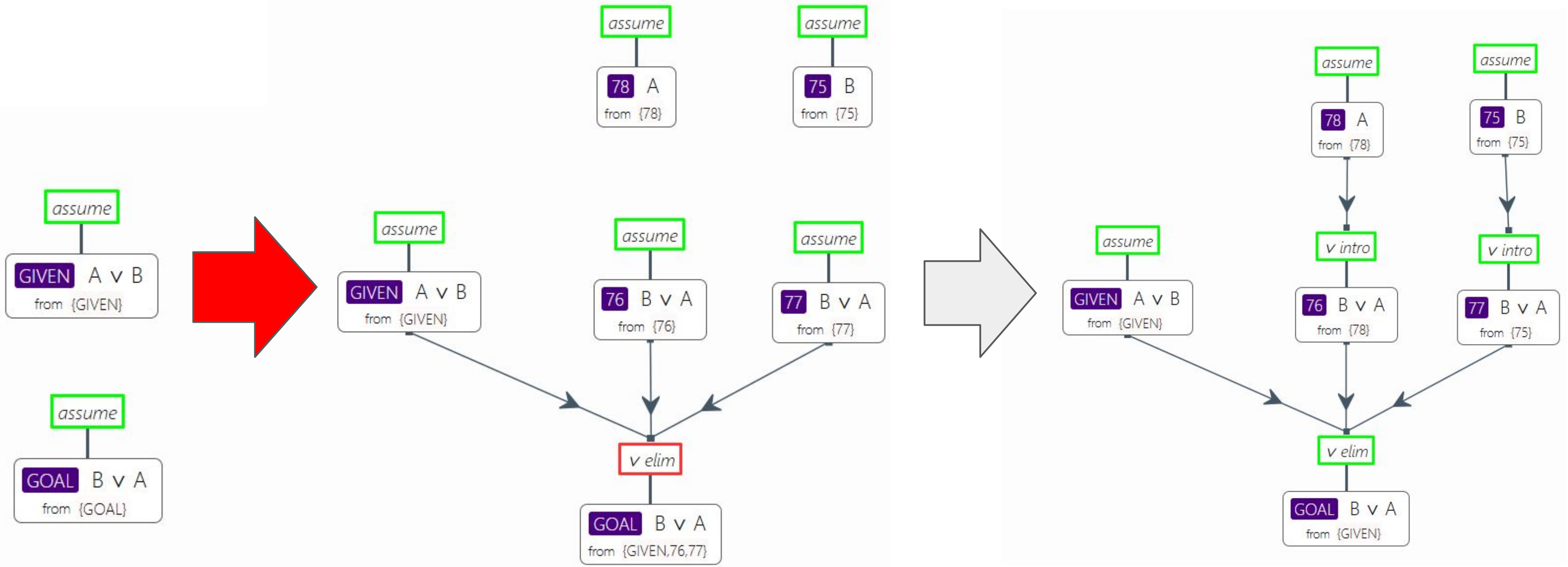
If you are given a disjunction or derive one while building a proof from the top down, there is a chance you will need to apply disjunction elimination and perform a proof by cases. These generally take the following form.



Given φ or ψ with a goal χ. Assume φ and prove χ. Then assume ψ and prove χ, use disjunction elimination with the conclusions and the original φ or ψ.

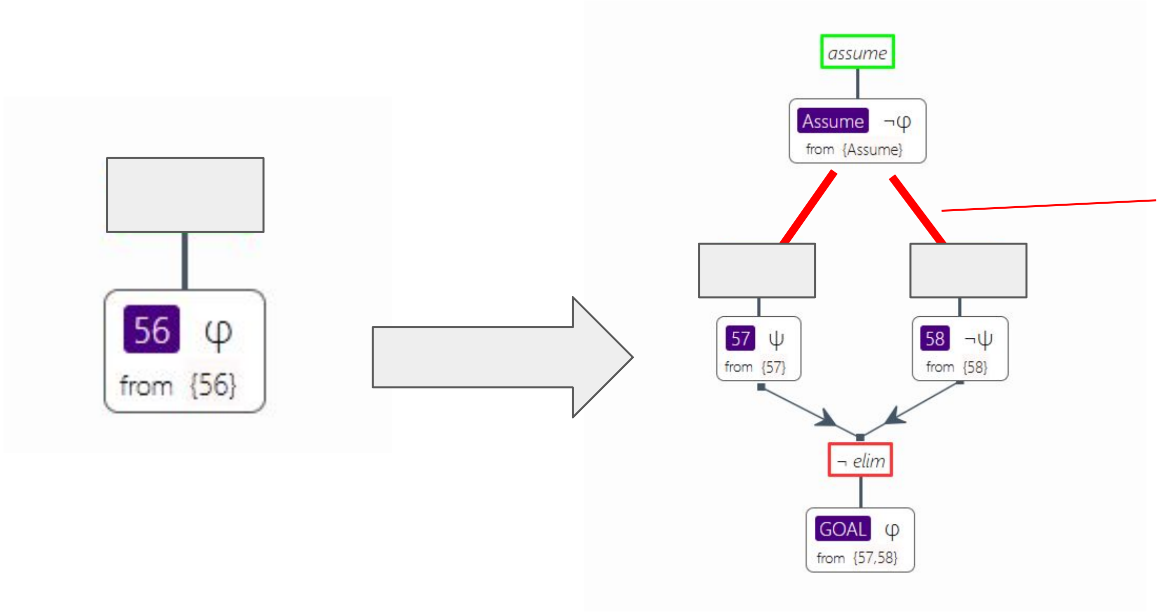
Proof By Cases Example

Swap disjunct order, Given A or B prove B or A. Start by noticing we have a disjunctive given! Trying proof by cases is a good idea!



Proof By Contradiction (Bottom Up + Top Down)

Only try this AFTER you have tried and failed with the other previous rules that split the goal based on its connective. If your goal is ANY formula φ, assume its negation (¬φ) and derive a contradiction with other givens to use not elimination.

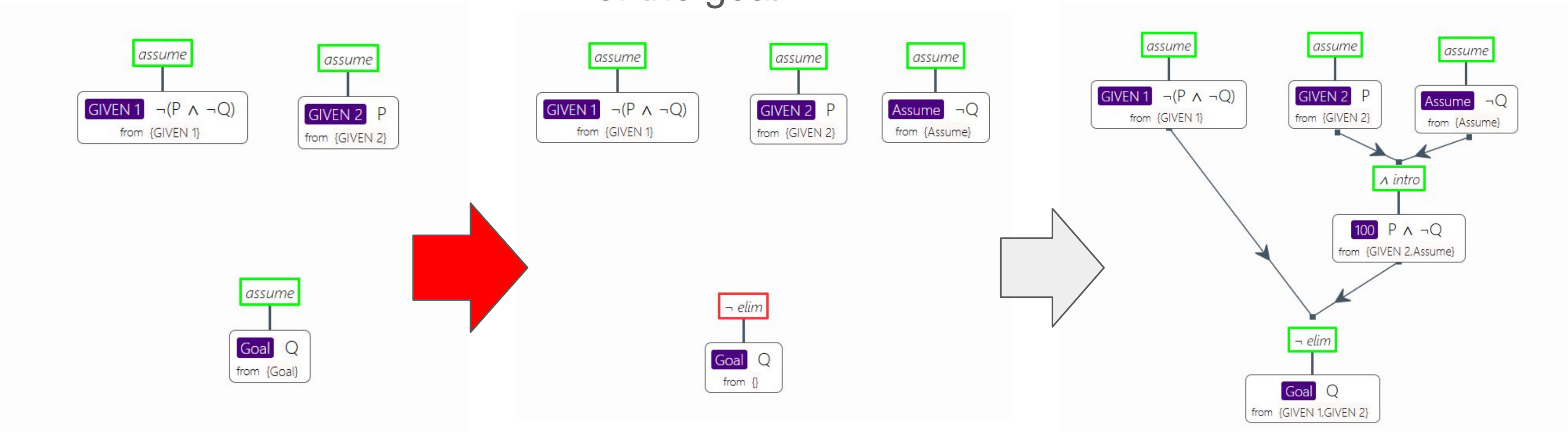


Now, use ¬φ and other givens to prove two formulae: ψ and ¬ψ, which you can use to apply not elimination.

Finding these formulae may require a bit of reasoning about what you can derive from ¬φ and other givens.

Proof By Contradiction Example

Assume the negation of the goal



Common Subproofs (Top Down + Bottom Up)

Often times your givens and goal will match a common subproofs, many of these are so common that they are considered derived inference rules and given special names. Since these rules are not base rules in hyperslate, you need to reconstruct the subproof each time.

Modus tollens

$$\frac{P \rightarrow Q, \neg Q}{\neg P}$$

Contraposition

$$\frac{P \rightarrow Q}{\neg Q \rightarrow \neg P}$$

Explosion (Ex Falso Quodlibet)

$$\frac{P \qquad \neg P}{Q}$$

De Morgan's Laws

$$\frac{\neg(P \vee Q)}{\neg P \wedge \neg Q} \qquad \frac{\neg P \wedge \neg Q}{\neg(P \vee Q)}$$

$$\frac{\neg(P \wedge Q)}{\neg P \vee \neg Q} \qquad \frac{\neg P \vee \neg Q}{\neg(P \wedge Q)}$$

Double Negation

$$\frac{P}{\neg\neg P}$$

$$\frac{\neg\neg P}{P}$$

Structural Weakening with Implication

$$\frac{Q}{P \rightarrow Q}$$

More Common Subproofs

Disjunctive Syllogism

$$\frac{P \vee Q, \neg P}{Q}$$

Constructive Dilemma

$$\frac{(P \to Q), (R \to S), P \vee R}{Q \vee S}$$

Absorption

$$\frac{P \to Q}{P \to (P \wedge Q)}$$

Hypothetical Syllogism

$$\frac{P \to Q, Q \to R}{P \to R}$$

Destructive Dilemma

$$\frac{P \to Q, R \to S, \neg Q \vee \neg S}{\neg P \vee \neg R}$$

Def of Material Implication

$$\frac{P \rightarrow Q}{\neg P \vee Q}$$

Def of Material Implication

$$\frac{\neg P \vee Q}{P \rightarrow Q}$$