“The Wager,” Overview of the Book, and Gödel’s Completeness Theorem

Selmer Bringsjord

Rensselaer AI & Reasoning (RAIR) Lab
Department of Cognitive Science
Department of Computer Science
Lally School of Management & Technology
Rensselaer Polytechnic Institute (RPI)
Troy, New York 12180 USA

IFLAI1 @ RPI
3/31/2025

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Note: This is a version of coverage of Gödel’s Completeness Theorem designed for those who’ve had at least one standard/standard-paced university-level course in formal logic.

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Background Context ...

Gödel's Great Theorems (OUP)

by Selmer Bringsjord

- Introduction ("The Wager")
- Brief Preliminaries (elementary discrete math, incl. ZOL, FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
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- Gödel's "God Theorem"
- Could a Machine Match Gödel's Genius?

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Some Timeline Points

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1906 Brünn, Austria-Hungary

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1923 Vienna

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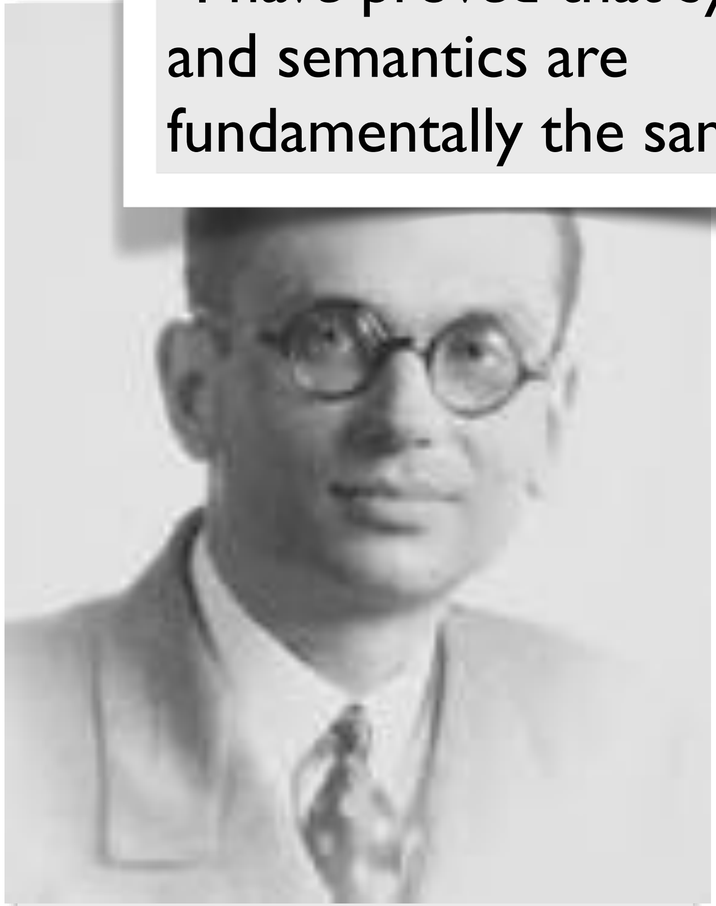

“I have proved that syntax and semantics are fundamentally the same.”

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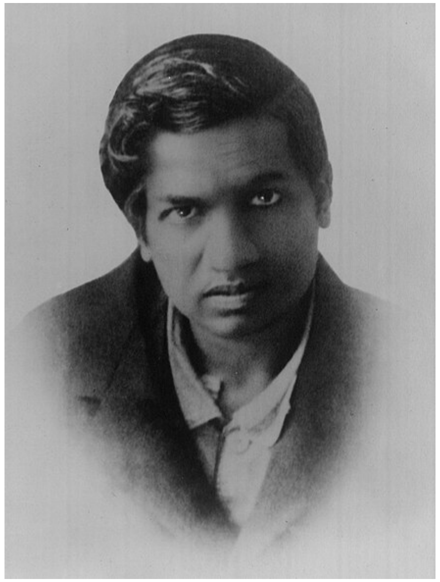
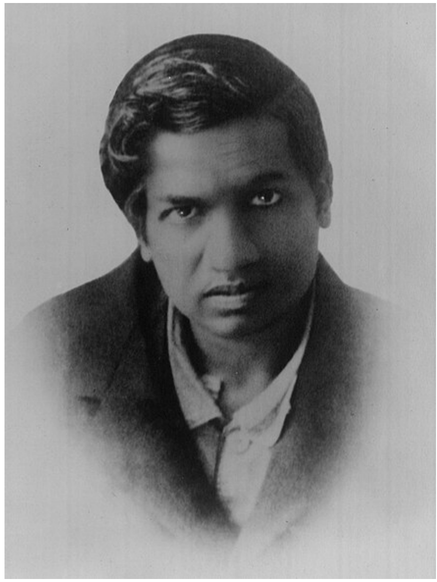
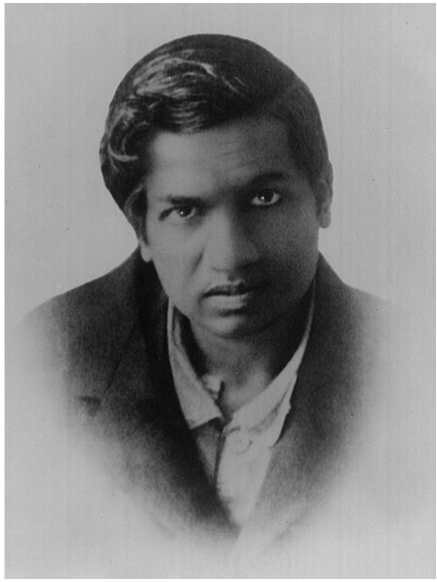
Ruminations on Srinivas Ramanujan

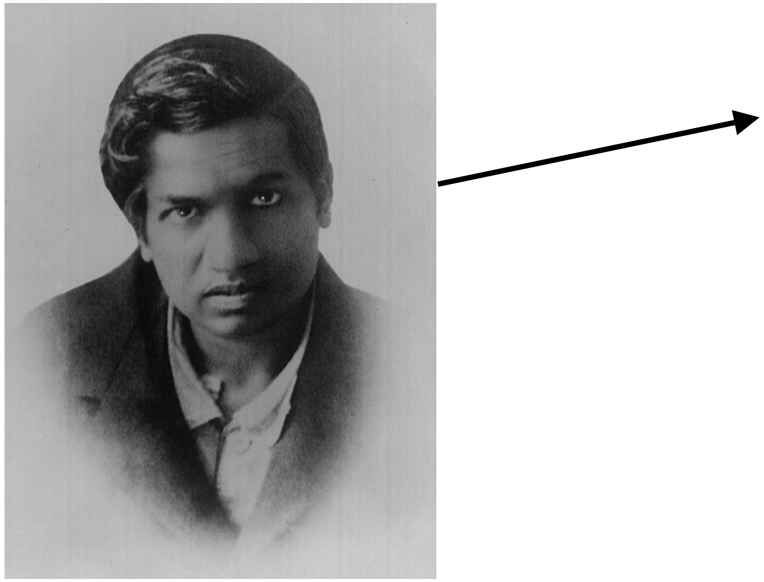
…





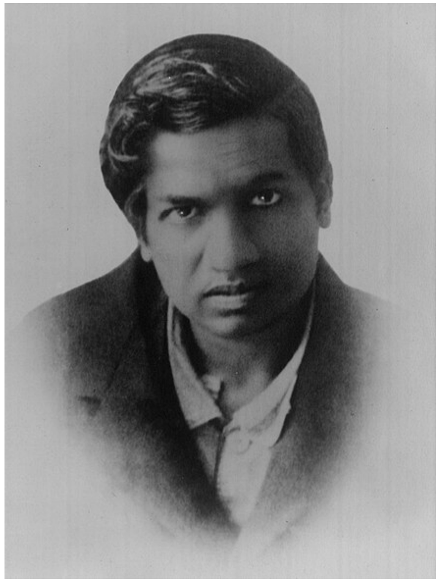


“ϕ is true!”



“ϕ is true!”

Should we trust R.?



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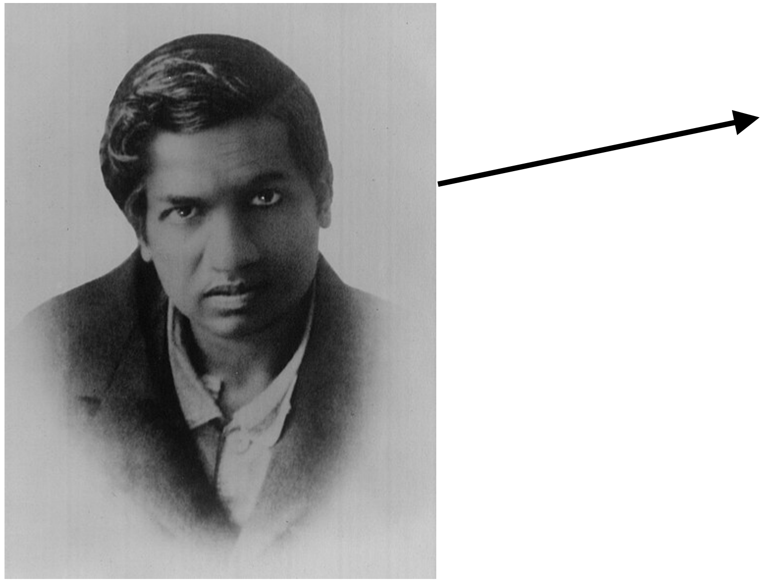

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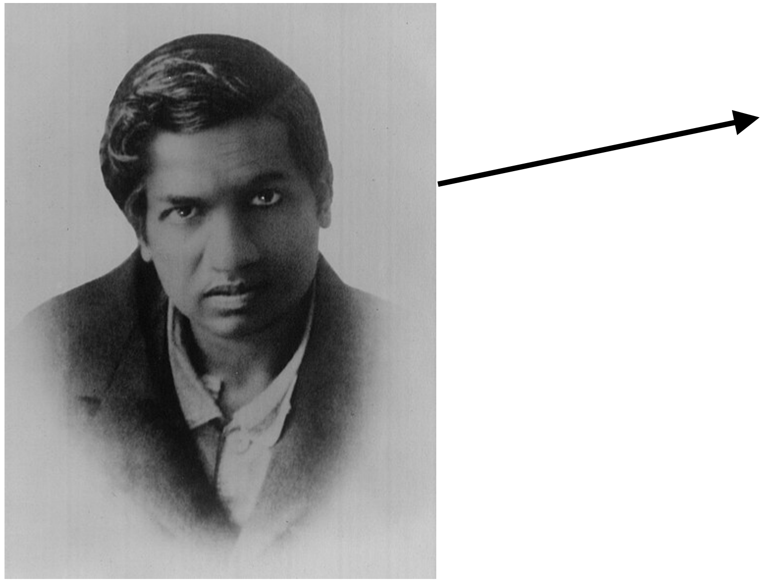
Let’s assume the affirmative.



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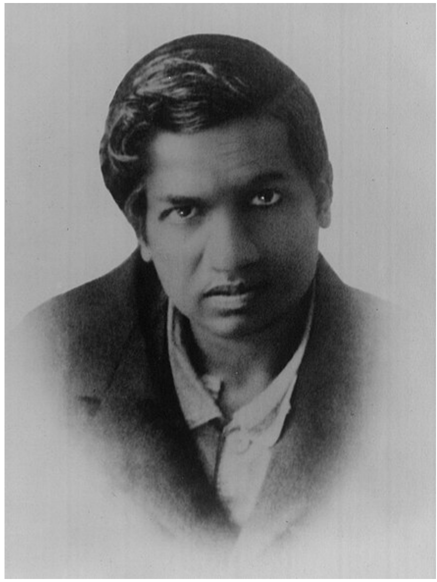
We still need … a proof!



“ϕ is true!”

Should we trust R.?

Should *everyone* trust R.?

Let’s assume the affirmative.

We still need … a proof!

Or at least we need to know there *is* a proof!

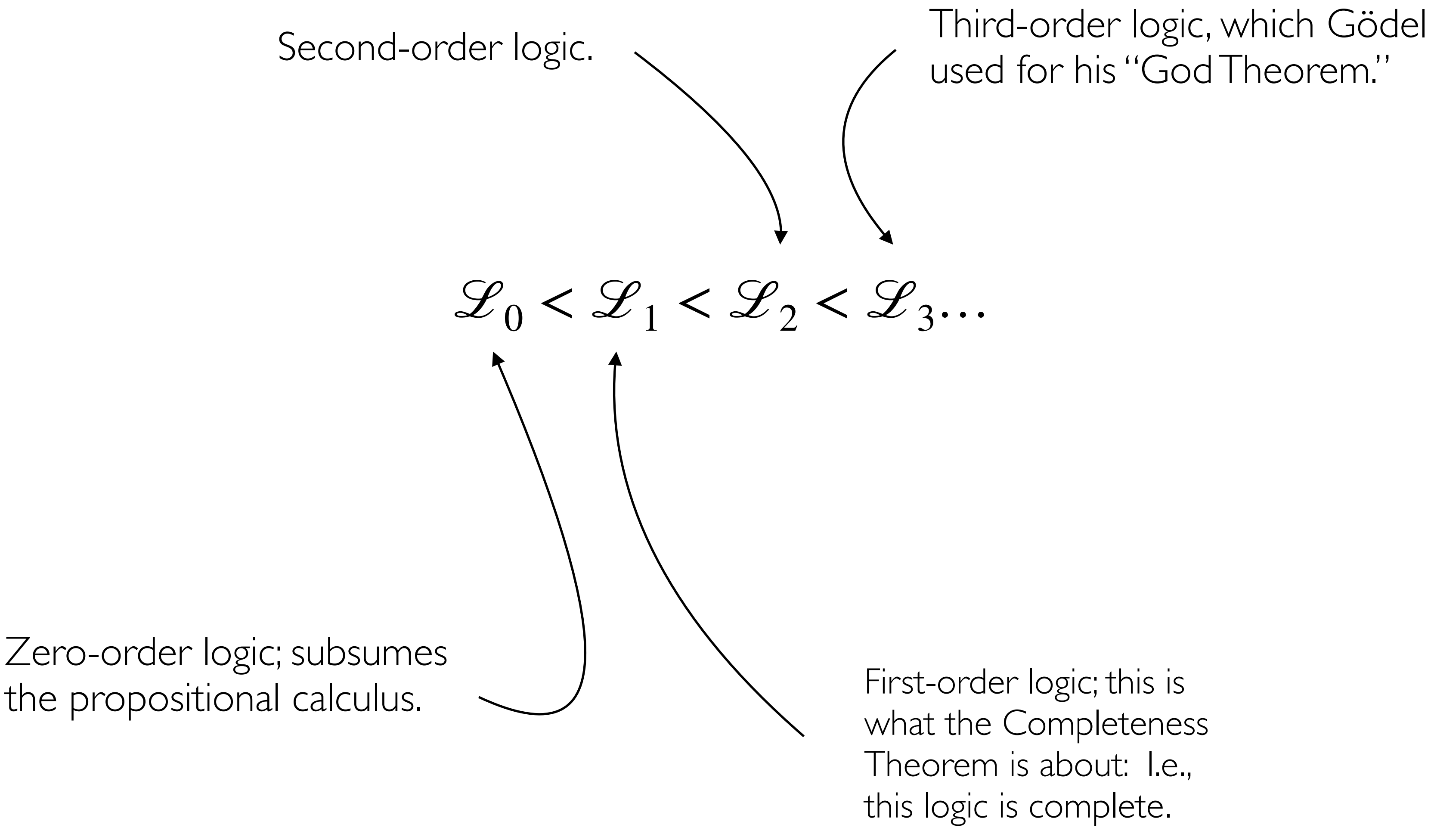
Preliminaries: Propositional Calculus & First-Order Logic

…

Actually …

$$\mathscr{L}_0 < \mathscr{L}_1 < \mathscr{L}_2 < \mathscr{L}_3\ldots$$

Actually …



Second-order logic.

Third-order logic, which Gödel used for his "God Theorem."

$$\mathscr{L}_0 < \mathscr{L}_1 < \mathscr{L}_2 < \mathscr{L}_3 \ldots$$

Zero-order logic; subsumes the propositional calculus.

First-order logic; this is what the Completeness Theorem is about: I.e., this logic is complete.

Actually …

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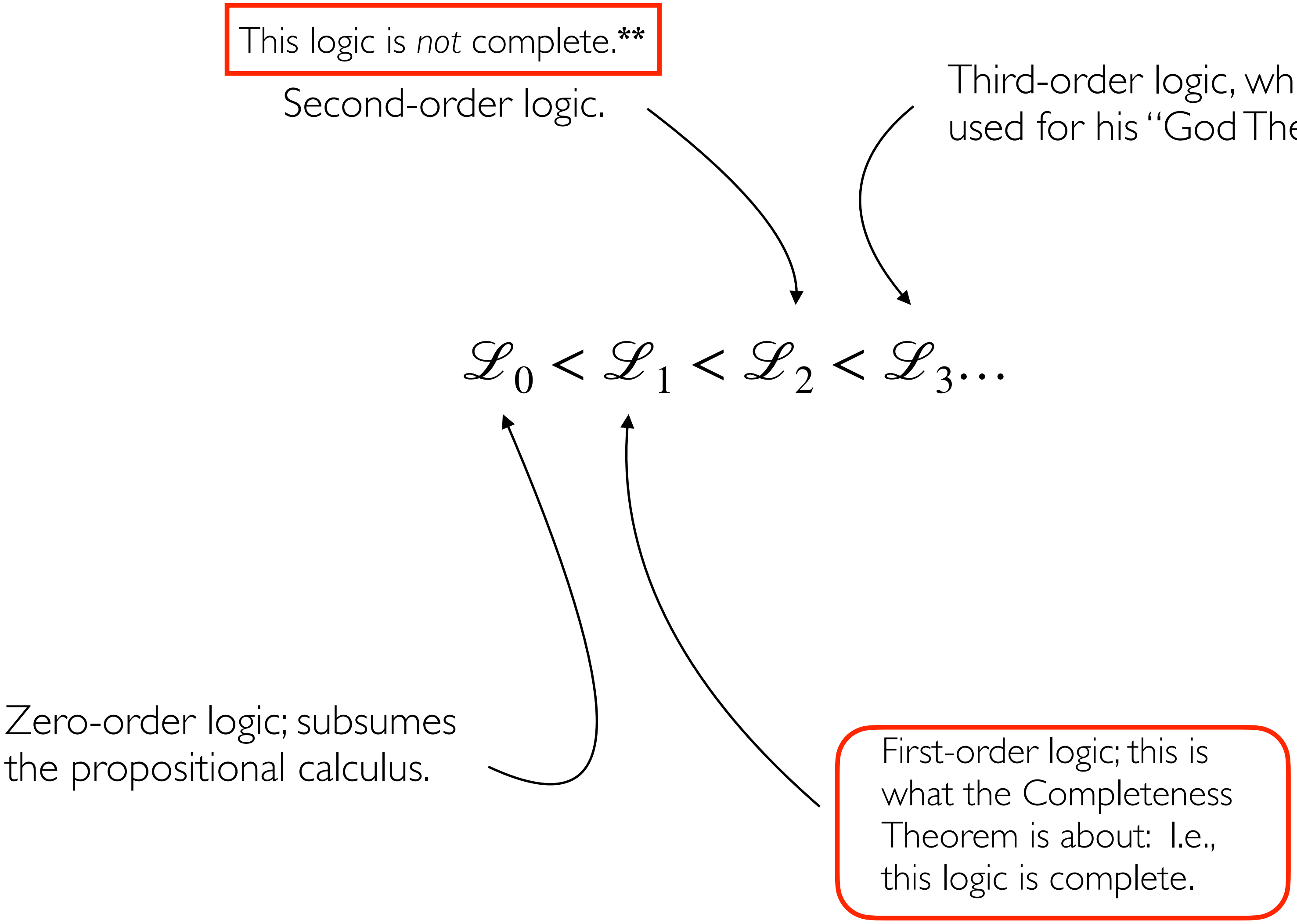
Actually . . .



This logic is *not* complete.**

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Zero-order logic; subsumes the propositional calculus.

First-order logic; this is what the Completeness Theorem is about: I.e., this logic is complete.

**At least in its "complete" form. There are complete *fragments* of $\mathscr{L}_2$.

R&W's Axiomatization of the Propositional Calculus

$$
\begin{array}{ll}
\text{A1} & (\phi \vee \phi) \to \phi \\
\text{A2} & \phi \to (\phi \vee \psi) \\
\text{A3} & (\phi \vee \psi) \to (\psi \vee \phi) \\
\text{A4} & (\psi \to \chi) \to ((\phi \vee \psi) \to (\phi \vee \chi))
\end{array}
$$

R&W's Axiomatization of the Propositional Calculus

$$\begin{array}{ll} \text{A1} & (\phi \vee \phi) \rightarrow \phi \\ \text{A2} & \phi \rightarrow (\phi \vee \psi) \\ \text{A3} & (\phi \vee \psi) \rightarrow (\psi \vee \phi) \\ \text{A4} & (\psi \rightarrow \chi) \rightarrow ((\phi \vee \psi) \rightarrow (\phi \vee \chi)) \end{array}$$

All instances of these schemata are true no matter what the input (true or false). (Agreed?) And indeed every single formula in the propositional calculus that is true no matter what the permutation (as shown in a truth table, e.g.), can be proved (somehow) from these four axioms (using any standard collection of inference schemata). This, Gödel (& later, Newell & Simon, when modern AI was born!) knew, and could use.

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\mathrm{A3} & (\phi \vee \psi) \rightarrow (\psi \vee \phi) \\
\mathrm{A4} & (\psi \rightarrow \chi) \rightarrow ((\phi \vee \psi) \rightarrow (\phi \vee \chi))
\end{array}
$$

```
(if (or \phi \phi) \phi)

(if \phi (or \phi \psi))

(if (or \phi \psi) (or \psi phi))

(if (if \psi \chi) (if (or \phi \psi) (or \phi \chi)))
```

All instances of these schemata are true no matter what the input (true or false). (Agreed?) And indeed every single formula in the propositional calculus that is true no matter what the permutation (as shown in a truth table, e.g.), can be proved (somehow) from these four axioms (using any standard collection of inference schemata). This, Gödel (& later, Newell & Simon, when modern AI was born!) knew, and could use.

Exercise 1:

Verify that these are true-no-matter what in a truth tree in HyperSlate®; then prove using our rules for the prop. calc.; or perhaps better yet, have the oracle prove in HyperSlate®.

$$(\phi \wedge \psi) \rightarrow (\psi \vee \chi)$$

$$\phi \rightarrow (\psi \rightarrow \phi)$$

Exercise 1:

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√

Build **truth tree** (**semantic hypergraph**, actually) showing this formula true no matter what the inputs.

Exercise 1:

Verify that these are true-no-matter what in a truth tree; then prove using our rules for the prop. calc.

$$(\phi \wedge \psi) \to (\psi \vee \chi)$$

√

Build **truth tree** (**semantic hypergraph**, actually) showing this formula true no matter what the inputs.

Proof:

Exercise 1:

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$$(\phi \wedge \psi) \rightarrow (\psi \vee \chi)$$

√

Build **truth tree** (**semantic hypergraph**, actually) showing this formula true no matter what the inputs.

Proof:

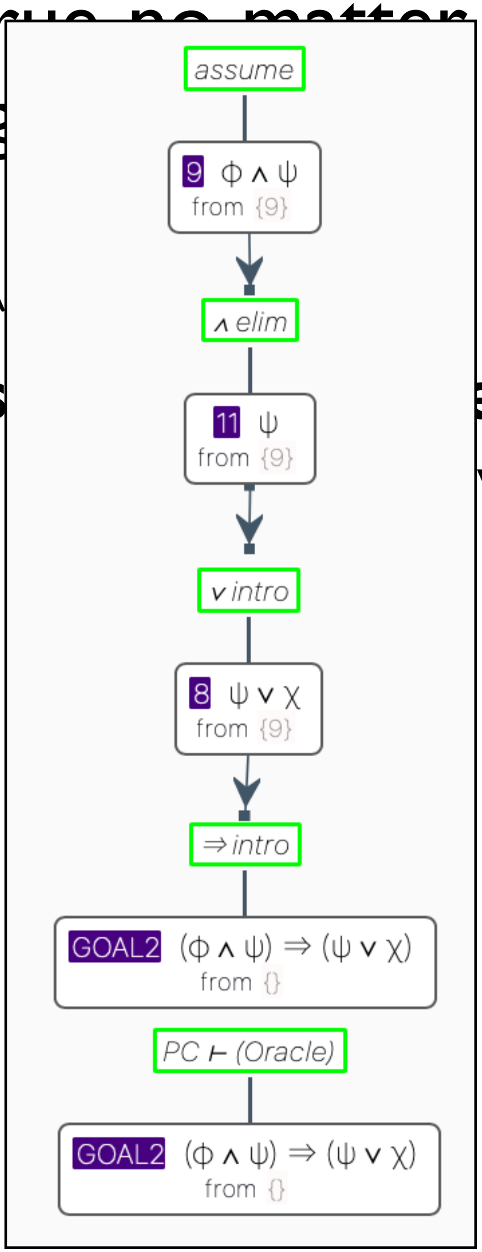
Exercise 1:

Verify that these are tr what in a truth tree;
then prove using the prop. calc.

$(\phi \wedge$ $\chi)$

√

Build **truth tree (s** **ergraph**, actually)
showing this formula what the inputs.



assume

9 ϕ ∧ ψ
from {9}

∧ elim

11 ψ
from {9}

∨ intro

8 ψ ∨ χ
from {9}

⇒ intro

GOAL2 (ϕ ∧ ψ) ⇒ (ψ ∨ χ)
from {}

PC ⊢ (Oracle)

GOAL2 (ϕ ∧ ψ) ⇒ (ψ ∨ χ)
from {}

Proof:

Exercise 1:

Verify that these are tr what in a truth tree;
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$$(\phi \wedge \qquad \chi)$$

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Build **truth tree (s** **ergraph**, actually)
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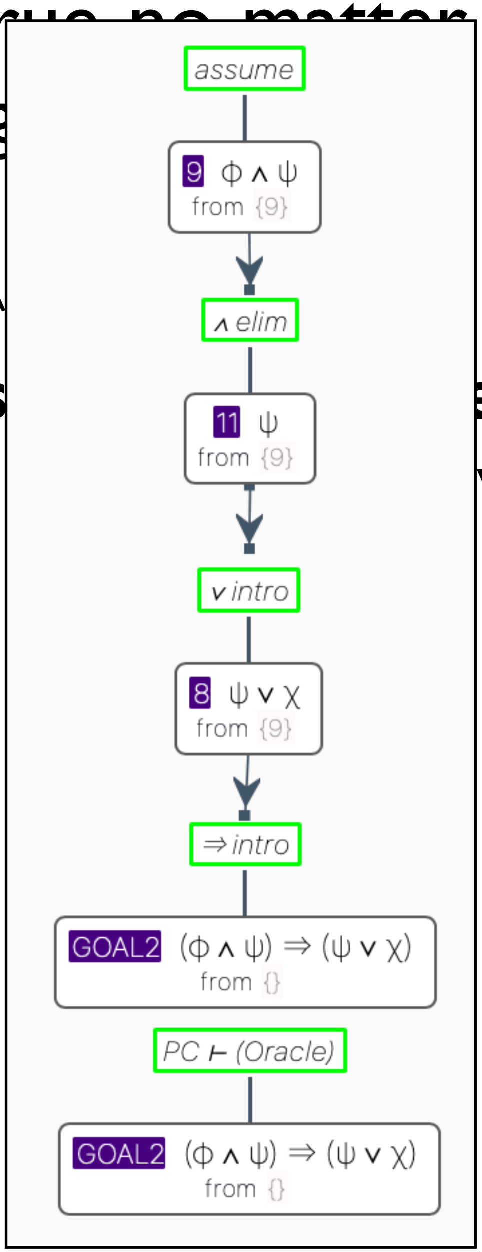


assume

9 $\phi \wedge \psi$ from {9}

$\wedge$ elim

11 ψ from {9}

$\vee$ intro

8 $\psi \vee \chi$ from {9}

$\Rightarrow$ intro

GOAL2 $(\phi \wedge \psi) \Rightarrow (\psi \vee \chi)$ from {}

PC $\vdash$ (Oracle)

GOAL2 $(\phi \wedge \psi) \Rightarrow (\psi \vee \chi)$ from {}

Proof:

resolution-based!

Exercise 1:

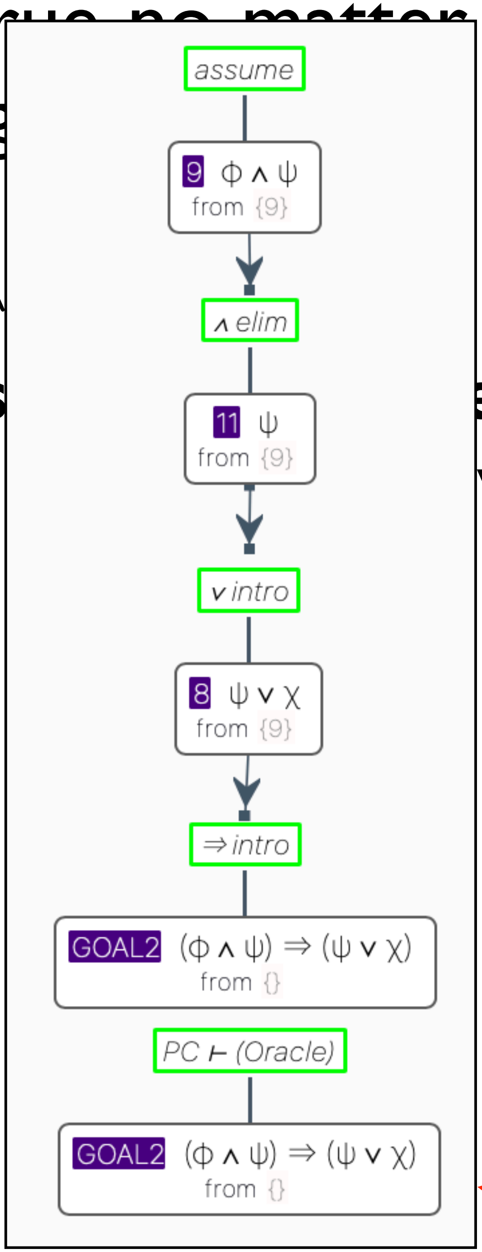
Verify that these are tr what in a truth tree;
then prove using the prop. calc.

$$(\phi \wedge \quad \chi)$$

√

Build **truth tree (s** **ergraph**, actually)
showing this formula what the inputs.

Proof:



assume

9 $\phi \wedge \psi$
from {9}

∧ elim

11 ψ
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∨ intro

8 $\psi \vee \chi$
from {9}

⇒ intro

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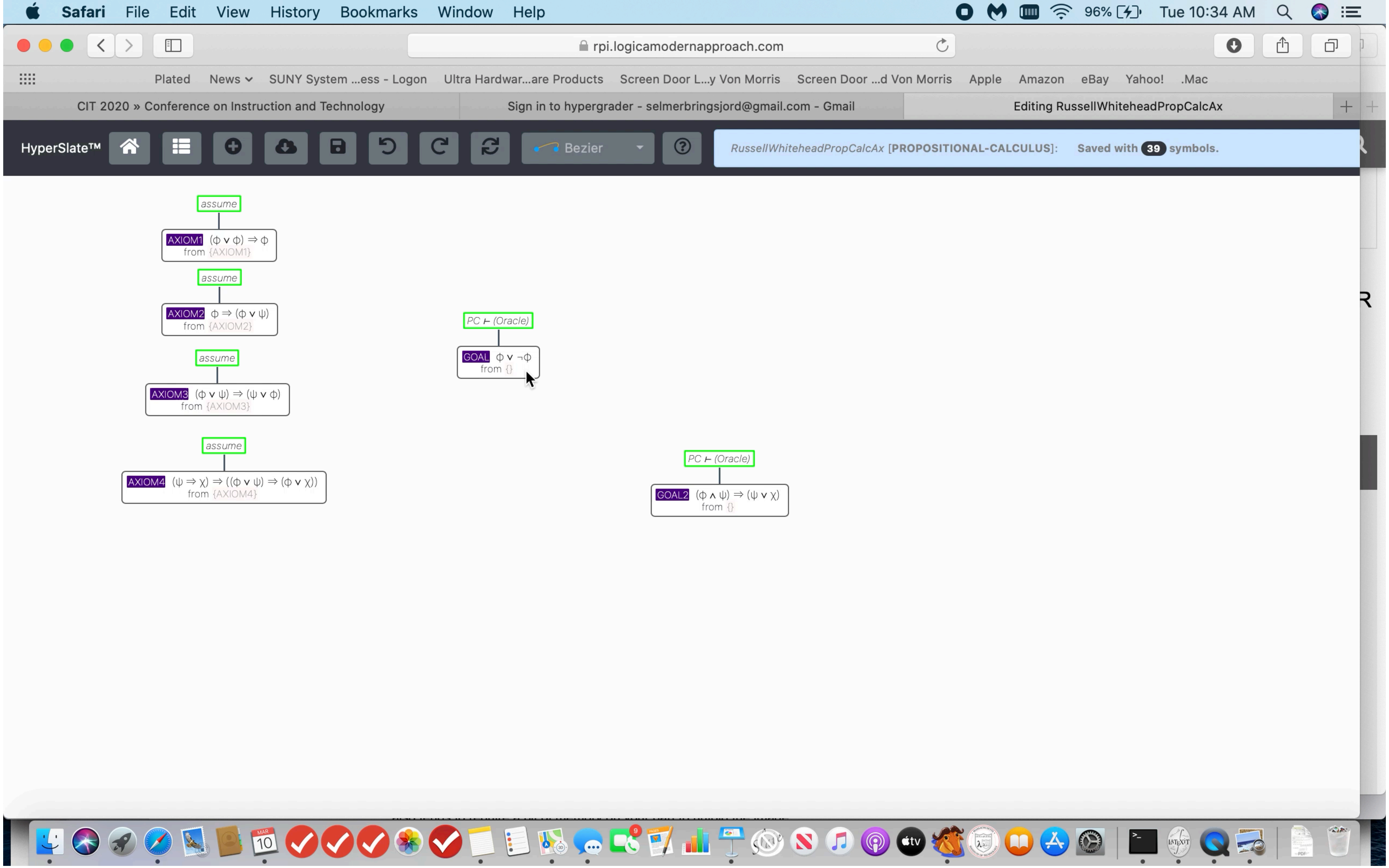
resolution-based!

As HyperSlate® Tutorial



RussellWhiteheadPropCalcAx **[PROPOSITIONAL-CALCULUS]**: **Saved with 39 symbols.**

assume

AXIOM1 $(\phi \lor \phi) \Rightarrow \phi$
from {AXIOM1}

assume

AXIOM2 $\phi \Rightarrow (\phi \lor \psi)$
from {AXIOM2}

assume

AXIOM3 $(\phi \lor \psi) \Rightarrow (\psi \lor \phi)$
from {AXIOM3}

assume

AXIOM4 $(\psi \Rightarrow \chi) \Rightarrow ((\phi \lor \psi) \Rightarrow (\phi \lor \chi))$
from {AXIOM4}

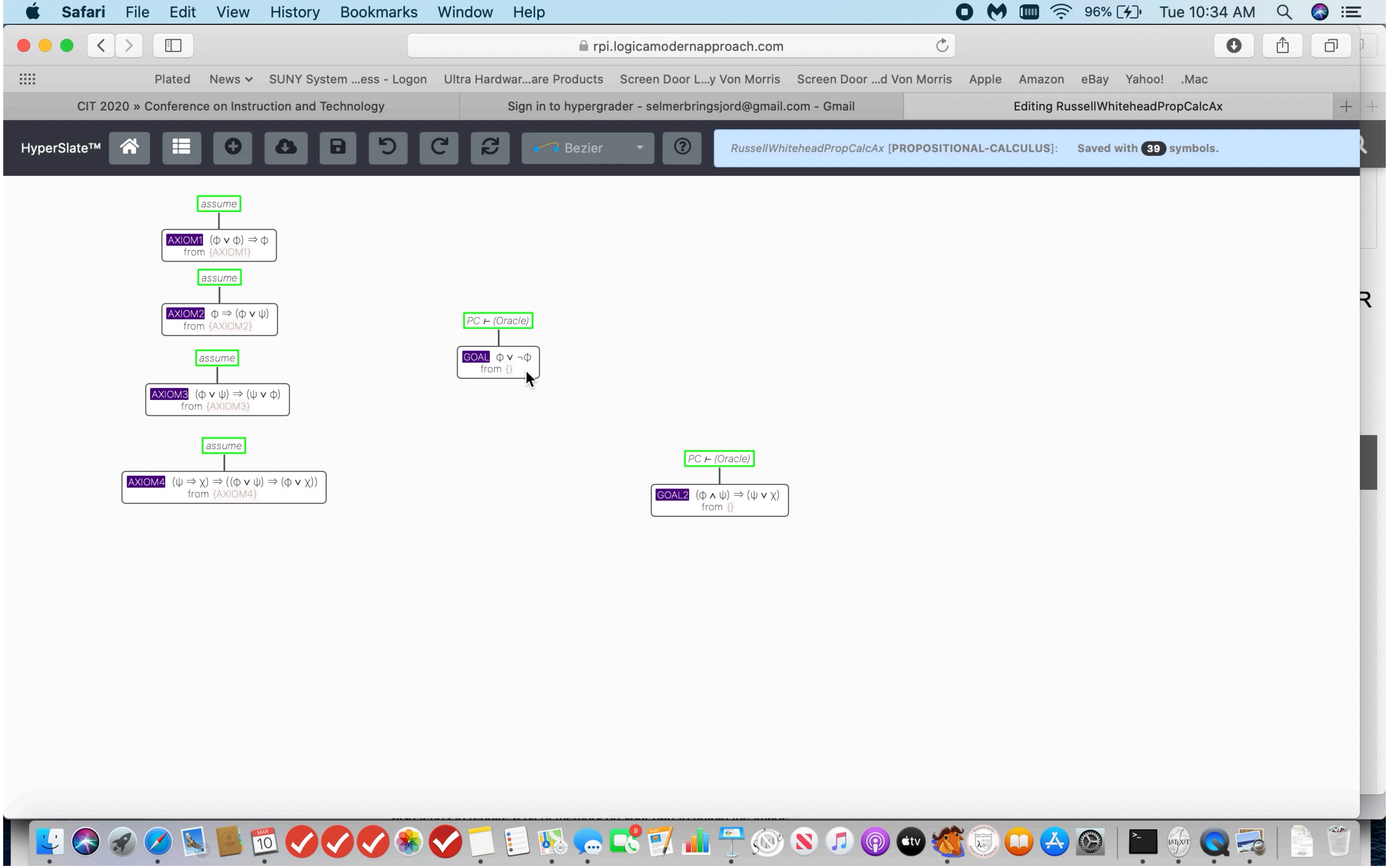
PC ⊢ (Oracle)

GOAL $\phi \lor \neg\phi$
from {}

PC ⊢ (Oracle)

GOAL2 $(\phi \land \psi) \Rightarrow (\psi \lor \chi)$
from {}

As HyperSlate® Tutorial



RussellWhiteheadPropCalcAx **[PROPOSITIONAL-CALCULUS]:** **Saved with 39 symbols.**

The Grammar of $\mathscr{L}_0$ = the Pure Predicate Calculus

$$
\begin{array}{lcl}
\mathit{Formula} & \Rightarrow & \mathit{AtomicFormula} \\
 & | & (\mathit{Formula}\ \ \mathit{Connective}\ \ \mathit{Formula}) \\
 & | & \neg\ \mathit{Formula}
\end{array}
$$

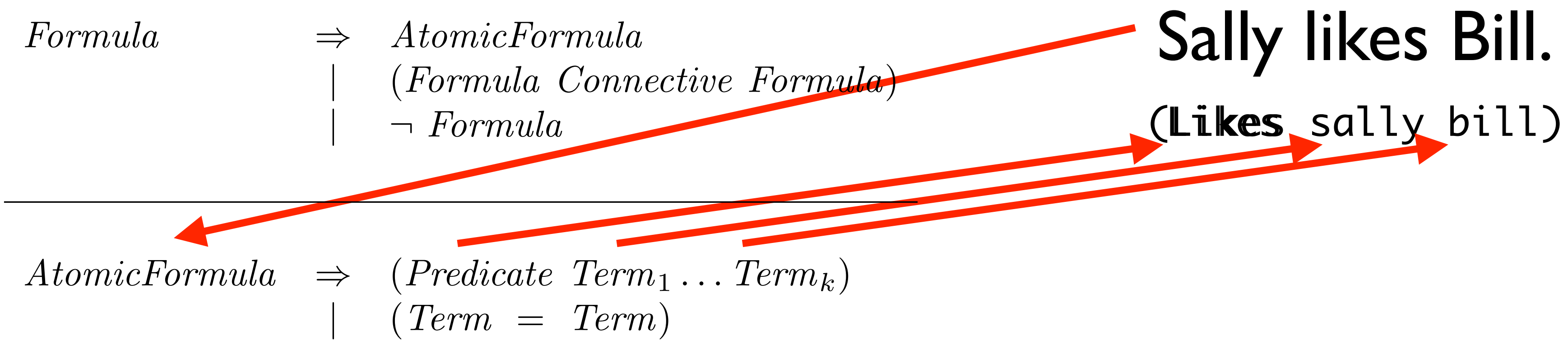
$$
\begin{array}{lcl}
\mathit{AtomicFormula} & \Rightarrow & (\mathit{Predicate}\ \ \mathit{Term}_1 \ldots \mathit{Term}_k) \\
 & | & (\mathit{Term}\ \ =\ \ \mathit{Term})
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{Term} & \Rightarrow & (\mathit{Function}\ \ \mathit{Term}_1\ \ \ldots\ \ \mathit{Term}_k) \\
 & | & \mathit{Constant}
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{Connective} & \Rightarrow & \wedge \mid \vee \mid \rightarrow \mid \leftrightarrow
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{Predicate} & \Rightarrow & P_1 \mid P_2 \mid P_3\ \ldots \\
\mathit{Constant} & \Rightarrow & c_1 \mid c_2 \mid c_3\ \ldots \\
\mathit{Function} & \Rightarrow & f_1 \mid f_2 \mid f_3\ \ldots
\end{array}
$$

Some Simple Examples



$Formula \Rightarrow AtomicFormula$
$\mid (Formula\ Connective\ Formula)$
$\mid \neg\ Formula$

$AtomicFormula \Rightarrow (Predicate\ Term_1 \ldots Term_k)$
$\mid (Term\ =\ Term)$

$Term \Rightarrow (Function\ Term_1\ \ldots\ Term_k)$
$\mid Constant$

$Connective \Rightarrow \wedge \mid \vee \mid \rightarrow \mid \leftrightarrow$

$Predicate \Rightarrow P_1 \mid P_2 \mid P_3\ \ldots$
$Constant \Rightarrow c_1 \mid c_2 \mid c_3\ \ldots$
$Function \Rightarrow f_1 \mid f_2 \mid f_3\ \ldots$

Lexicon

Sally likes Bill.

`(Likes sally bill)`

Sally likes Bill and Bill likes Sally.

Sally likes Bill's mother.

Sally likes Bill only if Bill's mother is tall.

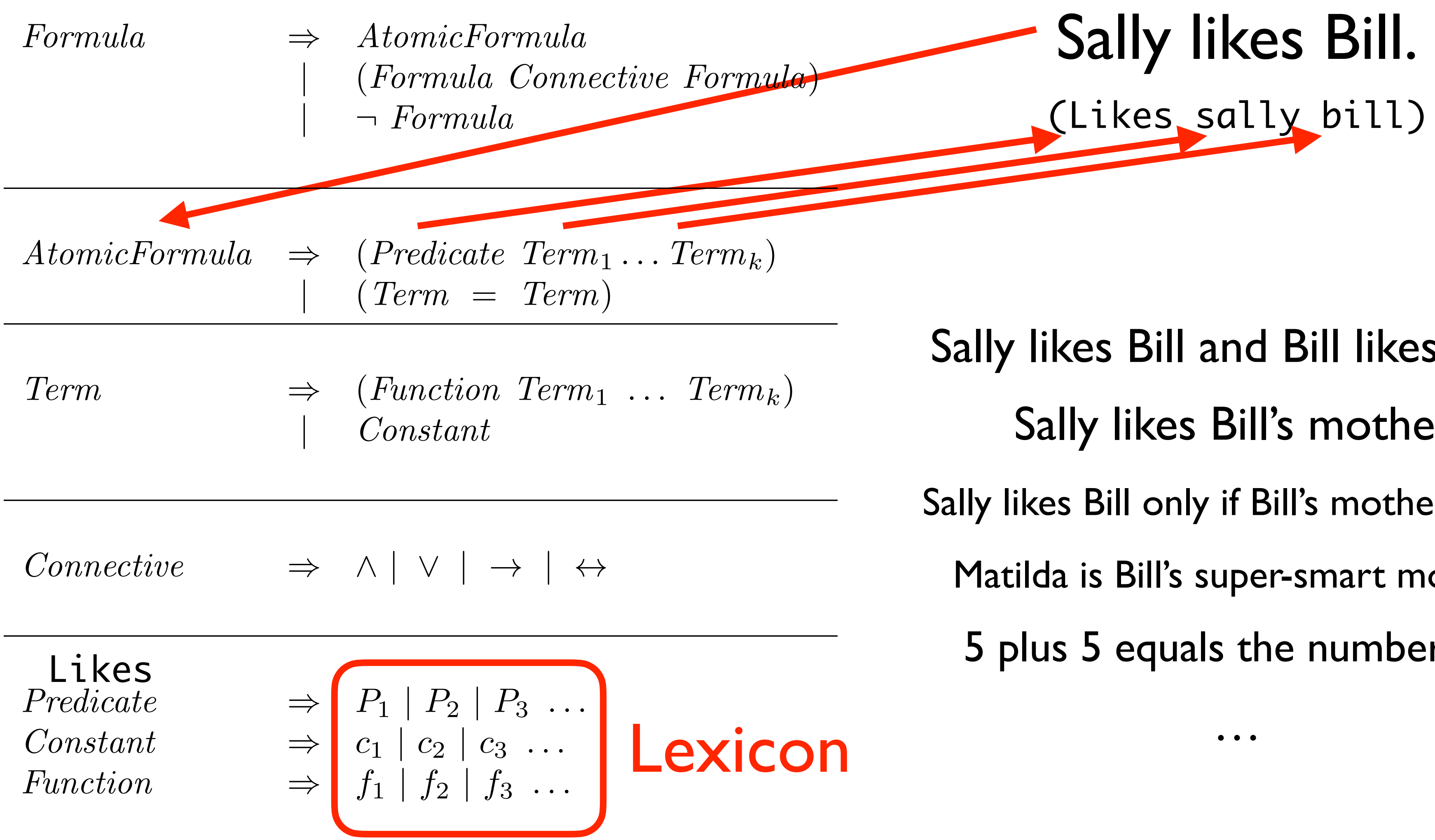
Matilda is Bill's super-smart mother.

5 plus 5 equals the number 10.

…

Can Roger be counted upon to declare: "Yes that sentence is okay!" whenever it's conforms to this grammar?

Some Simple Examples



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`Likes`

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These are all true, yes; but can they be proved?!

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Yes!

But Now the Deeper Challenge: Add Two Quantifiers to the Pure Predicate Calculus, Which Yields $\mathscr{L}_1$ First-order Logic = Predicate Calculus *simpliciter*

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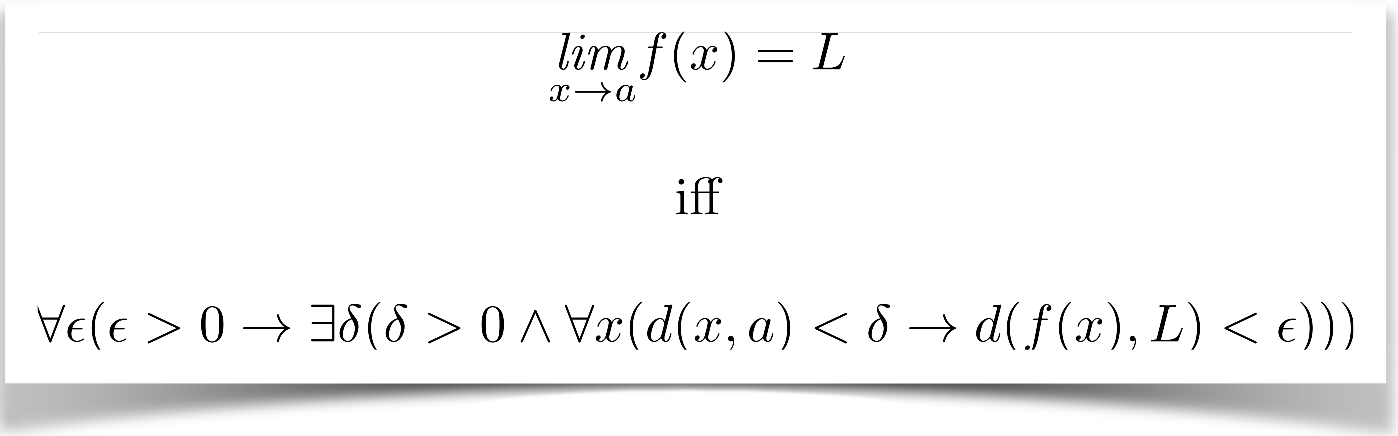

$$\lim_{x \to a} f(x) = L$$

iff

$$\forall \epsilon(\epsilon > 0 \rightarrow \exists \delta(\delta > 0 \wedge \forall x(d(x, a) < \delta \rightarrow d(f(x), L) < \epsilon)))$$

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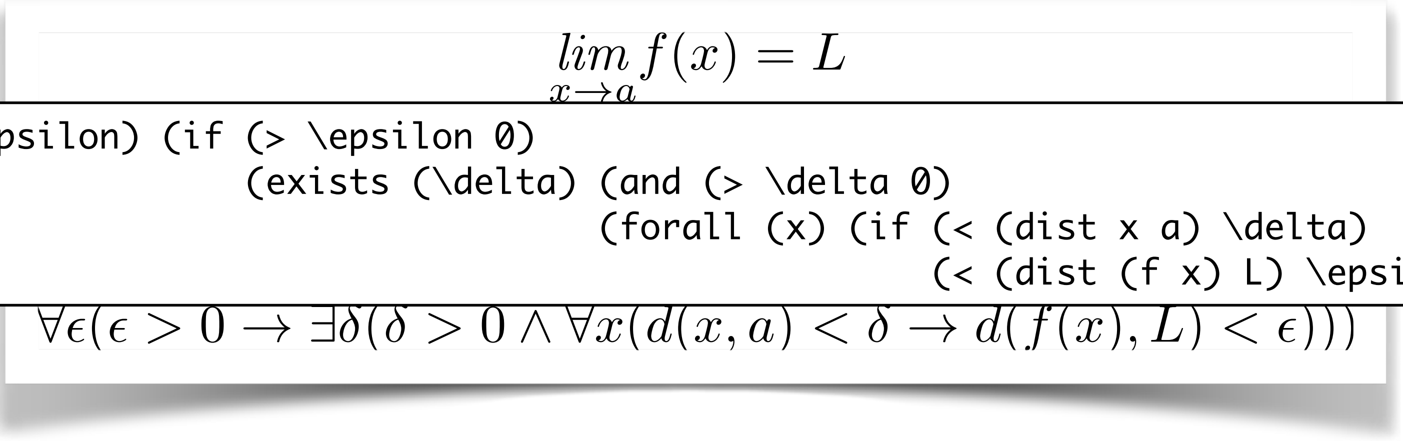

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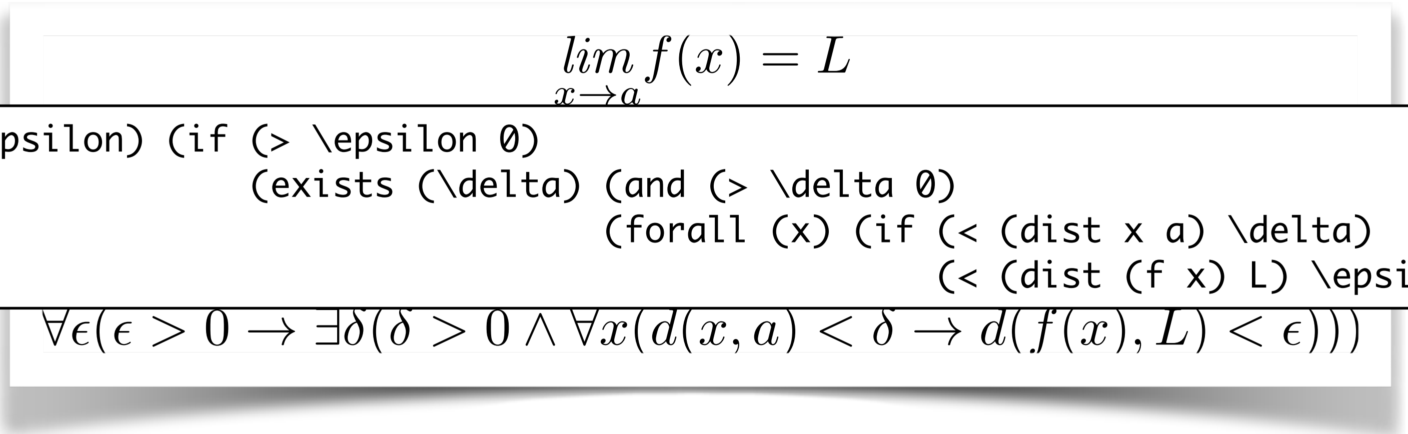

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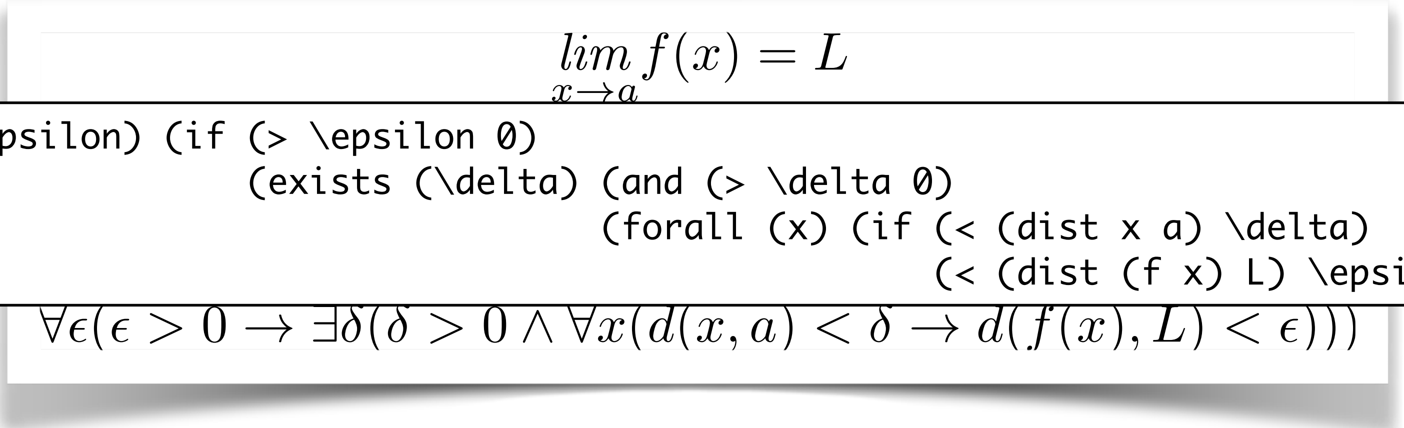

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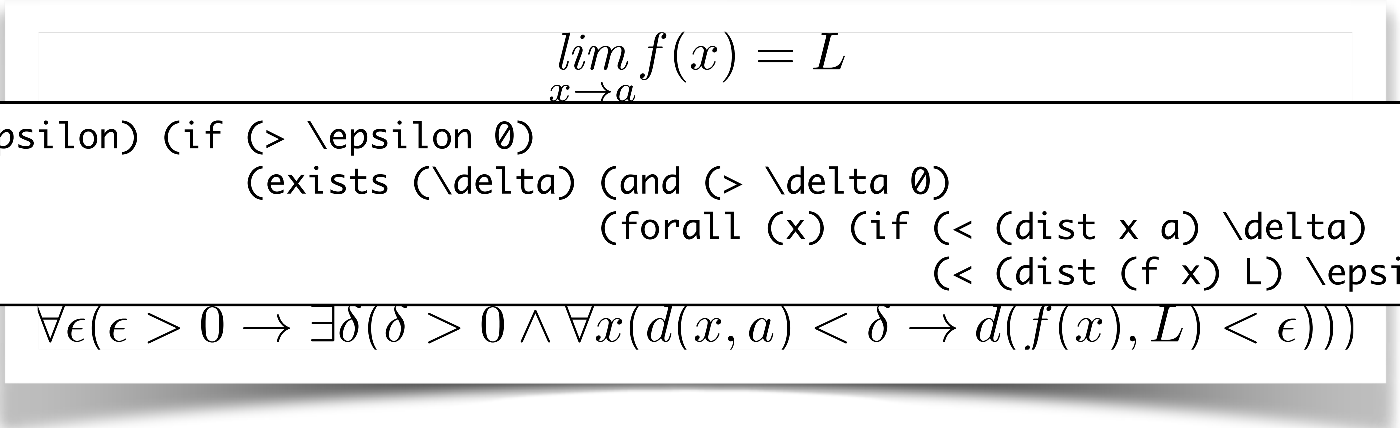

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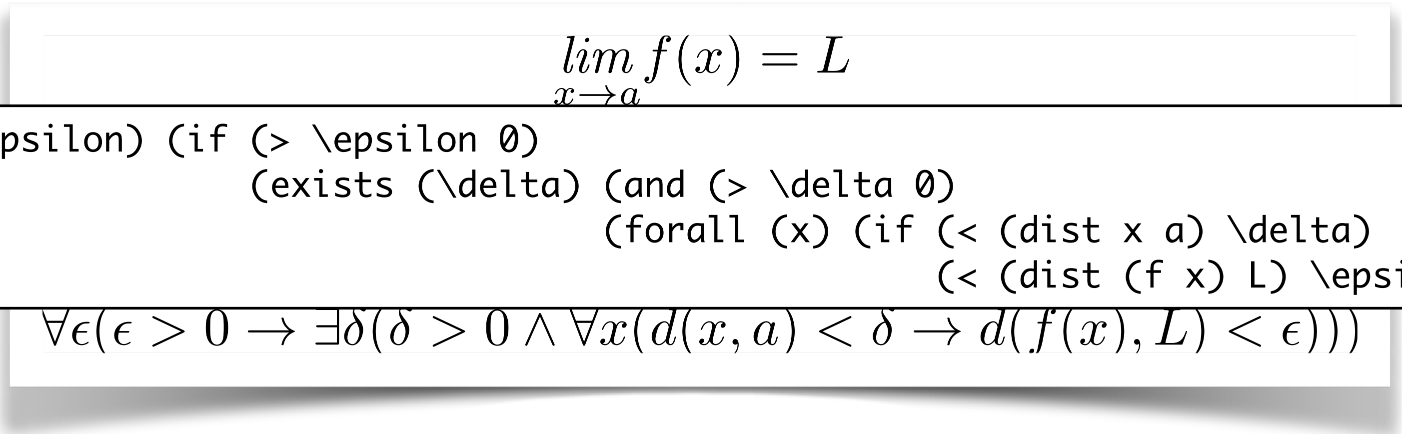

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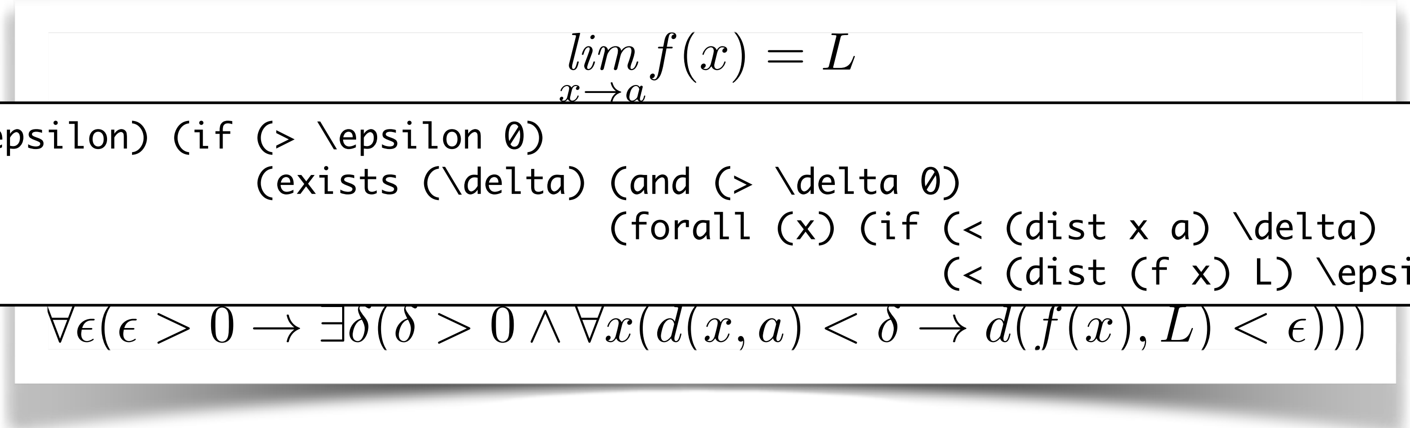

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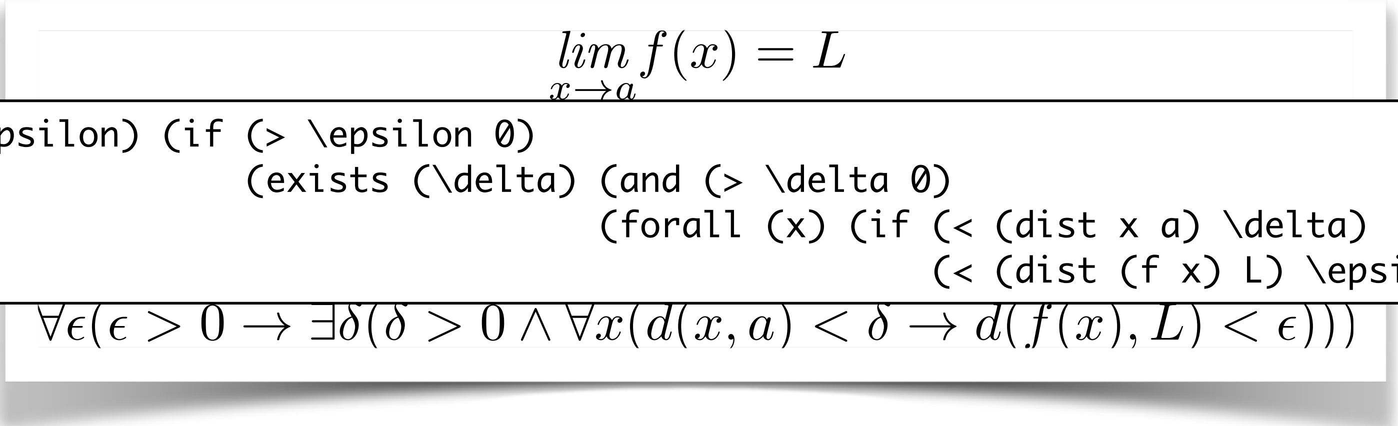

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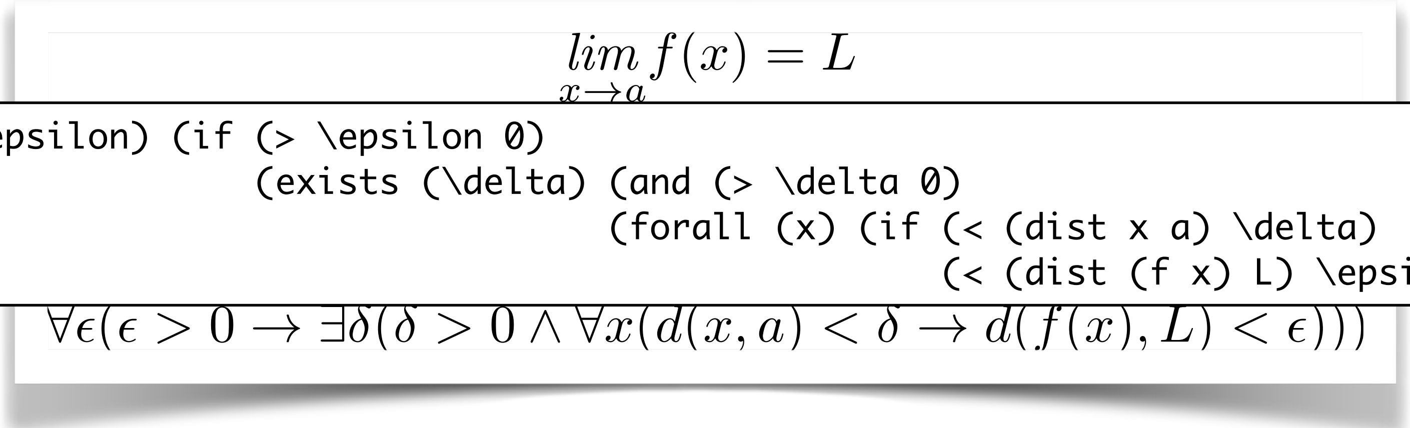

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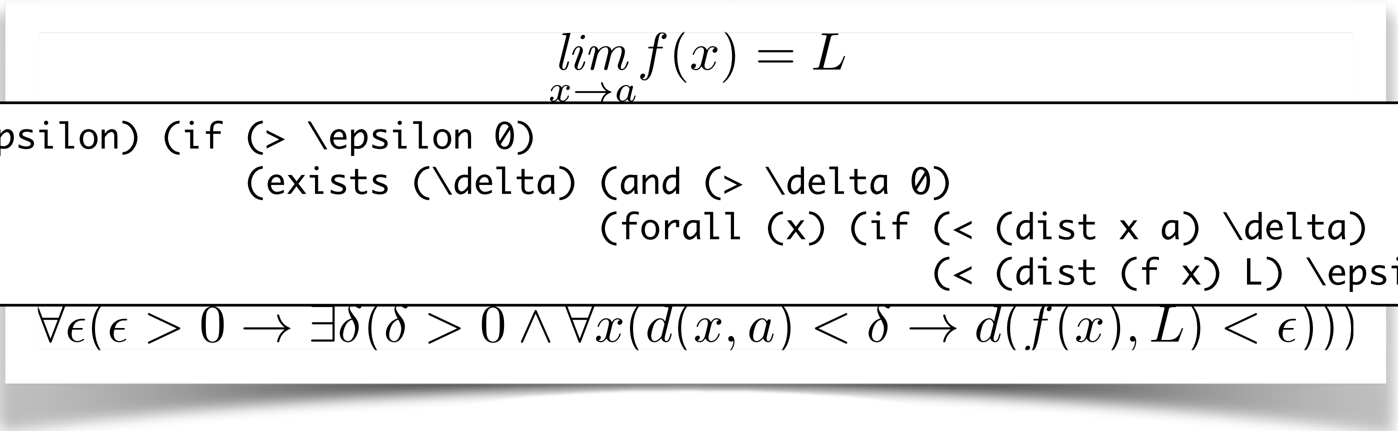

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$$\forall x \exists y (x \leq y) \qquad \forall x \exists y (\leq (x, y))$$

The Shoulders Available to Gödel for Standing Upon

…

Completeness Theorem for The Propositional Calculus

Let Γ be a set $\{\phi_1, \phi_2, \ldots\}$ of formulae in the the propositional calculus. Then either all of Γ are satisfiable, or the conjunction up to and including the point k (i.e. $\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_k$) of failure is refutable.

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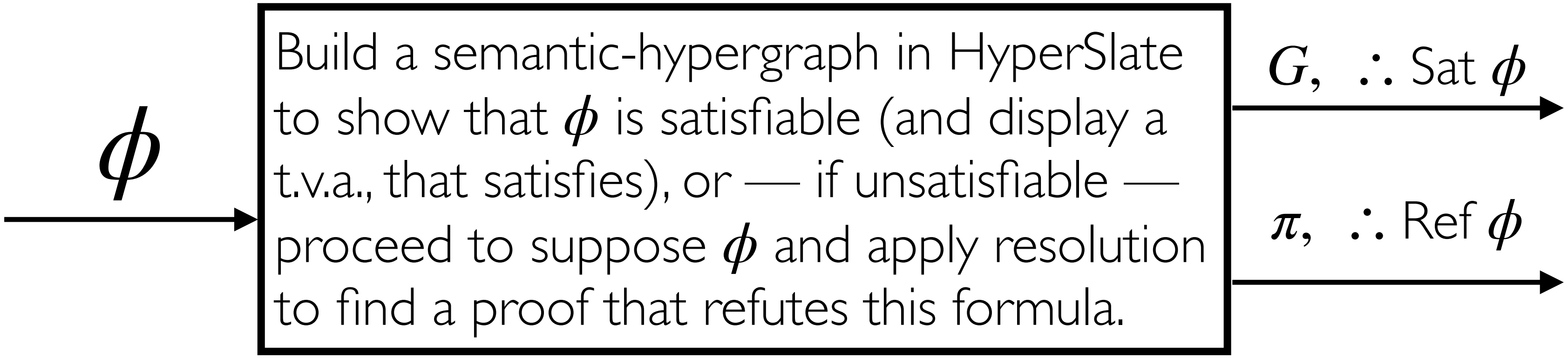
Let Γ be a set $\{\phi_1, \phi_2, \ldots\}$ of formulae in the the propositional calculus. Then either all of Γ can be simultaneously true in some scenario, or the conjunction up to and including the point k (i.e. $\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_k$) of failure is **refutable** (i.e. $\vdash \neg(\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_k)$).

Completeness Theorem (prop calc):
Either ϕ is satisfiable, or ϕ is refutable.



ϕ

Build a semantic-hypergraph in HyperSlate to show that ϕ is satisfiable (and display a t.v.a., that satisfies), or — if unsatisfiable — proceed to suppose ϕ and apply resolution to find a proof that refutes this formula.

G, $\therefore$ Sat ϕ

π, $\therefore$ Ref ϕ

What does the Completeness Theorem say?

. . .

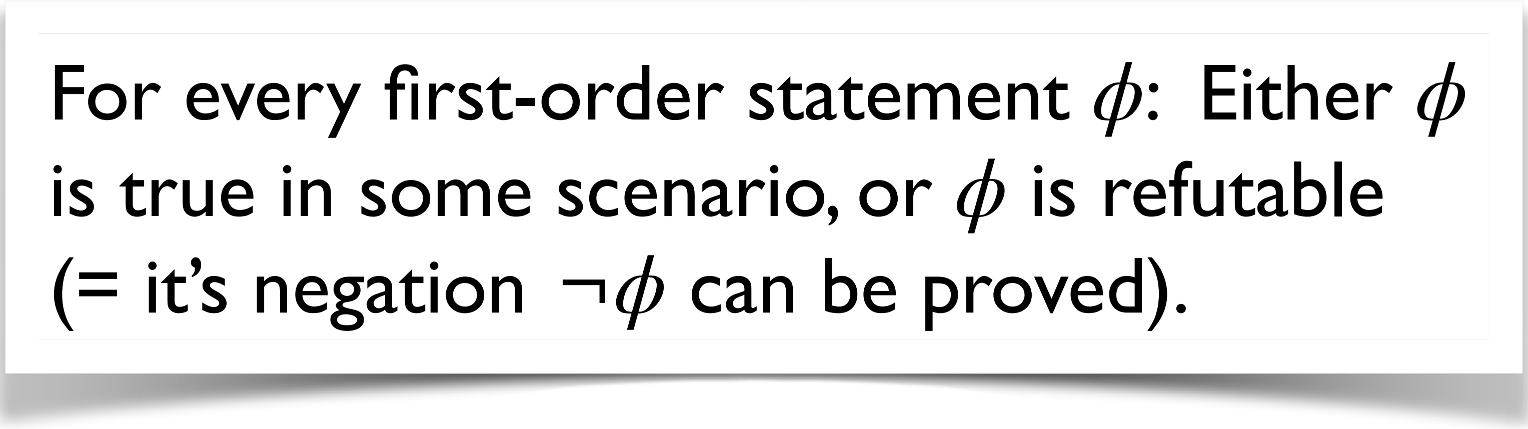
Completeness Theorem as an Equation

In first-order logic: NECESSARY TRUTH = PROVABILITY.

Completeness Theorem, More Precisely Put

For every first-order statement ϕ: if ϕ is a necessary or absolute truth (i.e. true in any scenario whatsoever), then ϕ is provable.

And the version Gödel targeted, and proved:



For every first-order statement ϕ: Either ϕ is true in some scenario, or ϕ is refutable (= it’s negation $\neg\phi$ can be proved).

GCT

The Proof-Sketch

Gödel's Version Equivalent

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$\vDash \phi$ implies $\vdash \phi$

iff

Either ϕ is true in some scenario (i.e. Sat ϕ),
or ϕ is refutable (i.e. $\vdash \neg\phi$).

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Prove the equivalence.

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$\models \phi$ implies $\vdash \phi$

iff

Either ϕ is true in some scenario (i.e. Sat ϕ),
or ϕ is refutable (i.e. $\vdash \neg\phi$).

Prove the equivalence.

Can you prove Gödel's version for the propositional calculus in HyperSlate?

The Proof-Sketch

To prove the theorem in the case of first-order logic ($= \mathscr{L}_1$), we need to show that given any set Γ of formulae in first-order logic, either there's a scenario on which every member of this set is true; otherwise, there is a refutation of the set, i.e. a proof from the set to an outright contradiction $\phi \land \neg\phi$. We can accomplish this by finding a procedure $\mathscr{P}$ that first takes the set in question and goes hunting for a scenario that does the trick. If the scenario is found, we're done. But, if such a scenario *can't* be found, then our procedure moves on to find a proof of a contradiction from Γ!

How?! The procedure $\mathscr{P}$ is the building out of a semantic hypergraph! If all the branches in the graph close, then the finding of a proof of a contradiction uses **resolution**, and the **resolution guarantee**. The guarantee is that if you have a set of formulae that can’t be true in any scenario, resolution applied to the set finds a contradiction $\perp = \zeta \land \neg\zeta = \{\}$. **QED**

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See LAMA-BDLAHGHS.

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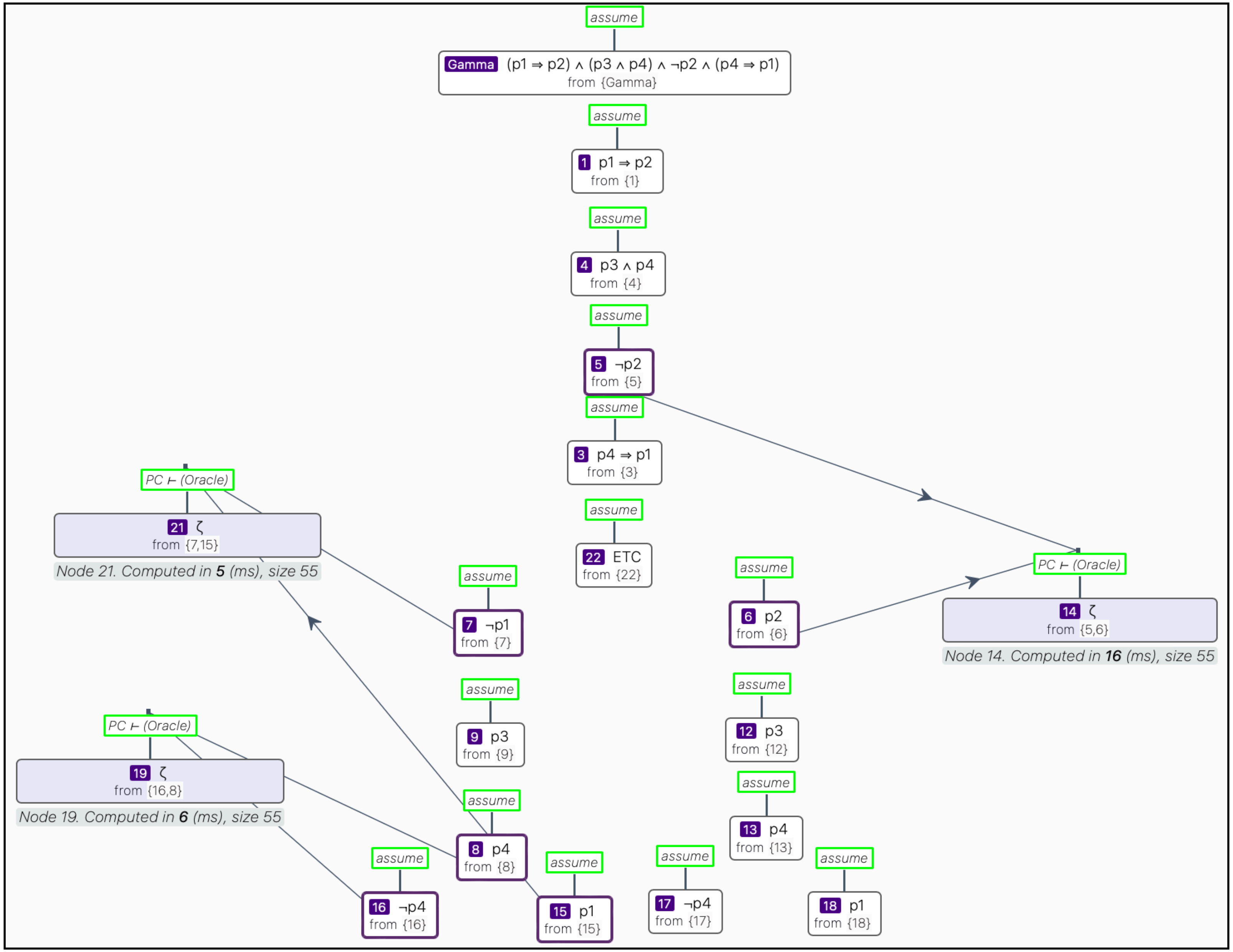
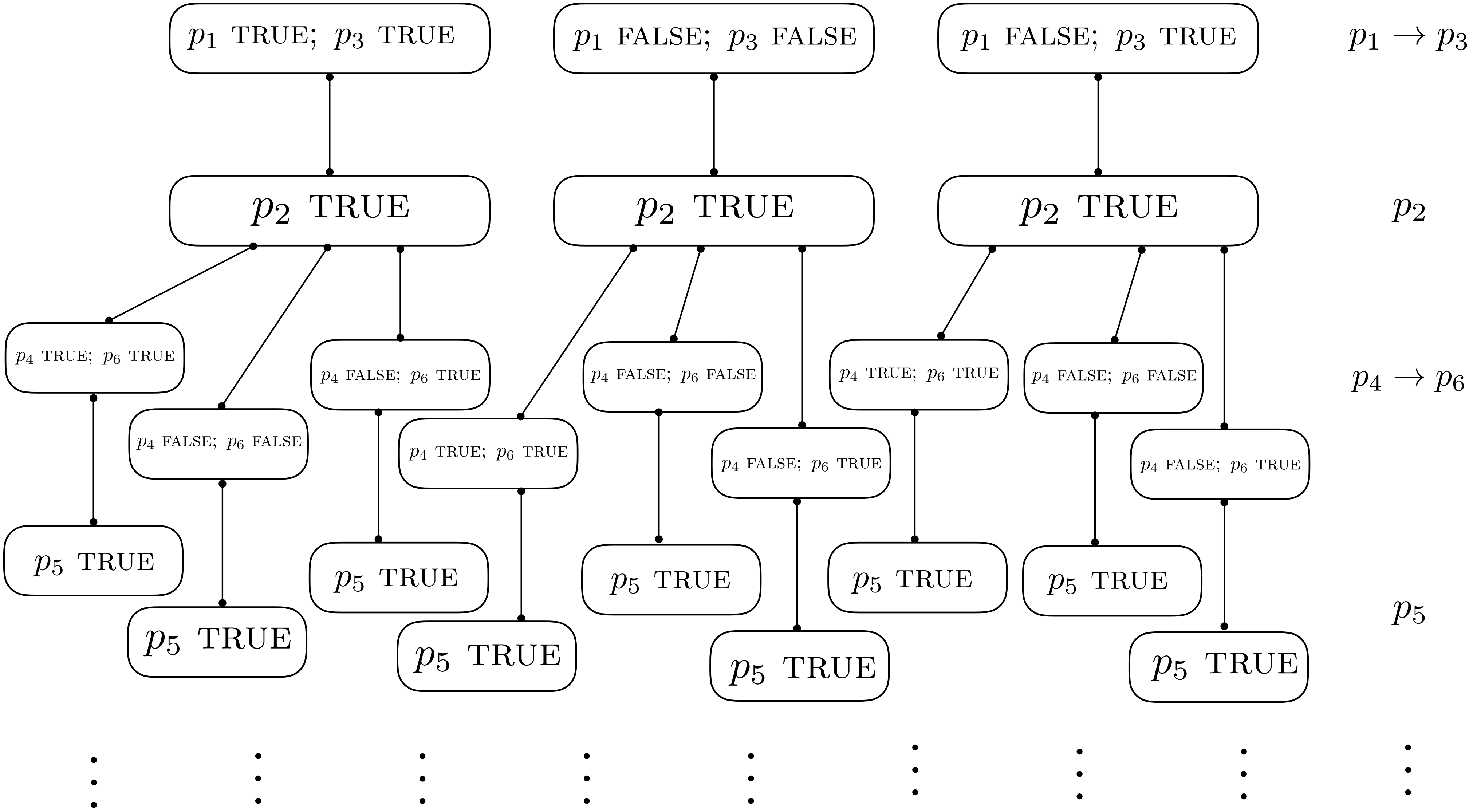
Semantic Graph For $\mathscr{P}$

$$\Gamma := \{p_1 \to p_2, p_3 \wedge p_4, \neg p_2, p_4 \to p_1, \ldots\}$$



$$\Gamma := \{p_1 \to p_3,\ p_2,\ p_4 \to p_6,\ p_5,\ p_7 \to p_9,\ p_8,\ \ldots\}$$



p_1 TRUE; p_3 TRUE

p_1 FALSE; p_3 FALSE

p_1 FALSE; p_3 TRUE

$p_1 \to p_3$

p_2 TRUE

p_2 TRUE

p_2 TRUE

p_2

p_4 TRUE; p_6 TRUE

p_4 FALSE; p_6 FALSE

p_4 FALSE; p_6 TRUE

p_4 TRUE; p_6 TRUE

p_4 FALSE; p_6 FALSE

p_4 FALSE; p_6 TRUE

p_4 TRUE; p_6 TRUE

p_4 FALSE; p_6 FALSE

p_4 FALSE; p_6 TRUE

$p_4 \to p_6$

p_5 TRUE

p_5 TRUE

p_5 TRUE

p_5 TRUE

p_5 TRUE

p_5 TRUE

p_5 TRUE

p_5 TRUE

p_5 TRUE

p_5

Therefore, since we can travel to infinity, there *is* a scenario in which all of the formulae are true: any infinite path down will do.

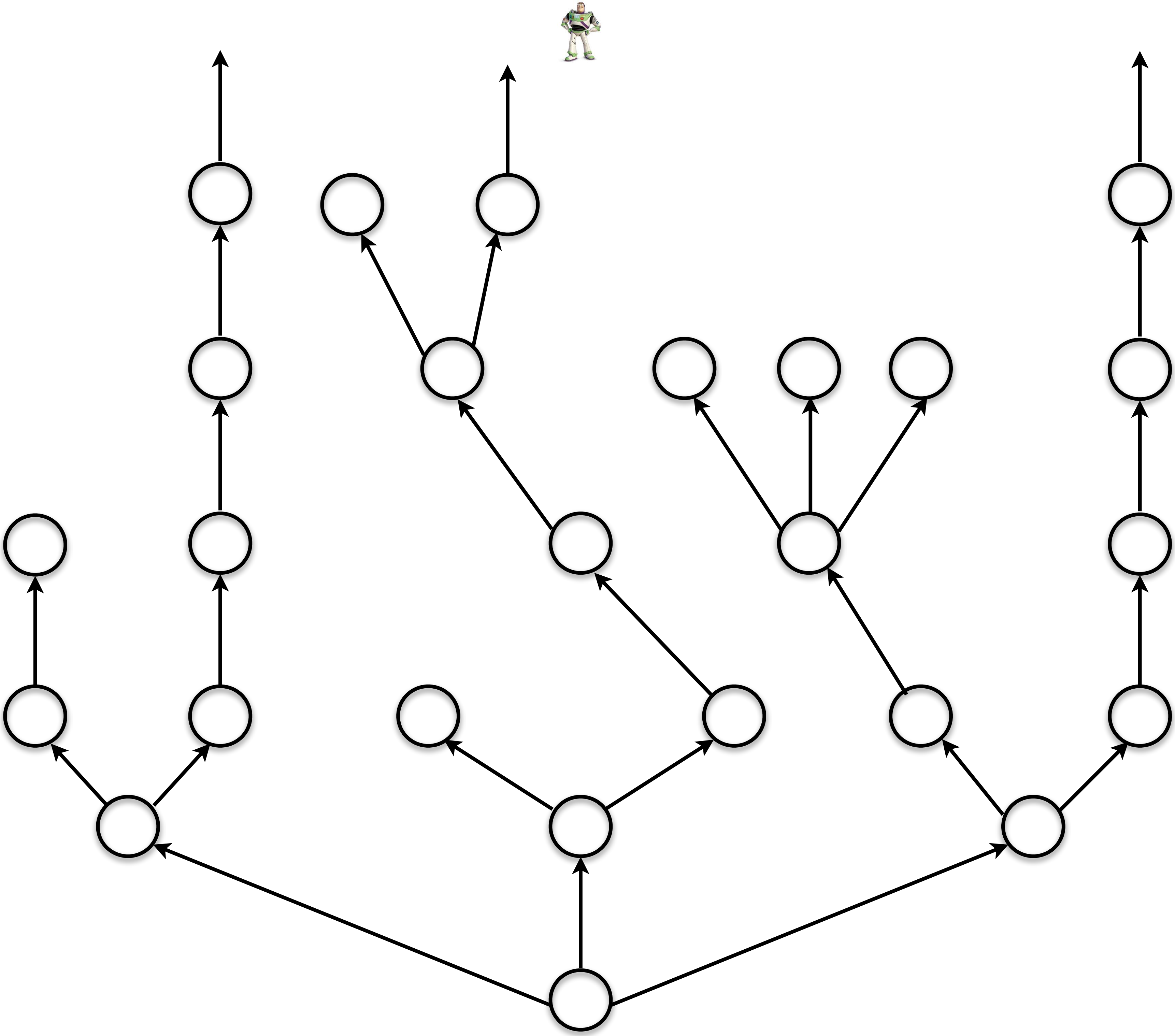
Ah, but can we travel to infinity? The assumption that there *is* an infinite branch here is based on König's Lemma …

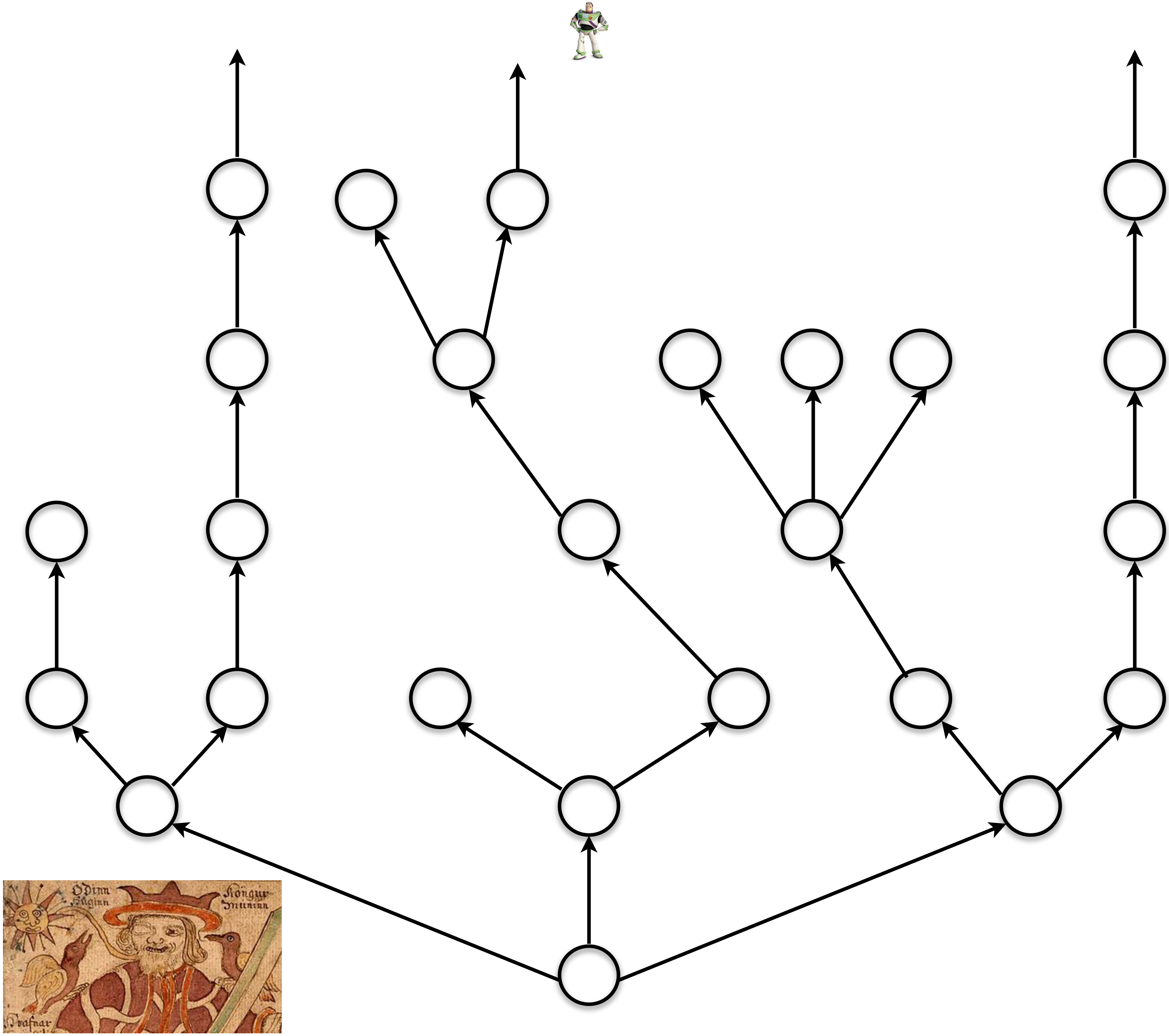
Toward König's Lemma as Train Travel

“To infinity and beyond!”

König's Lemma (train-travel version)

In a horizontalized one-way train-travel map extending infinitely upwards into Euclid's Quadrants I & II in which each stop at level n (y-axis) allows passage to level $n+1$ by at least one train option, there is an infinite *trip* (= a trip "to infinity").

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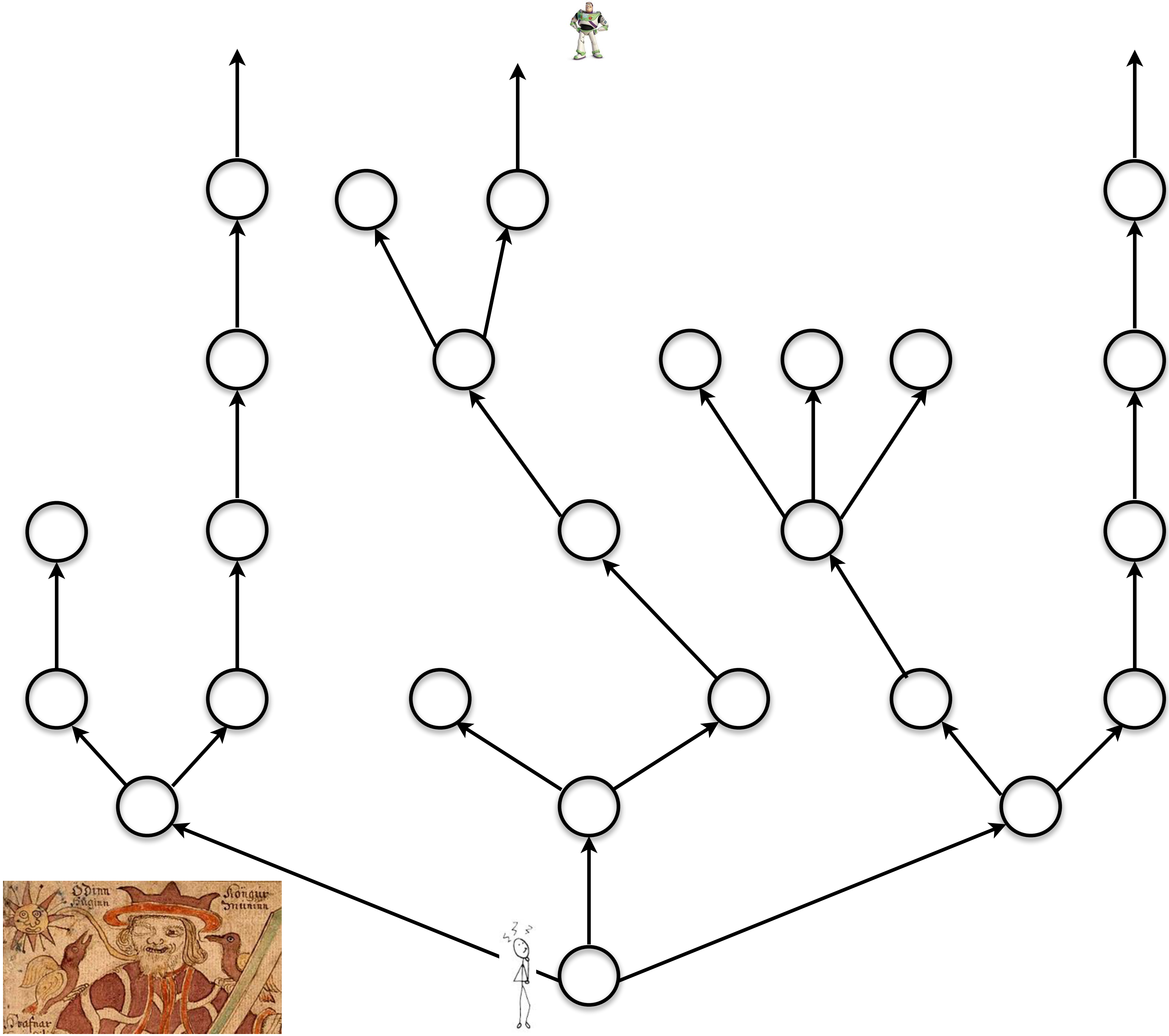
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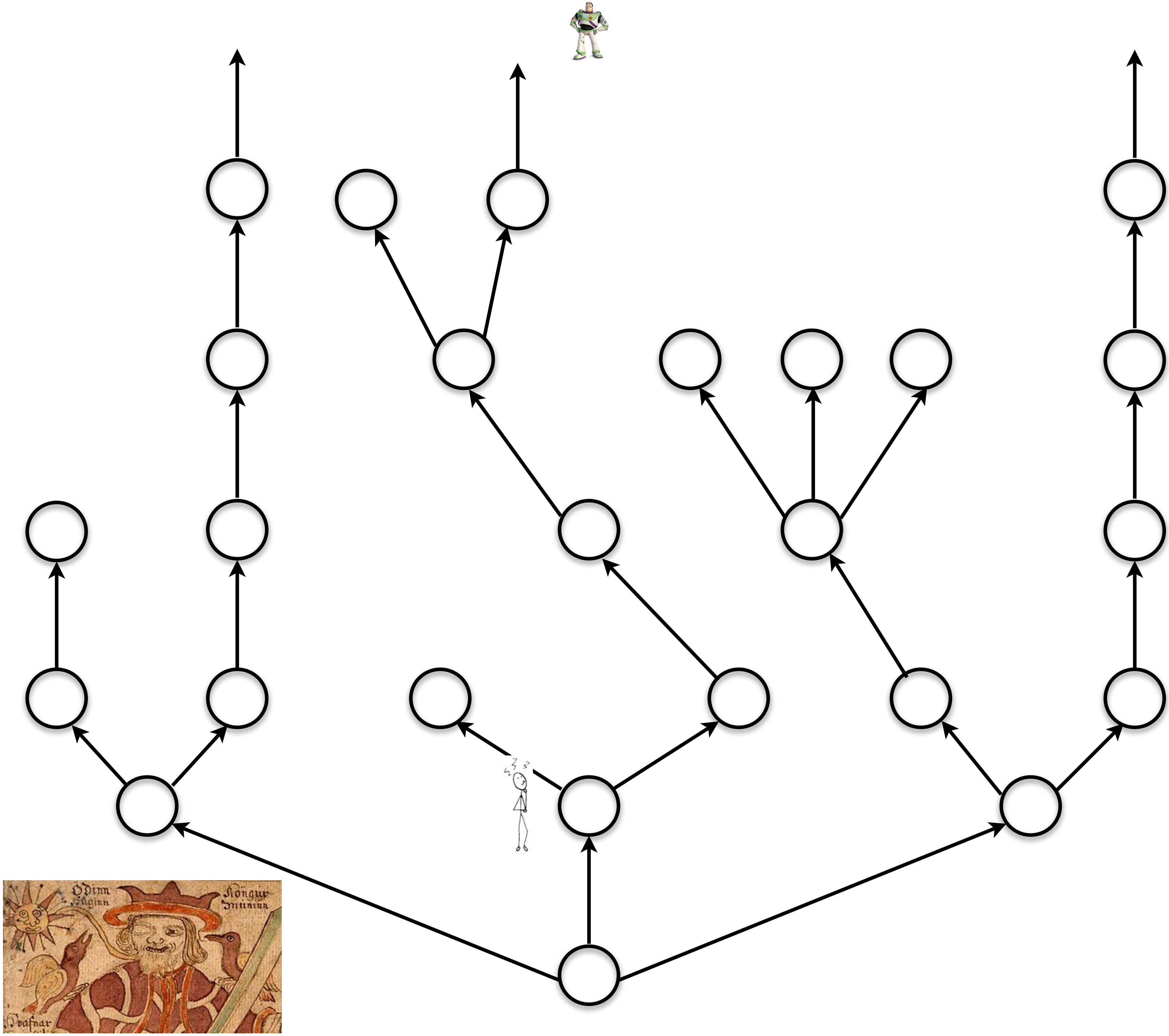
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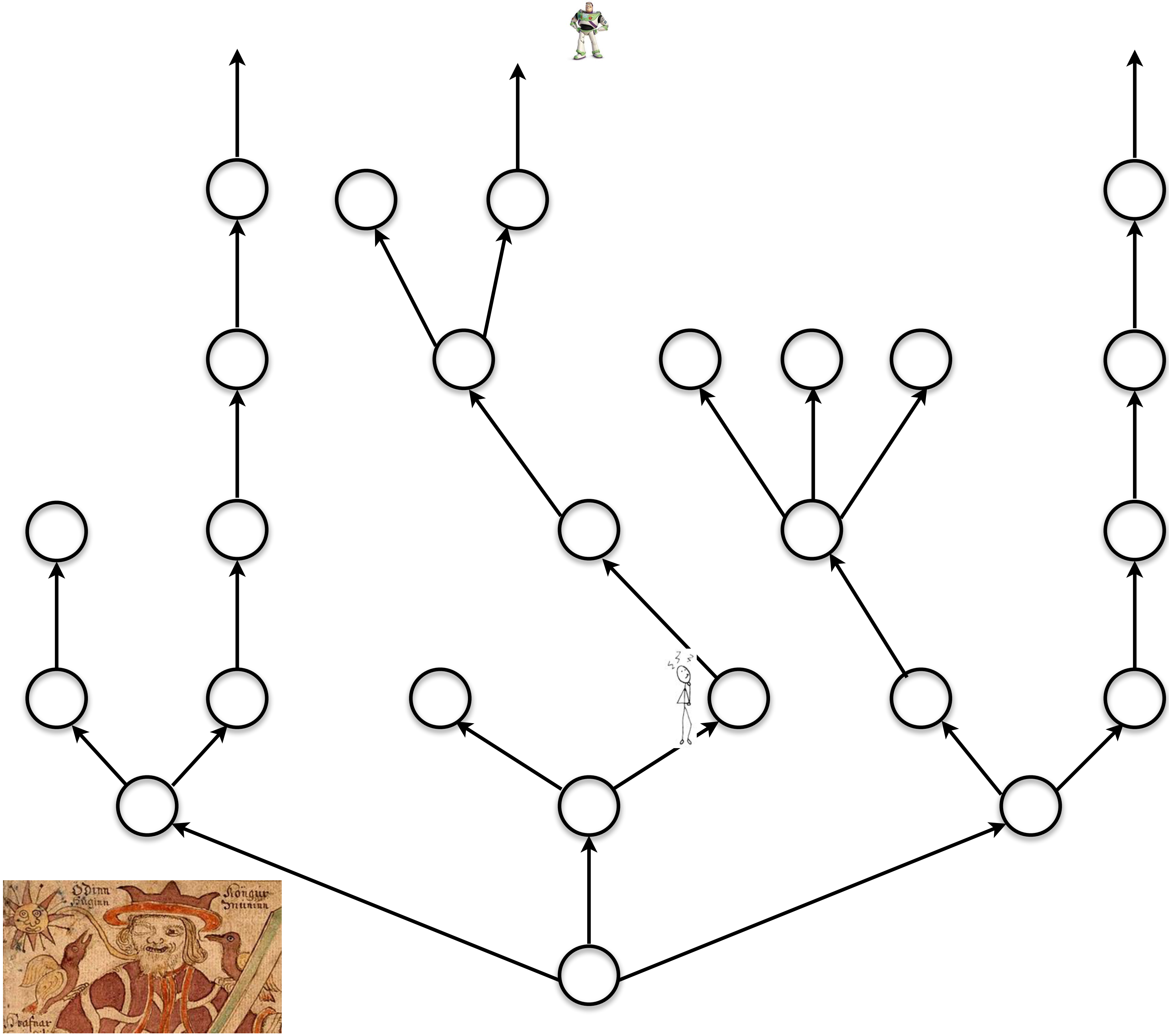
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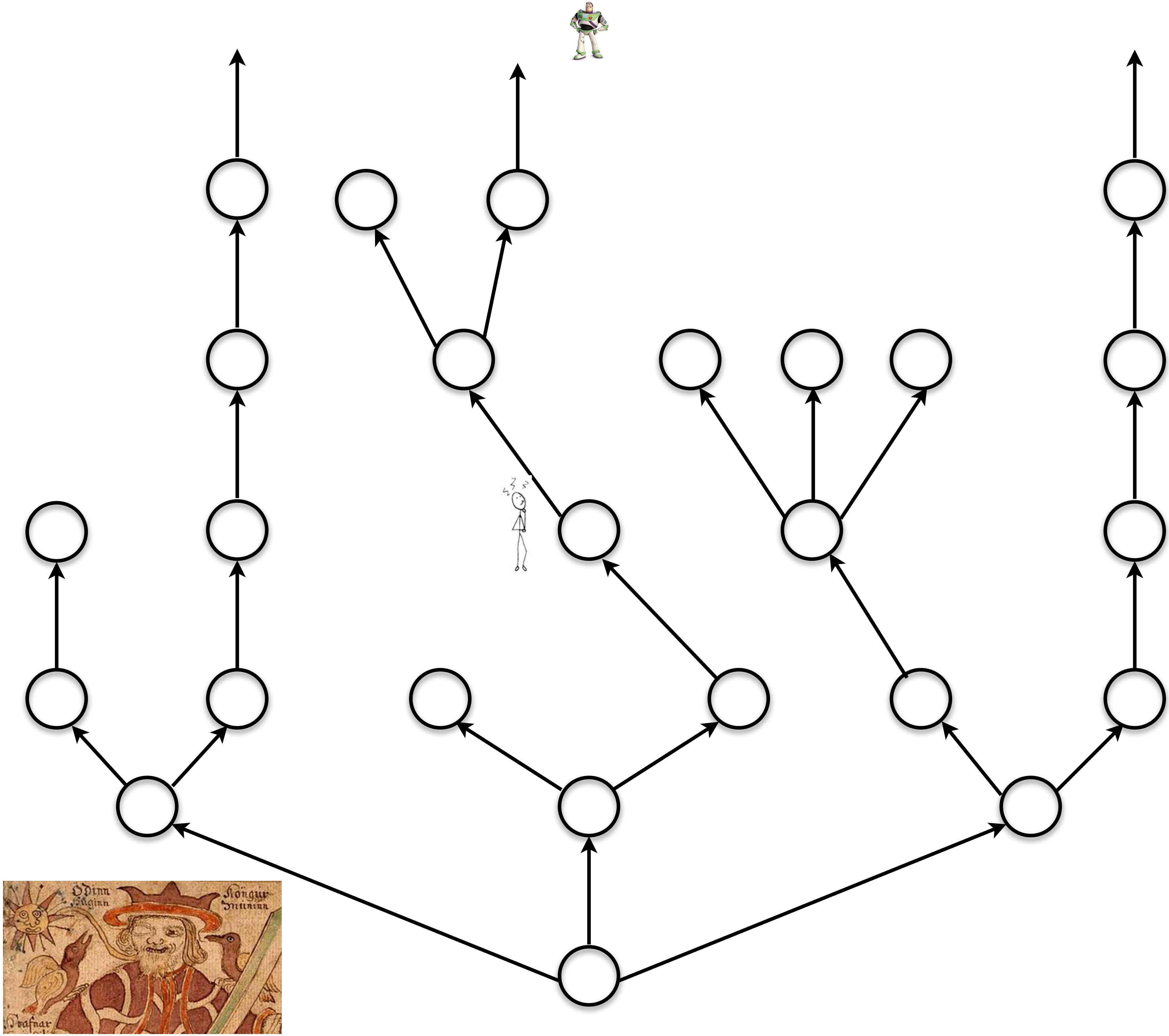

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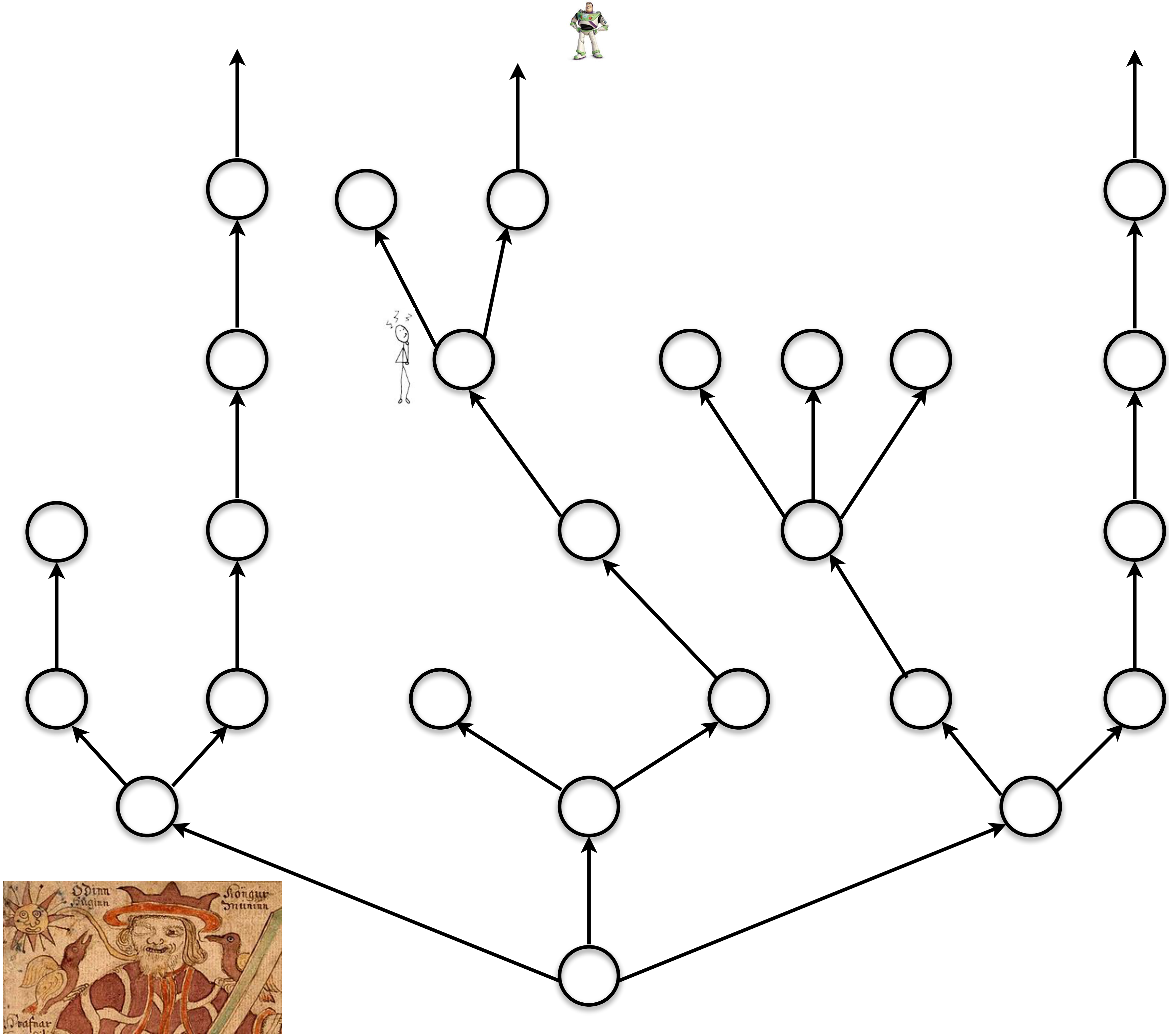
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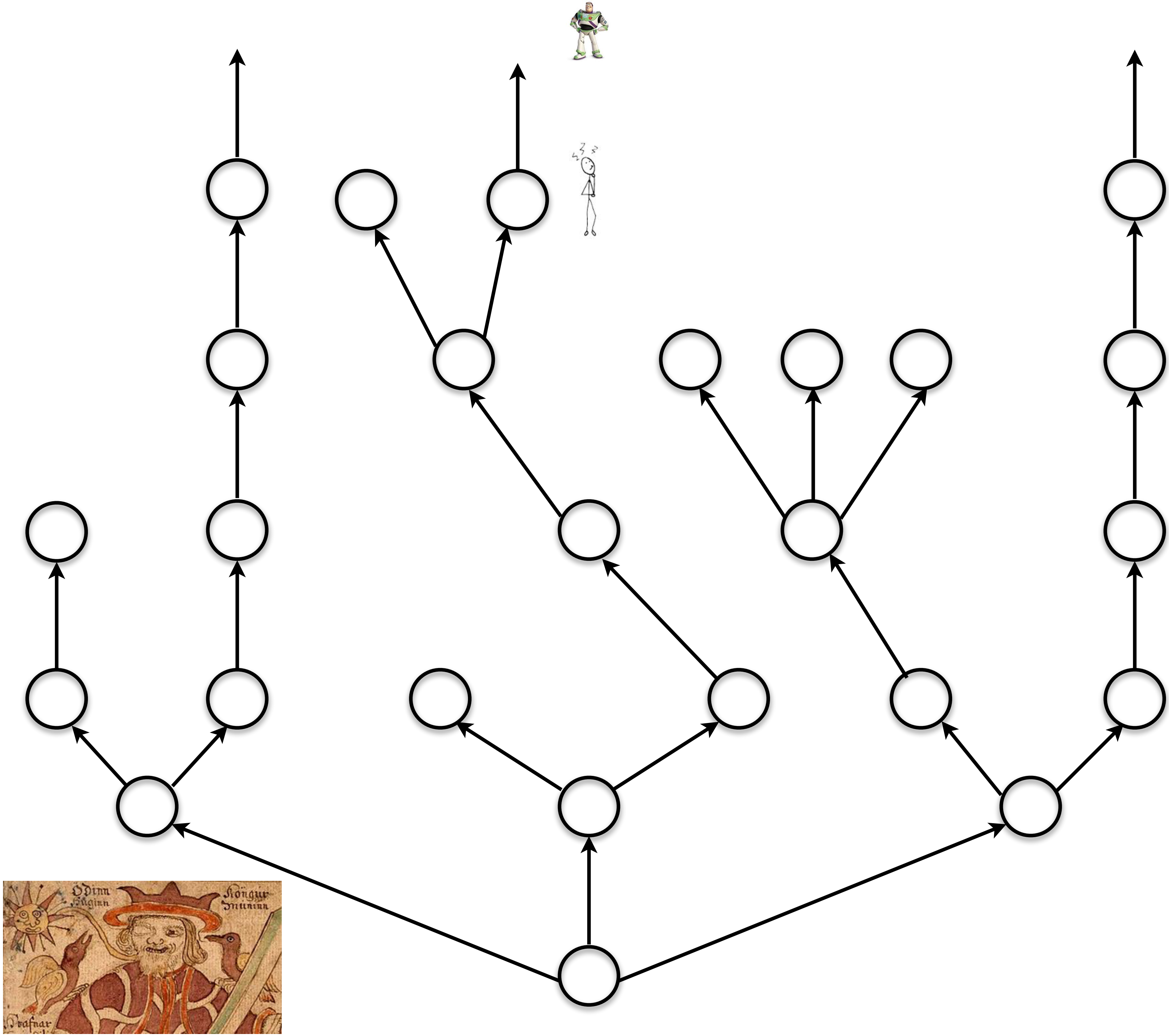
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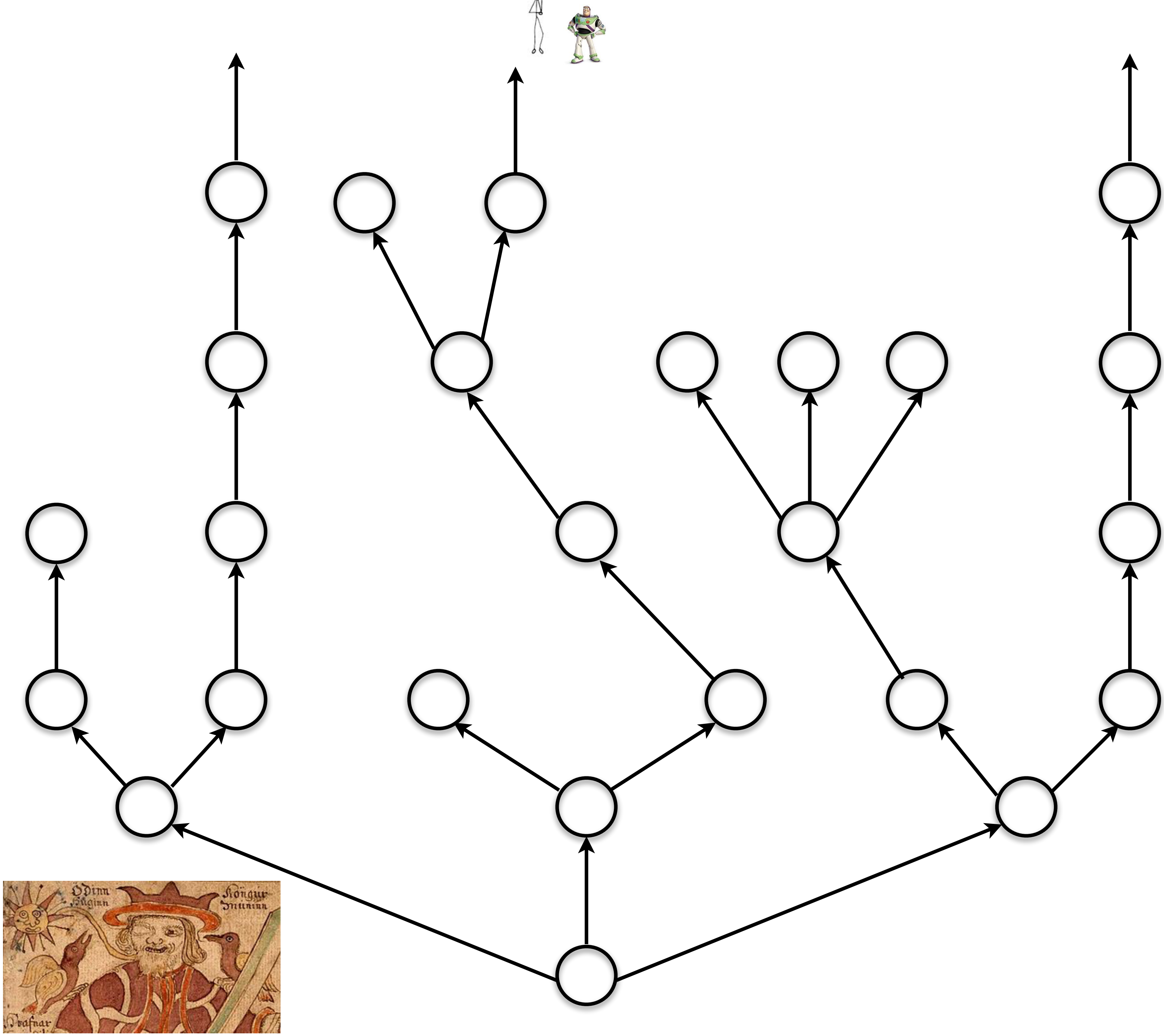
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Exercise 2:
Is there an algorithm for traveling this way?

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No. This strategy for travel is beyond the reach of constructive mathematics/standard computation.

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No. This strategy for travel is beyond the reach of constructive mathematics/standard computation.

(Does it not then follow, assuming that humans can find and "use" a provably correct strategy for this travel, that humans can't be fundamentally computing machines?)

NY-Centric Proof of the Lemma

(that there is an infinite branch)

Proof: We are seeking to prove that there is an infinite path (= that you can keep going forward forever = that the number of your stops forward are the size of $\mathbf{Z}^+$).

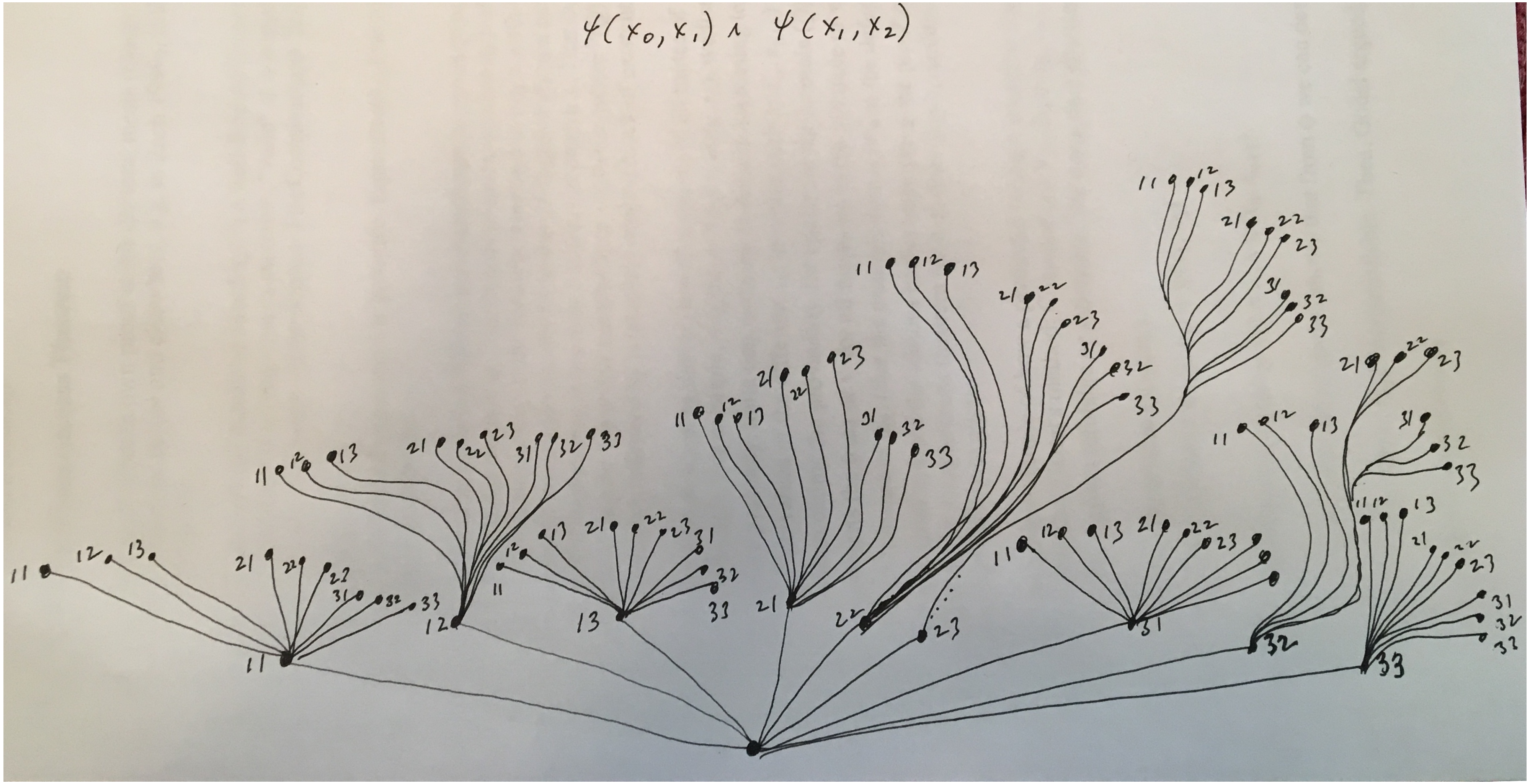
To begin, assume the antecedent of the theorem (i.e. that, (1), there are finitely many options leading from each station, and that, (2), in the map there are partial paths forward of every finite size).

Now, you are standing at Penn Station (S_1), facing k options. At least one of these options must lead to partial paths of arbitrary size (the size of any m in $\mathbf{Z}^+$). (**Sub-Proof**: Suppose otherwise for indirect proof. Then there is some positive integer n that places a ceiling on the size of partial paths that can be reached. But this violates (2) — contradiction.) Proceed to choose one of these options that lead to partial paths of arbitrary size. You are now standing at a new station (S_2), one stop after Penn Station. At least one of these options must lead to partial parts of arbitrary size (the size of any m in $\mathbf{Z}^+$). (**Sub-Proof**: Suppose otherwise for indirect proof …)

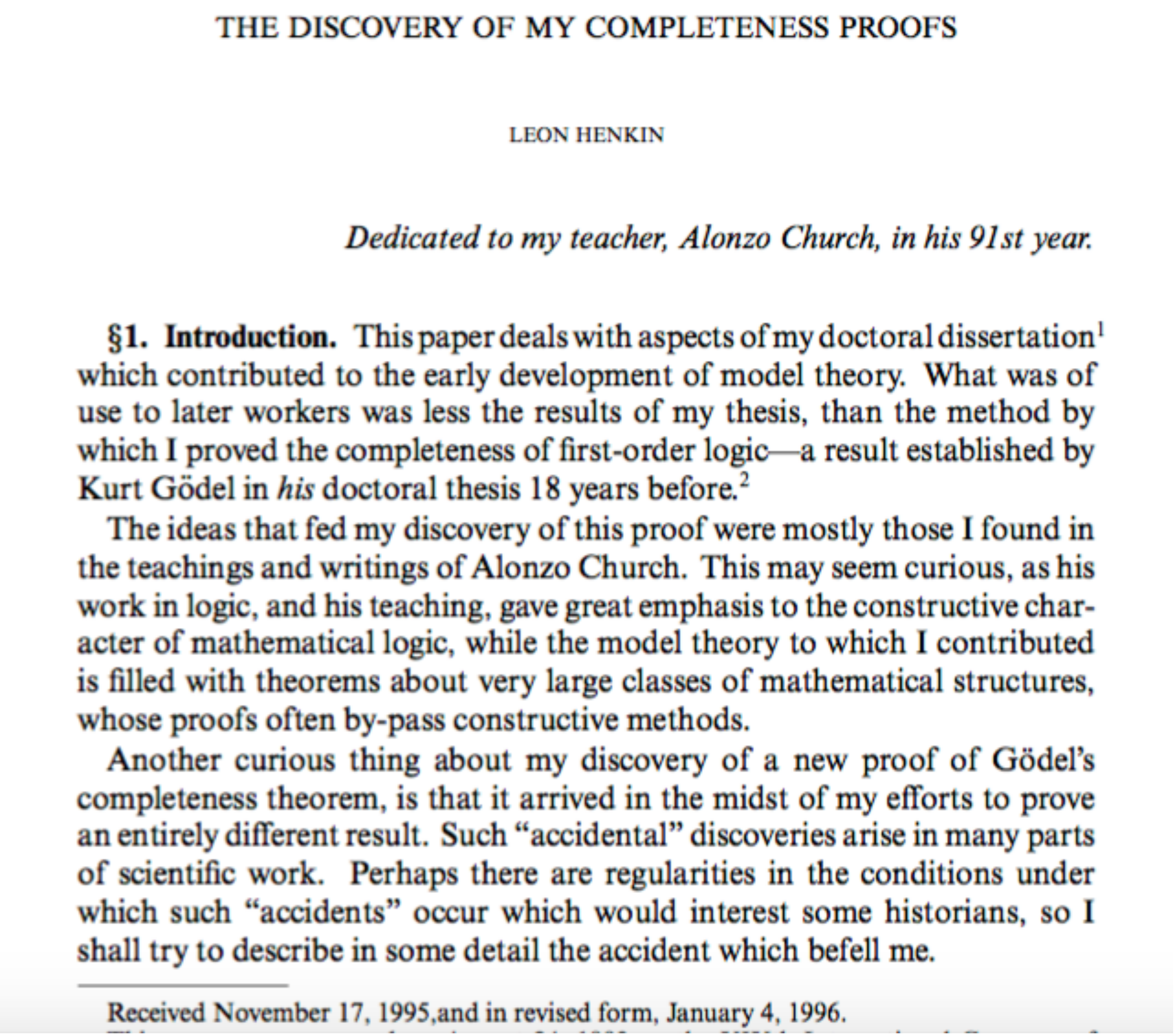
Since you can iterate this forever, you'll be on an infinite trip to infinity! Buzz will be happy.

QED

Simple Buzz-Lightyear-Like Branch



Gödel as Giant: Some Evidence



THE DISCOVERY OF MY COMPLETENESS PROOFS

LEON HENKIN

Dedicated to my teacher, Alonzo Church, in his 91st year.

§1. Introduction. This paper deals with aspects of my doctoral dissertation[1] which contributed to the early development of model theory. What was of use to later workers was less the results of my thesis, than the method by which I proved the completeness of first-order logic—a result established by Kurt Gödel in *his* doctoral thesis 18 years before.[2]

The ideas that fed my discovery of this proof were mostly those I found in the teachings and writings of Alonzo Church. This may seem curious, as his work in logic, and his teaching, gave great emphasis to the constructive character of mathematical logic, while the model theory to which I contributed is filled with theorems about very large classes of mathematical structures, whose proofs often by-pass constructive methods.

Another curious thing about my discovery of a new proof of Gödel's completeness theorem, is that it arrived in the midst of my efforts to prove an entirely different result. Such "accidental" discoveries arise in many parts of scientific work. Perhaps there are regularities in the conditions under which such "accidents" occur which would interest some historians, so I shall try to describe in some detail the accident which befell me.

Received November 17, 1995,and in revised form, January 4, 1996.

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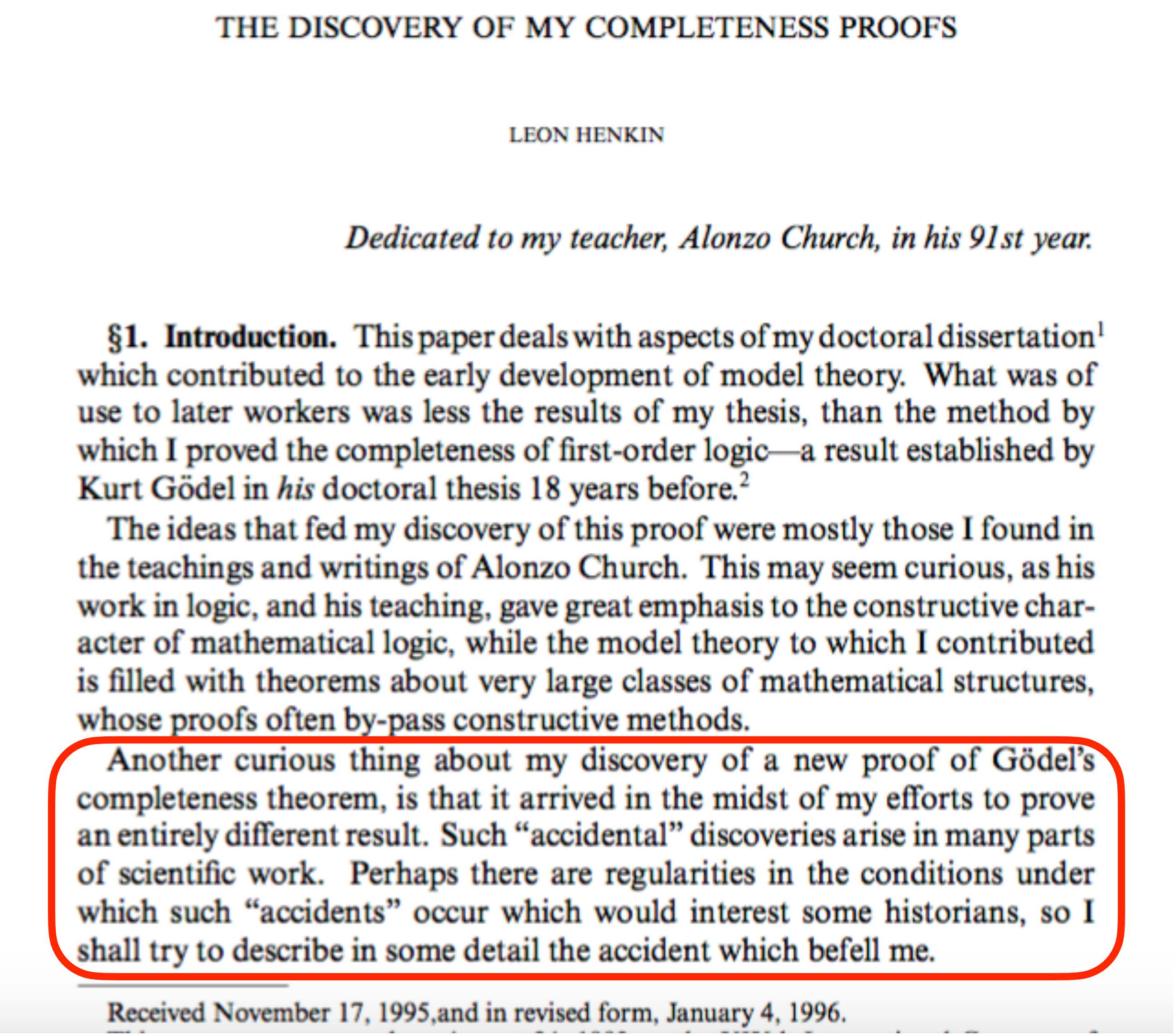

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$$\forall x_1, x_2, \ldots, x_k \exists y_1, y_2, \ldots, y_m \phi(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_m)$$

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How? By ingenious tree-building, which starts by creating an enumeration of new constants $\mathfrak{c} = c_1, c_2, \ldots$that becomes our "universe of discourse"/"domain of quantification." Note that from $\mathfrak{c}$ we can algorithmically generate an enumeration of tuples $\mathfrak{c}^t = \langle c \rangle_1, \langle c \rangle_2, \ldots$ of any finite size. (Those of size *k* will work for the *x*-variables, and those of size *n* will work for the *y*-variables.) And now we can build a BIG tree at the level of the pure predicate calculus, looking for either a scenario that makes our formula true by traveling with Buzz to infinity, or getting all branches closed, in which case we turn back to the resolution guarantee! Let's make sense of this by hand on paper …

Gödel's Completeness Theorem, Degree-1 Formulae

3/8/20

l.B.

$$\psi := \forall x_1, x_2, \dots, x_K \, \exists y_1, y_2, \dots, y_m \, \phi(x_1, x_2, \dots, x_K; y_1, y_2, \dots, y_m)$$

$\mathcal{C} := c_1, c_2, c_3, \dots \quad \omega$

Then tuples of the size of the x_i variables (i.e. size k) in this progression,

$\mathcal{C}^{t\text{-}K} := \langle c \rangle^K_1, \langle c \rangle^K_2, \langle c \rangle^K_3, \dots \quad \omega$

<u>And</u>, tuples of the size of the y_i variables (i.e. size m) in this progression

$\mathcal{C}^{t\text{-}m} := \langle c \rangle^m_1, \langle c \rangle^m_2, \langle c \rangle^m_3, \dots \quad \omega$

And here's my enumeration

$$\langle c \rangle^K_1 ; \langle c \rangle^m_1 , \ \langle c \rangle^K_1 ; \langle c \rangle^m_2 , \ \langle c \rangle^K_1 ; \langle c \rangle^K_3 , \dots \omega$$
$$\langle c \rangle^K_2 ; \langle c \rangle^m_1 , \ \langle c \rangle^K_2 ; \langle c \rangle^m_2 , \ \langle c \rangle^K_2 ; \langle c \rangle^K_3 , \dots \omega$$
$$\vdots$$

In streamlined form:

11	12	13	14	15	...	ω
21	22	23	24	25	...	ω
31	32	33	34	35	...	
41	42	43	44	45	...	ω
⋮	⋮	⋮	⋮	⋮		
ω	ω	ω	ω	ω		

Do you see a way to travel to ∞ in a tree based on some enumeration of this infinite array?

Gödel's Completeness Theorem, Degree-1 Formulae

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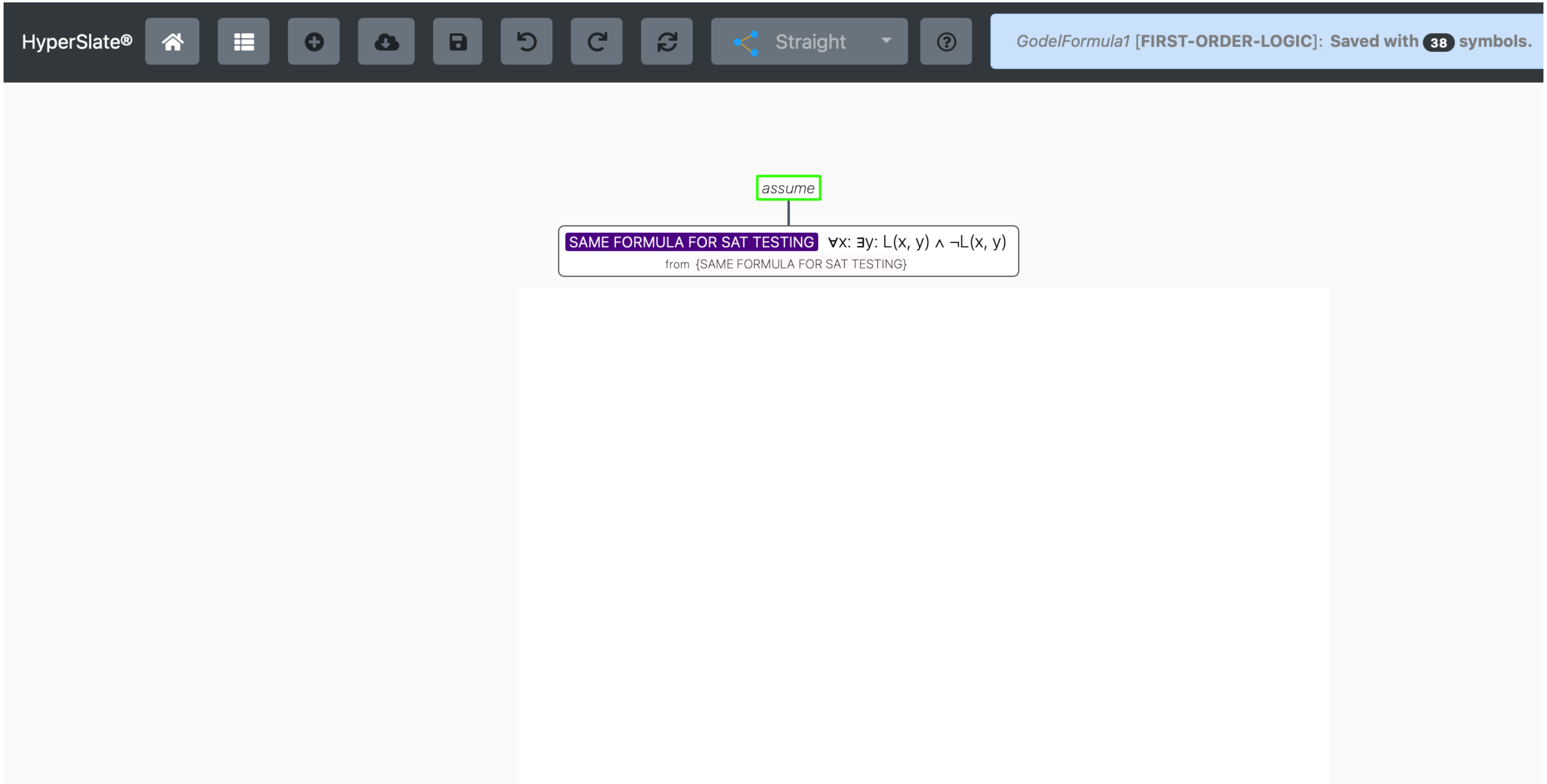
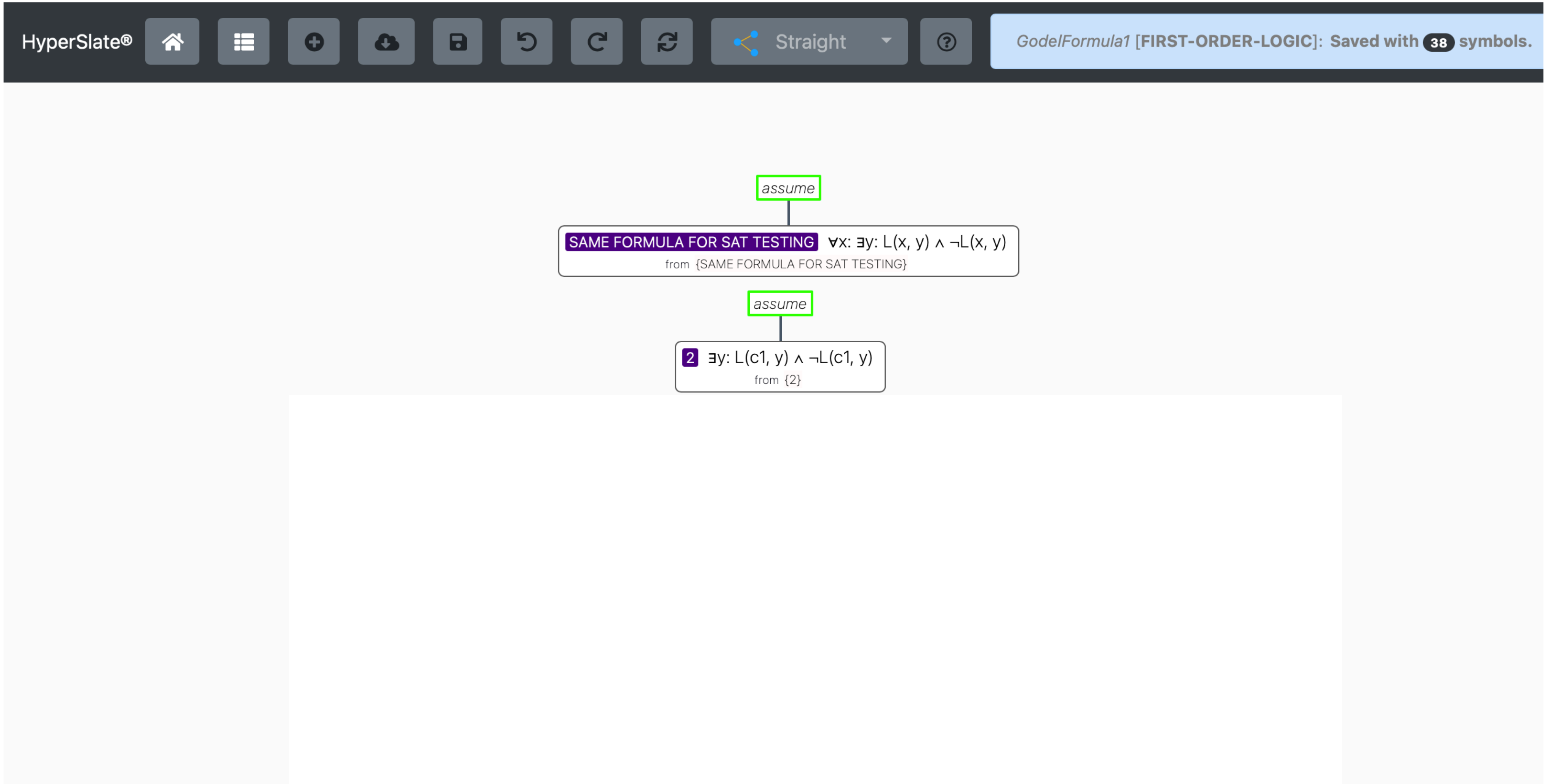


GodelFormula1 [FIRST-ORDER-LOGIC]: Saved with 38 symbols.

SAME FORMULA FOR SAT TESTING ∀x: ∃y: L(x, y) ∧ ¬L(x, y)

from {SAME FORMULA FOR SAT TESTING}

Making this Concrete
Courtesy of HyperSlate®



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assume

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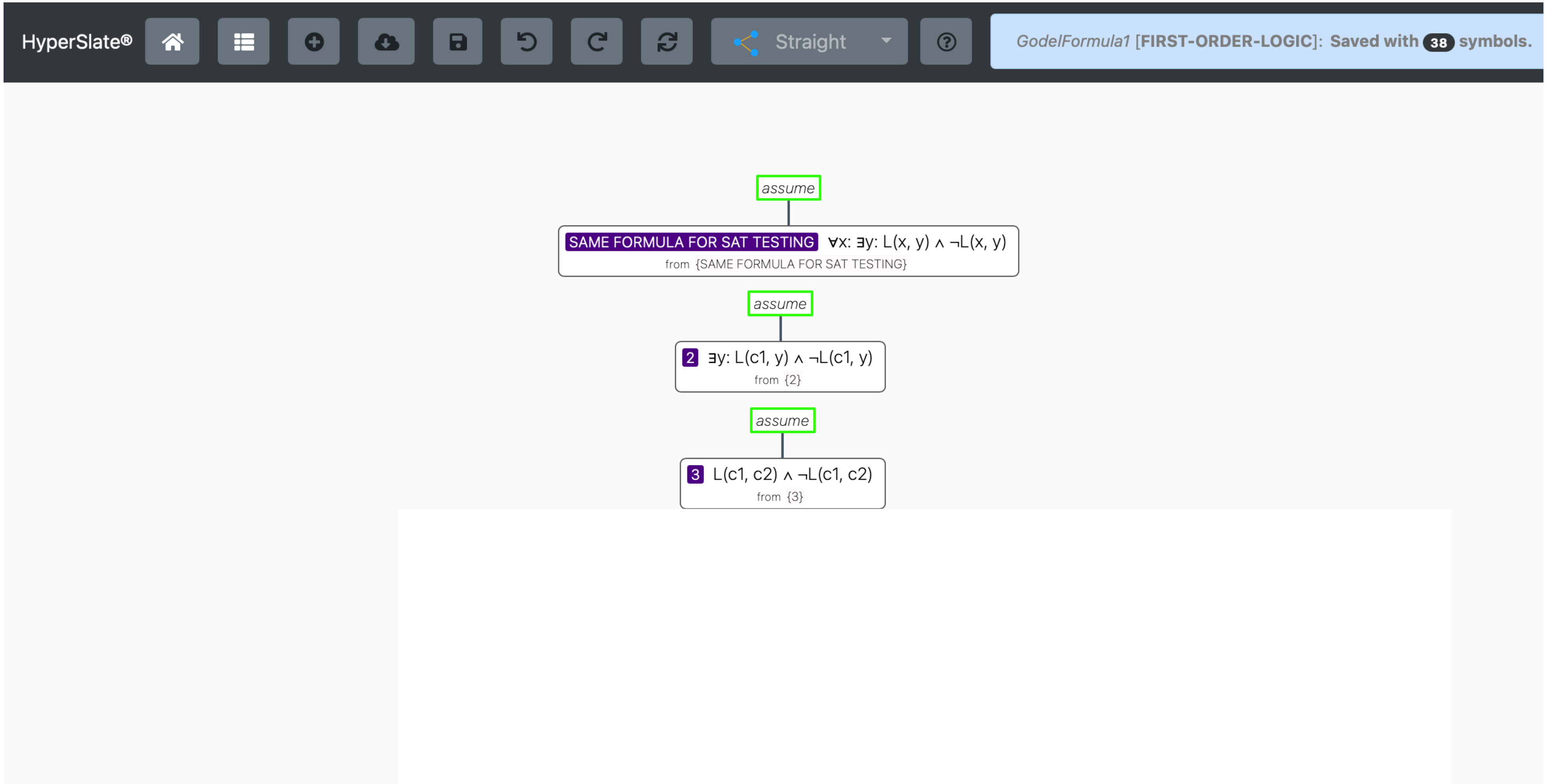
from {SAME FORMULA FOR SAT TESTING}

assume

2 ∃y: L(c1, y) ∧ ¬L(c1, y)
from {2}

Making this Concrete
Courtesy of HyperSlate®



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from {3}

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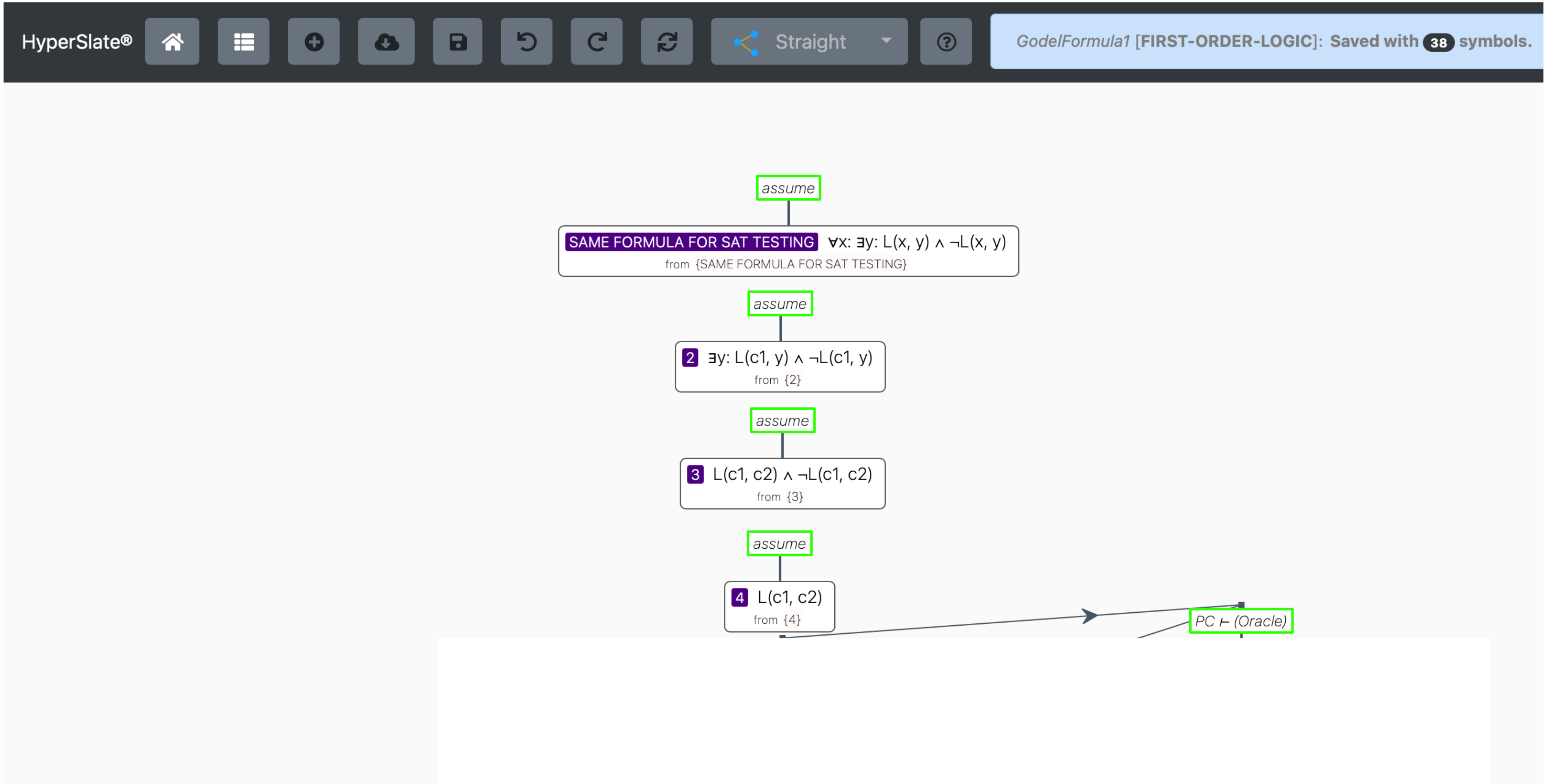

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4 L(c1, c2)
from {4}

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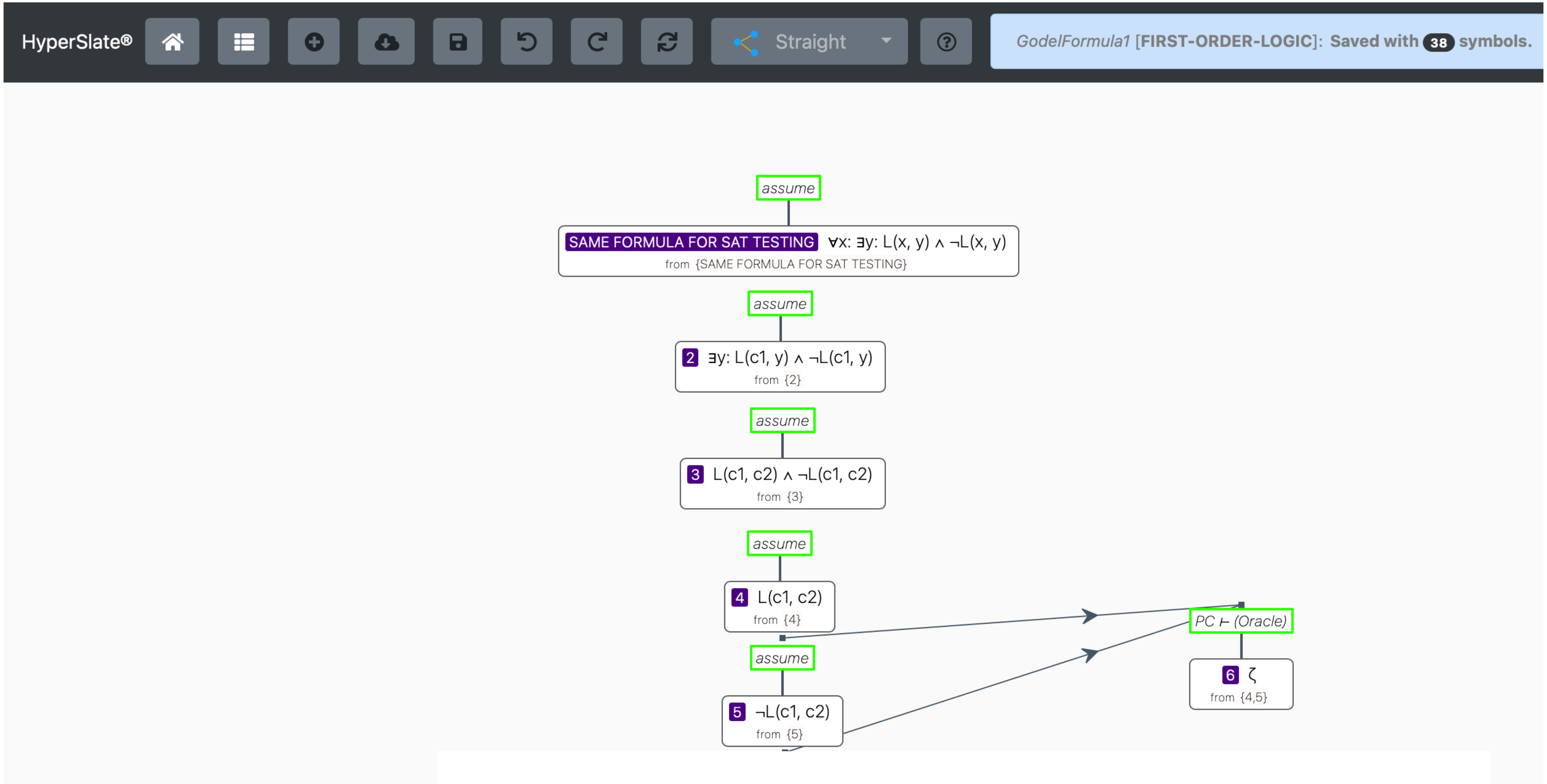

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assume
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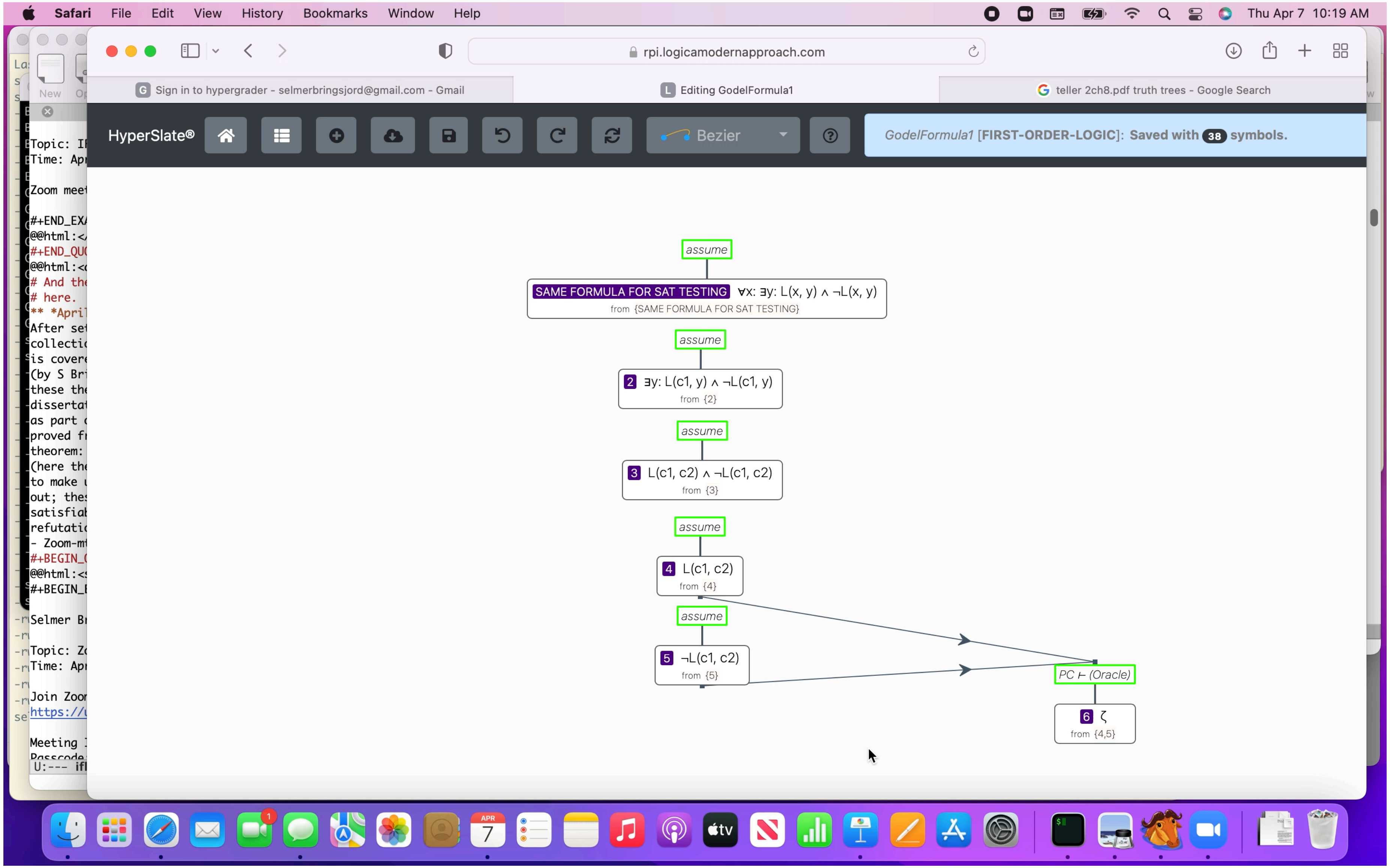
assume
3 L(c1, c2) ∧ ¬L(c1, c2)
from {3}

assume
4 L(c1, c2)
from {4}

assume
5 ¬L(c1, c2)
from {5}

PC ⊢ (Oracle)
6 ζ
from {4,5}

Making this Concrete Courtesy of HyperSlate®



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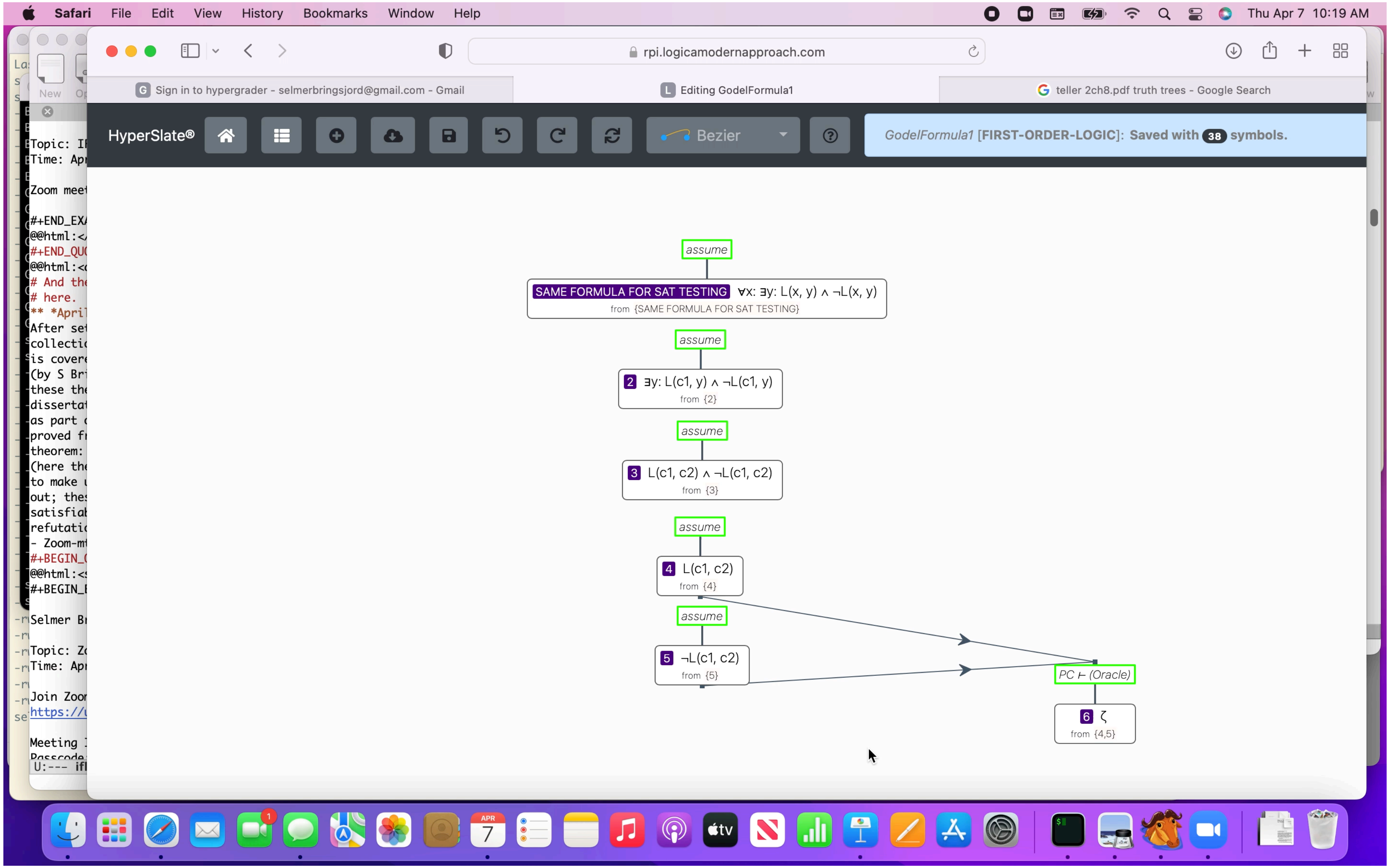

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assume
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from {3}

assume
4 L(c1, c2)
from {4}

assume
5 ¬L(c1, c2)
from {5}

PC ⊢ (Oracle)
6 ζ
from {4,5}

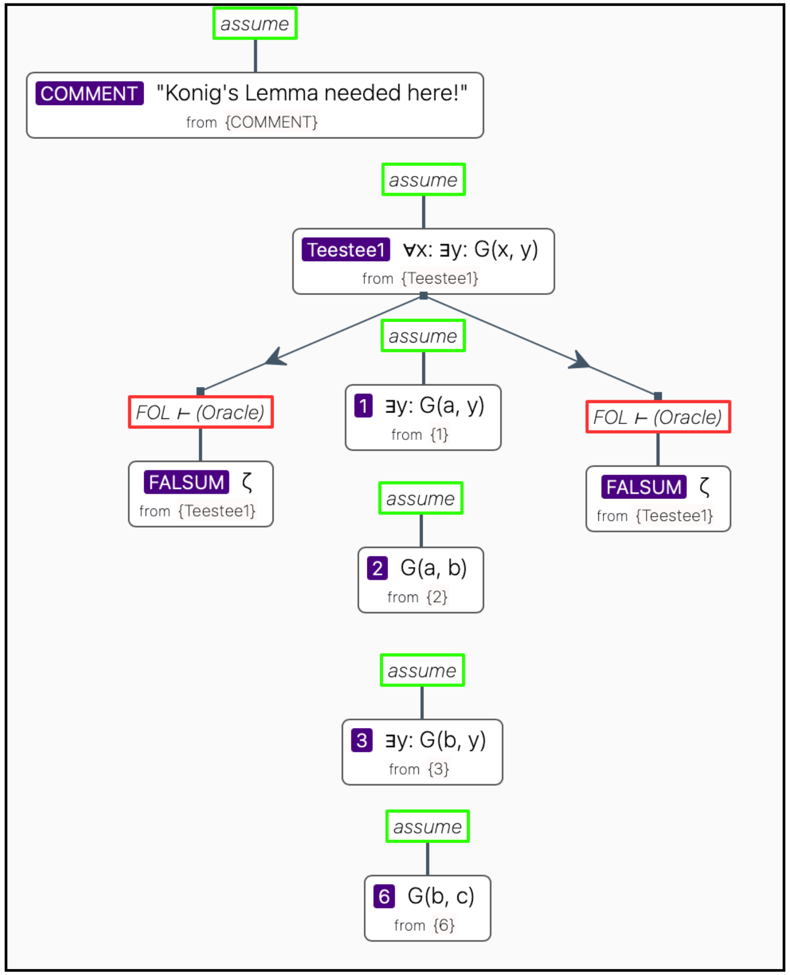
Making this Concrete Courtesy of HyperSlate®

Volunteers to do the following Buzz-Lightyear-relevant formula, from earlier in the present slide deck?

$$\forall x \exists y \; y > x$$

Making this Concrete Courtesy of HyperSlate®

E.g.



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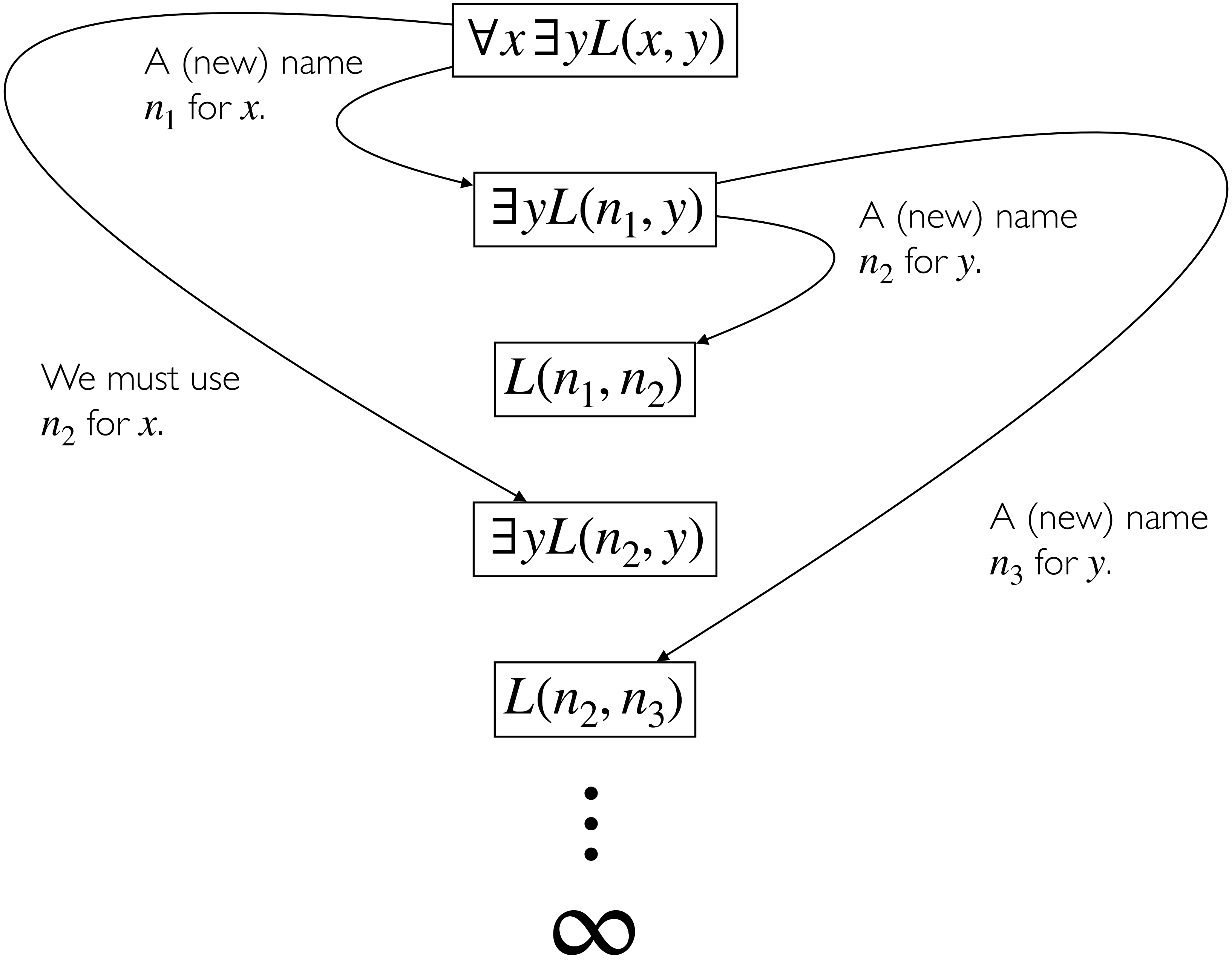


$\forall x \, \exists y L(x, y)$

A (new) name n_1 for x.

$\exists y L(n_1, y)$

A (new) name n_2 for y.

$L(n_1, n_2)$

We must use n_2 for x.

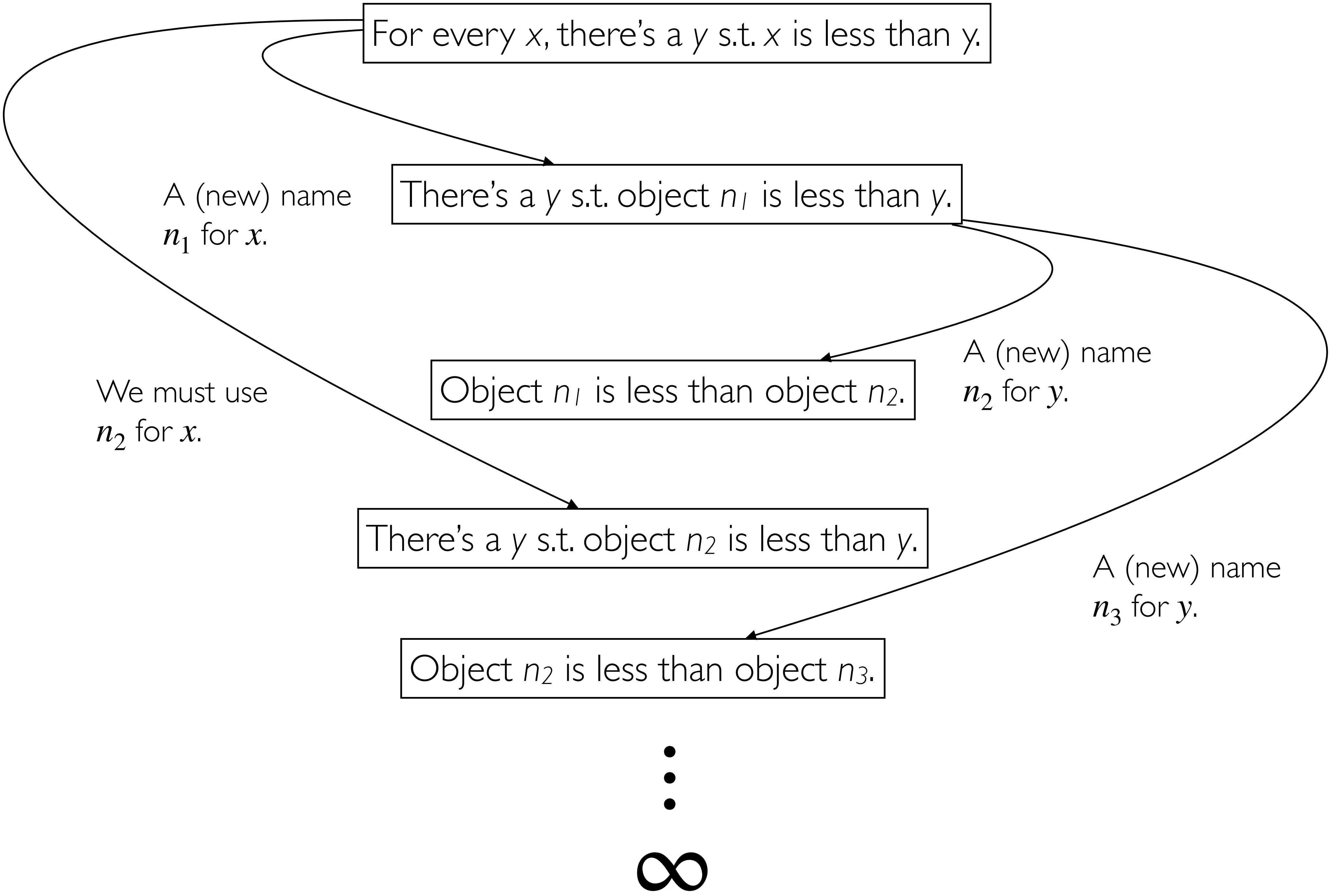
$\exists y L(n_2, y)$

A (new) name n_3 for y.

$L(n_2, n_3)$

$\vdots$

∞



For every x, there's a y s.t. x is less than y.

A (new) name n_1 for x.

There's a y s.t. object n_1 is less than y.

A (new) name n_2 for y.

Object n_1 is less than object n_2.

We must use n_2 for x.

There's a y s.t. object n_2 is less than y.

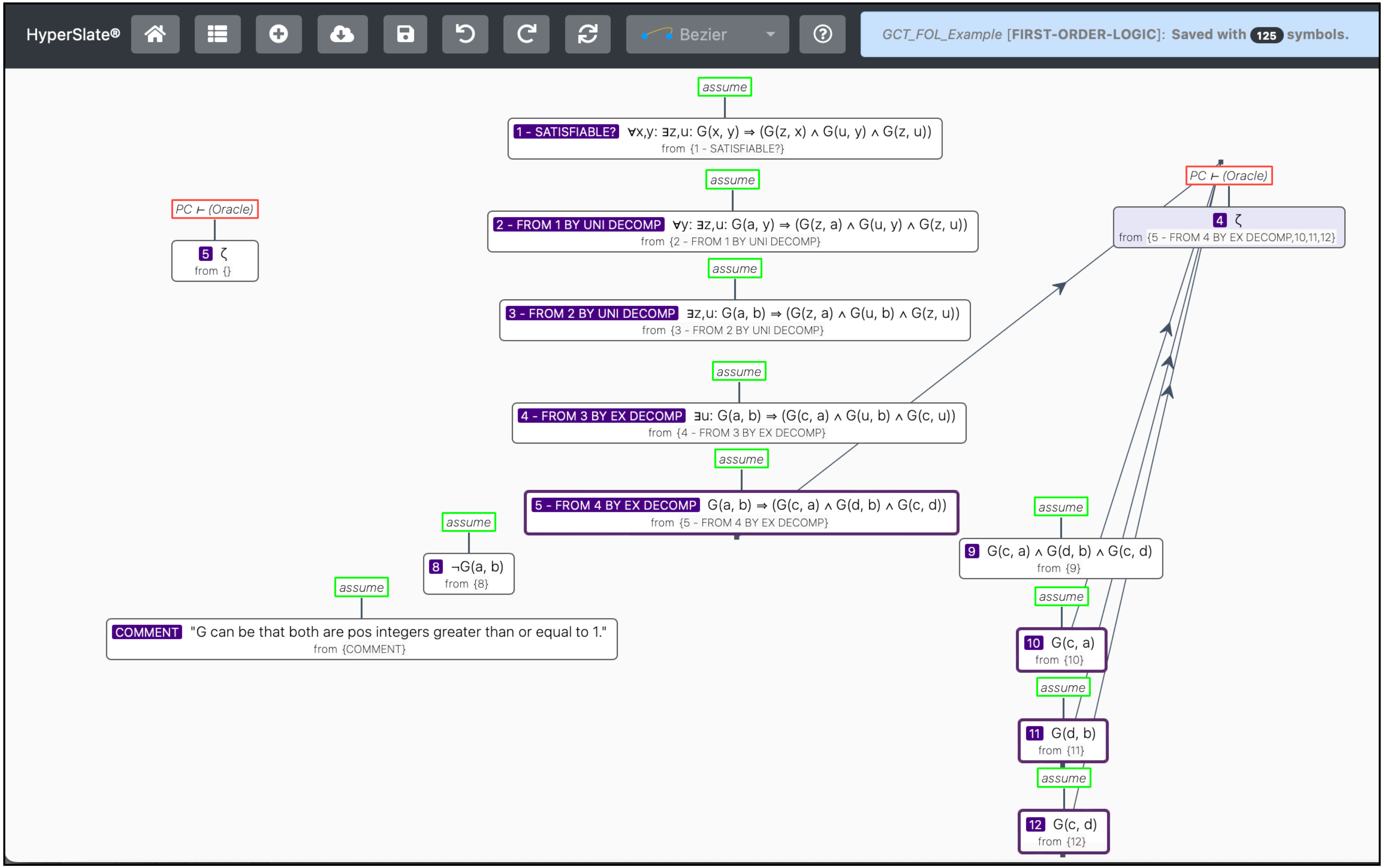
A (new) name n_3 for y.

Object n_2 is less than object n_3.

$\vdots$

∞

Infinite Train Trip in HyperSlate?



GCT_FOL_Example **[FIRST-ORDER-LOGIC]: Saved with 125 symbols.**

1 - SATISFIABLE? $\forall x,y\colon \exists z,u\colon G(x, y) \Rightarrow (G(z, x) \wedge G(u, y) \wedge G(z, u))$
from {1 - SATISFIABLE?}

2 - FROM 1 BY UNI DECOMP $\forall y\colon \exists z,u\colon G(a, y) \Rightarrow (G(z, a) \wedge G(u, y) \wedge G(z, u))$
from {2 - FROM 1 BY UNI DECOMP}

3 - FROM 2 BY UNI DECOMP $\exists z,u\colon G(a, b) \Rightarrow (G(z, a) \wedge G(u, b) \wedge G(z, u))$
from {3 - FROM 2 BY UNI DECOMP}

4 - FROM 3 BY EX DECOMP $\exists u\colon G(a, b) \Rightarrow (G(c, a) \wedge G(u, b) \wedge G(c, u))$
from {4 - FROM 3 BY EX DECOMP}

5 - FROM 4 BY EX DECOMP $G(a, b) \Rightarrow (G(c, a) \wedge G(d, b) \wedge G(c, d))$
from {5 - FROM 4 BY EX DECOMP}

5 ζ
from {}

4 ζ
from {5 - FROM 4 BY EX DECOMP,10,11,12}

8 $\neg G(a, b)$
from {8}

COMMENT "G can be that both are pos integers greater than or equal to 1."
from {COMMENT}

9 $G(c, a) \wedge G(d, b) \wedge G(c, d)$
from {9}

10 $G(c, a)$
from {10}

11 $G(d, b)$
from {11}

12 $G(c, d)$
from {12}

slutten