FOL III

Selmer Bringsjord

Rensselaer AI & Reasoning (RAIR) Lab
Department of Cognitive Science
Department of Computer Science
Lally School of Management & Technology
Rensselaer Polytechnic Institute (RPI)
Troy, New York 12180 USA

Intro to Formal Logic (& AI)
2/20/2025

Melvin Fitting anecdote re Hao Wang, logic-based AI (re automated reasoning "for" $\mathscr{L}_1$, specifically), and Aristotle — stunningly relevant to this course …

$\mathscr{L}_1$

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Let's prove the 2nd of these in HS, and also …
a Selmer AI-generated ZOL problem on Test 1, and …
explore the concept of URC^3L.

Part I: *Slutten — for i dag.*

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e.g. $\{ \neg(\phi \lor \psi) \} \vdash \neg\phi \land \neg\psi$

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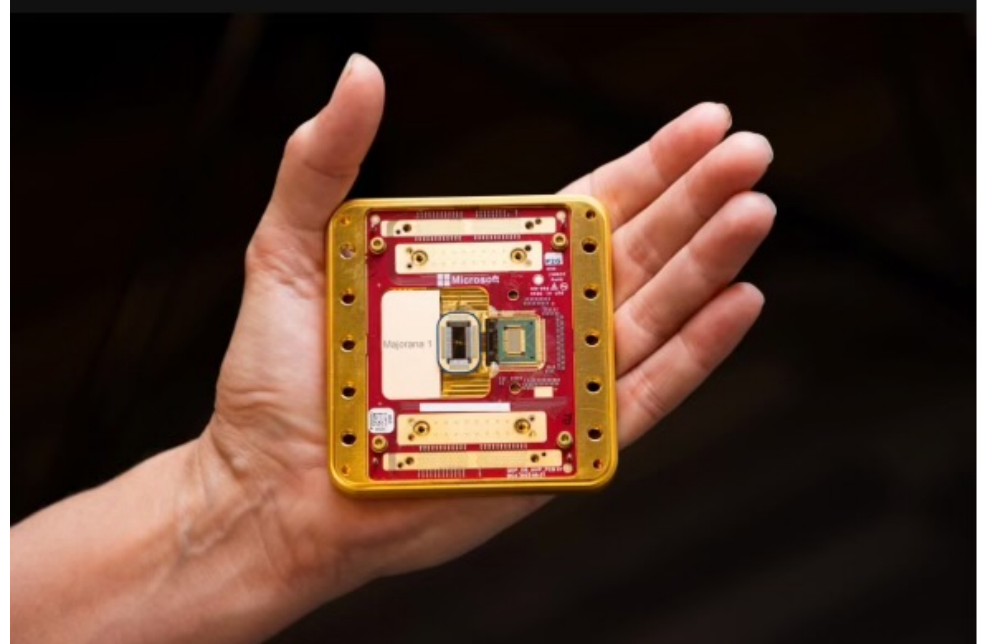
What does formal logic & AI say about this?

Microsoft Claims Quantum Computing Breakthrough by Creating New State of Matter

Tech giant's researchers say advance could lead to vastly more powerful computers



Microsoft's 'Majorana 1' quantum computing chip in a handout photo from the company. (PHOTO: COURTESY OF MICROSOFT/REUTERS)

Live-action on HyperGrader …

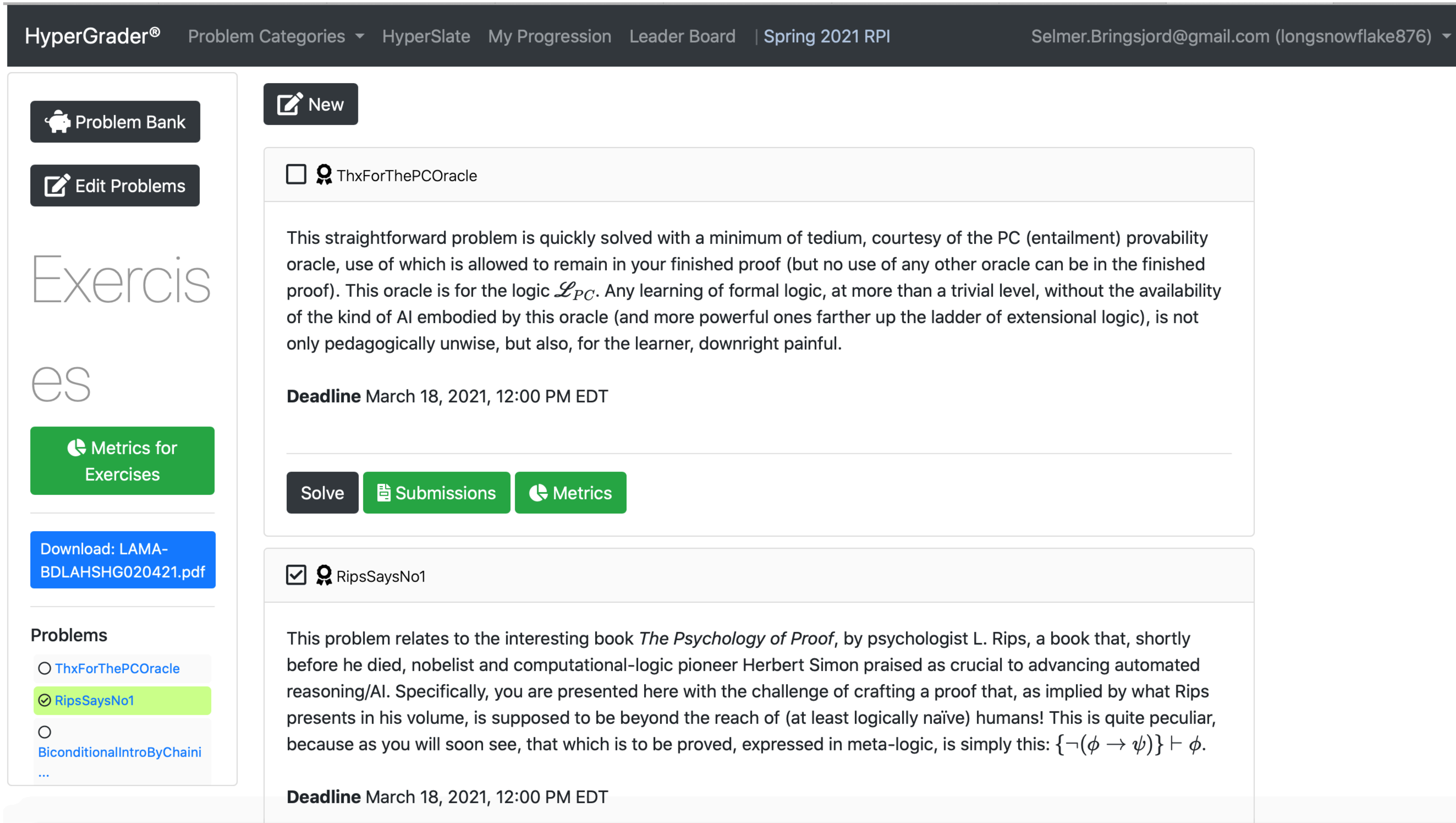
ThxForThePCOracle

ThxForThePCOracle

Please attempt that now-ish; thx.

ThxForThePCOracle

Please attempt that now-ish; thx.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Selmer.Bringsjord@gmail.com (longsnowflake876)
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Problems
ThxForThePCOracle
RipsSaysNo1
BiconditionalIntroByChaini ...
New
ThxForThePCOracle
This straightforward problem is quickly solved with a minimum of tedium, courtesy of the PC (entailment) provability oracle, use of which is allowed to remain in your finished proof (but no use of any other oracle can be in the finished proof). This oracle is for the logic $\mathscr{L}_{PC}$. Any learning of formal logic, at more than a trivial level, without the availability of the kind of AI embodied by this oracle (and more powerful ones farther up the ladder of extensional logic), is not only pedagogically unwise, but also, for the learner, downright painful.
Deadline March 18, 2021, 12:00 PM EDT
Solve
Submissions
Metrics
RipsSaysNo1
This problem relates to the interesting book The Psychology of Proof, by psychologist L. Rips, a book that, shortly before he died, nobelist and computational-logic pioneer Herbert Simon praised as crucial to advancing automated reasoning/AI. Specifically, you are presented here with the challenge of crafting a proof that, as implied by what Rips presents in his volume, is supposed to be beyond the reach of (at least logically naïve) humans! This is quite peculiar, because as you will soon see, that which is to be proved, expressed in meta-logic, is simply this: $\{\neg(\phi \rightarrow \psi)\} \vdash \phi$.
Deadline March 18, 2021, 12:00 PM EDT

HyperGrader® Problem Categories HyperSlate My Progression Leader Board | Spring 2021 RPI Selmer.Bringsjord@gmail.com (longsnowflake876)

Open in HyperSlate

☐ ThxForThePCOracle

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Deadline March 18, 2021, 12:00 PM EDT

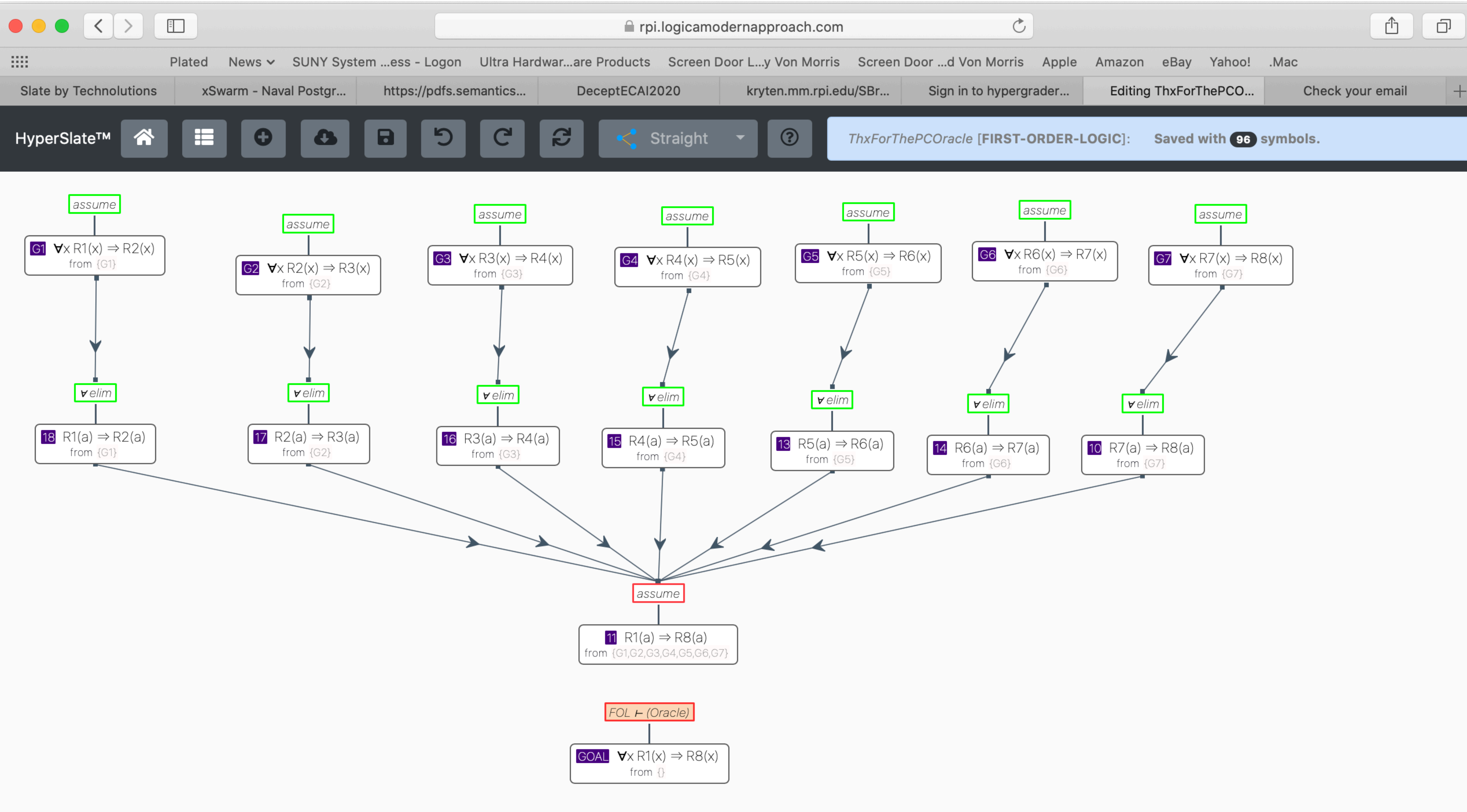
Days	Hours	Minutes	Seconds
03	19	39	00

Problem Type: SIMPLE

Difficulty: 1

Points: 10

Leaderboard



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Screen Door ...d Von Morris
Apple
Amazon
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Yahoo!
.Mac
Slate by Technolutions
xSwarm - Naval Postgr...
https://pdfs.semantics...
DeceptECAI2020
kryten.mm.rpi.edu/SBr...
Sign in to hypergrader...
Editing ThxForThePCO...
Check your email
HyperSlate™
Straight
ThxForThePCOracle [FIRST-ORDER-LOGIC]: Saved with 96 symbols.
assume
G1 ∀x R1(x) ⇒ R2(x) from {G1}
assume
G2 ∀x R2(x) ⇒ R3(x) from {G2}
assume
G3 ∀x R3(x) ⇒ R4(x) from {G3}
assume
G4 ∀x R4(x) ⇒ R5(x) from {G4}
assume
G5 ∀x R5(x) ⇒ R6(x) from {G5}
assume
G6 ∀x R6(x) ⇒ R7(x) from {G6}
assume
G7 ∀x R7(x) ⇒ R8(x) from {G7}
∀elim
18 R1(a) ⇒ R2(a) from {G1}
∀elim
17 R2(a) ⇒ R3(a) from {G2}
∀elim
16 R3(a) ⇒ R4(a) from {G3}
∀elim
15 R4(a) ⇒ R5(a) from {G4}
∀elim
13 R5(a) ⇒ R6(a) from {G5}
∀elim
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∀elim
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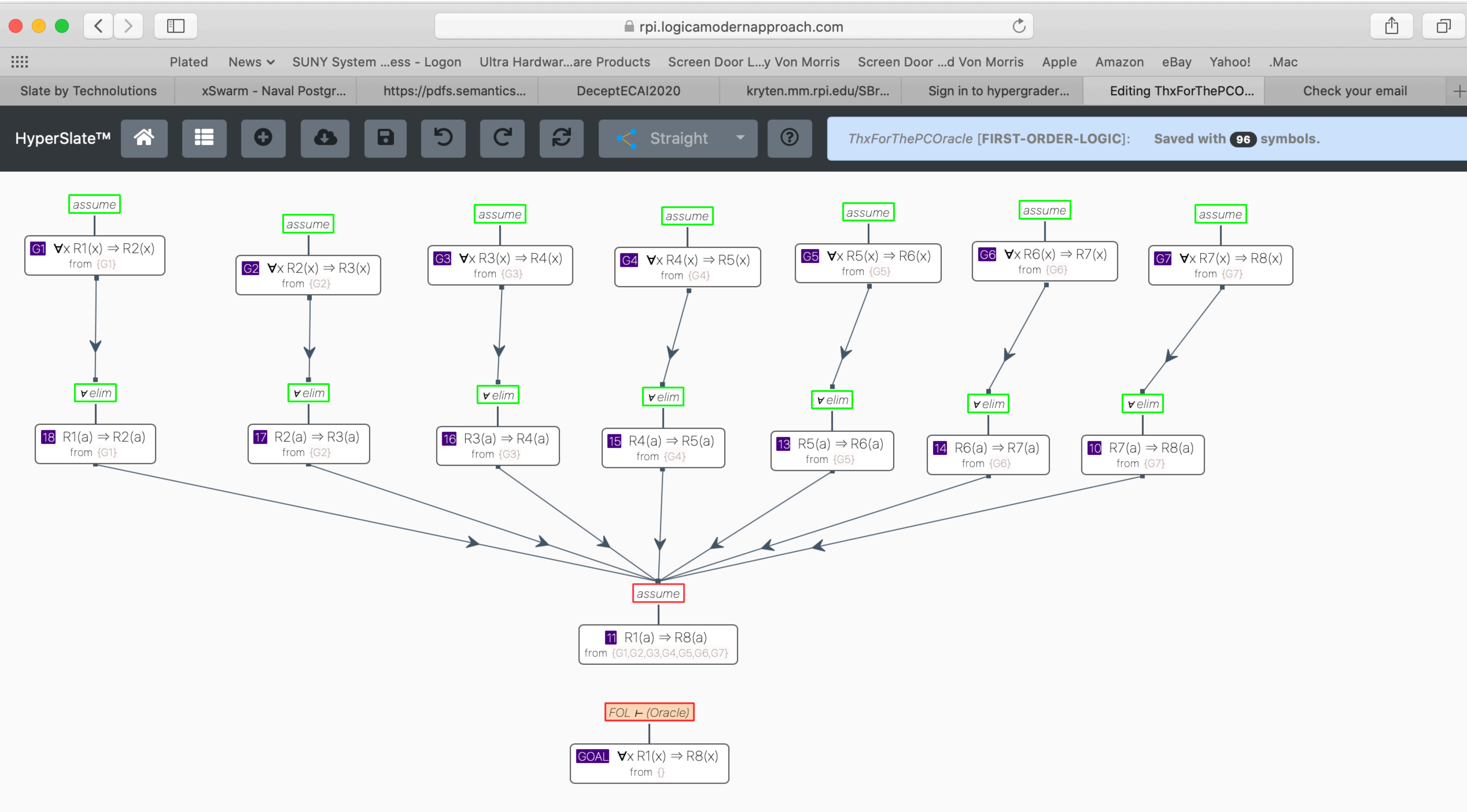
assume
11 R1(a) ⇒ R8(a) from {G1,G2,G3,G4,G5,G6,G7}
FOL ⊢ (Oracle)
GOAL ∀x R1(x) ⇒ R8(x) from {}


rpi.logicamodernapproach.com
HyperSlate™
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Selmer's Third = Hardest AI-generated Test-1 Problem

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Student Success on the New Knight-Knave THLPE #1

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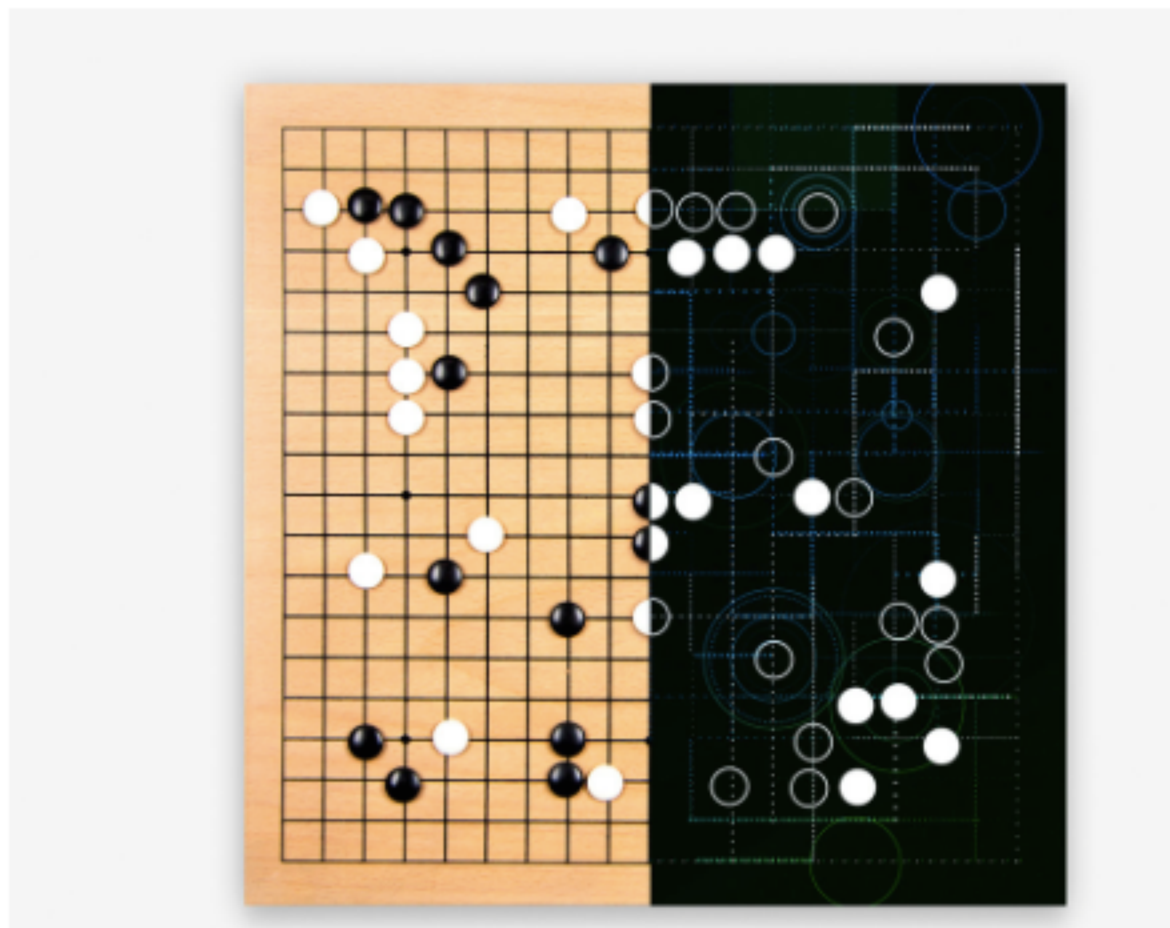
Interlude re
Formal Logic & Games …

IN A HUGE BREAKTHROUGH, GOOGLE'S AI BEATS A TOP PLAYER AT THE GAME OF GO



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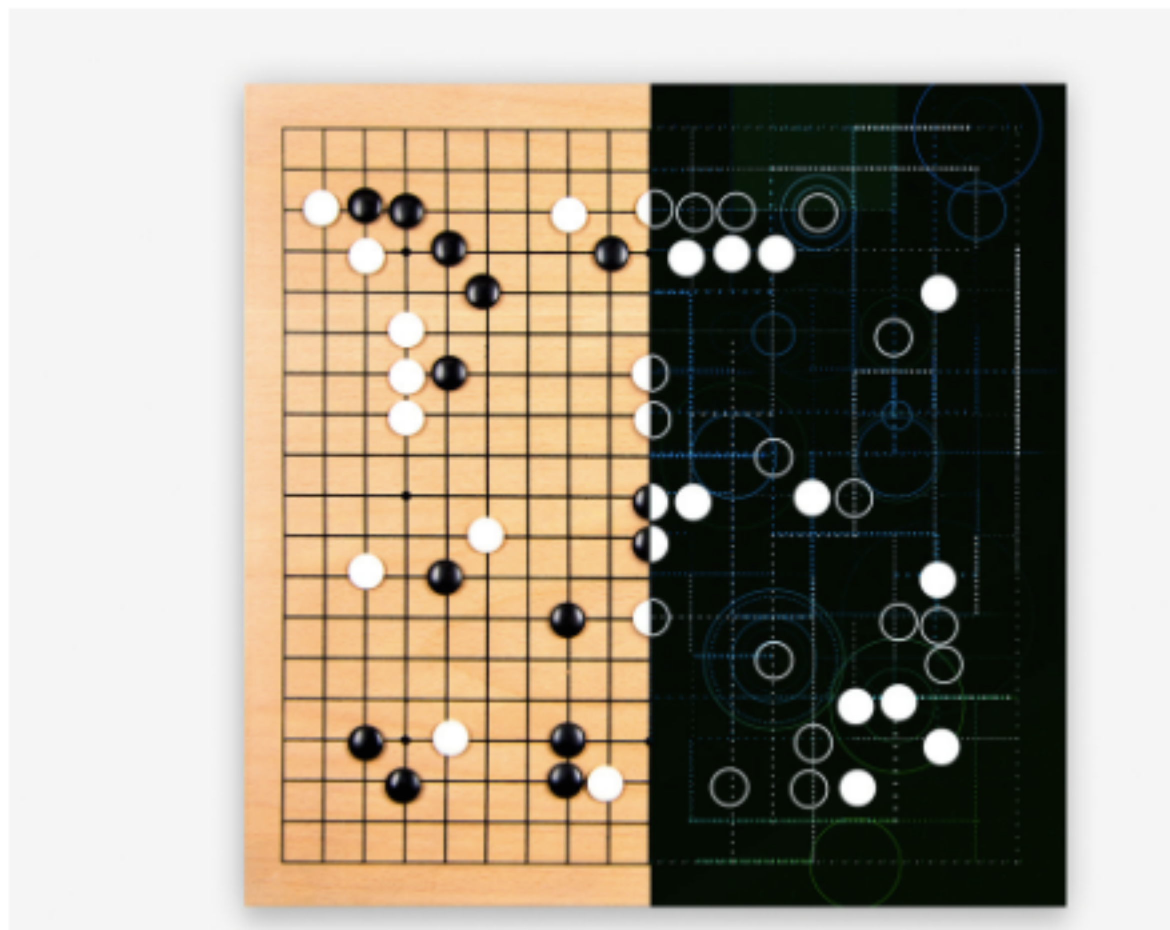

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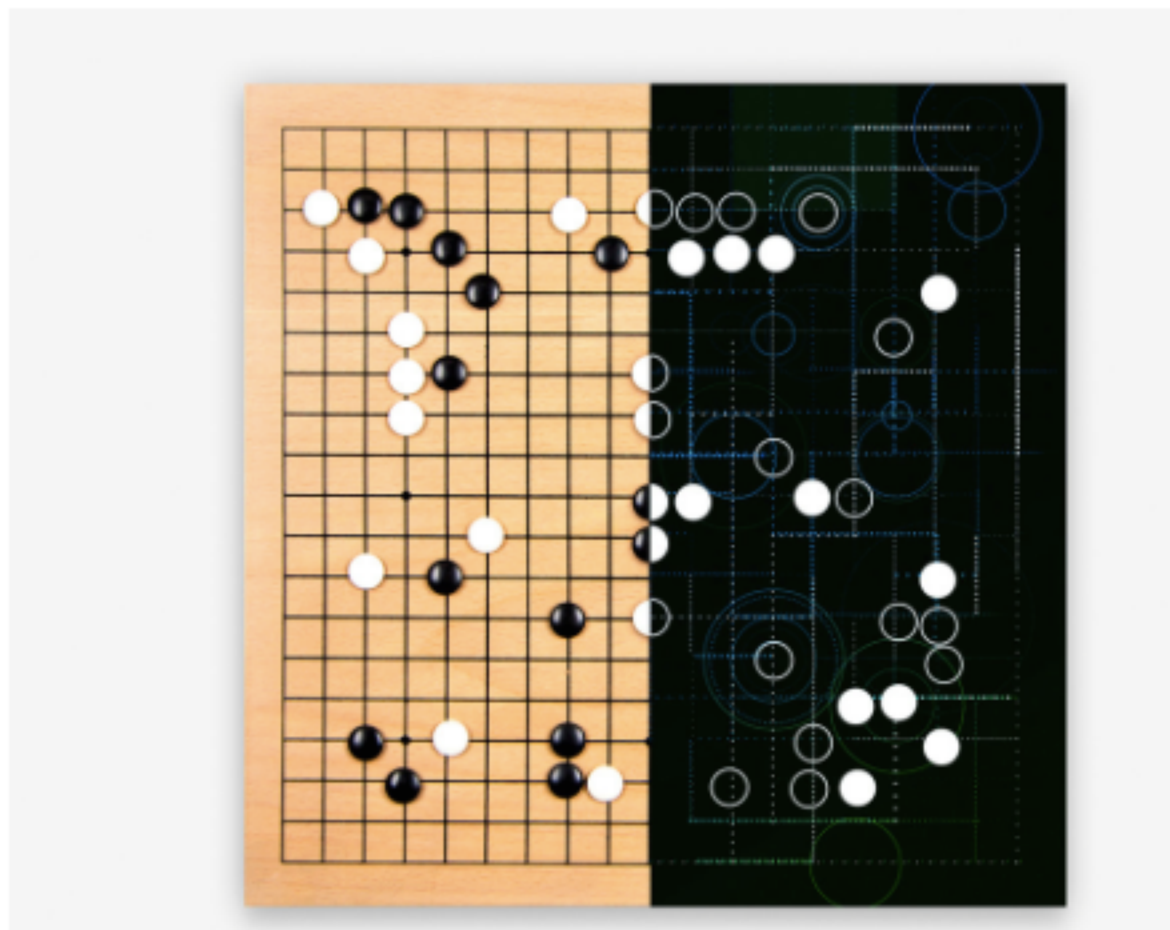

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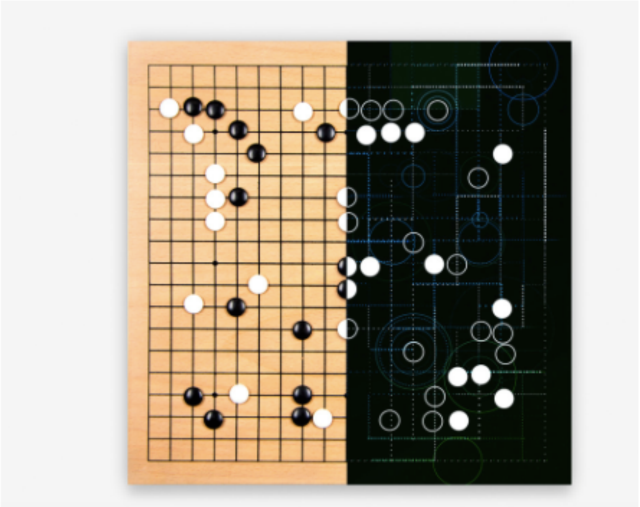

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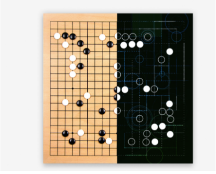

The *Entscheidungsproblem*

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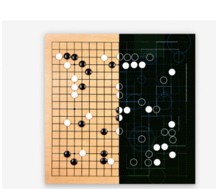

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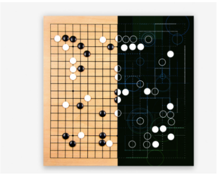
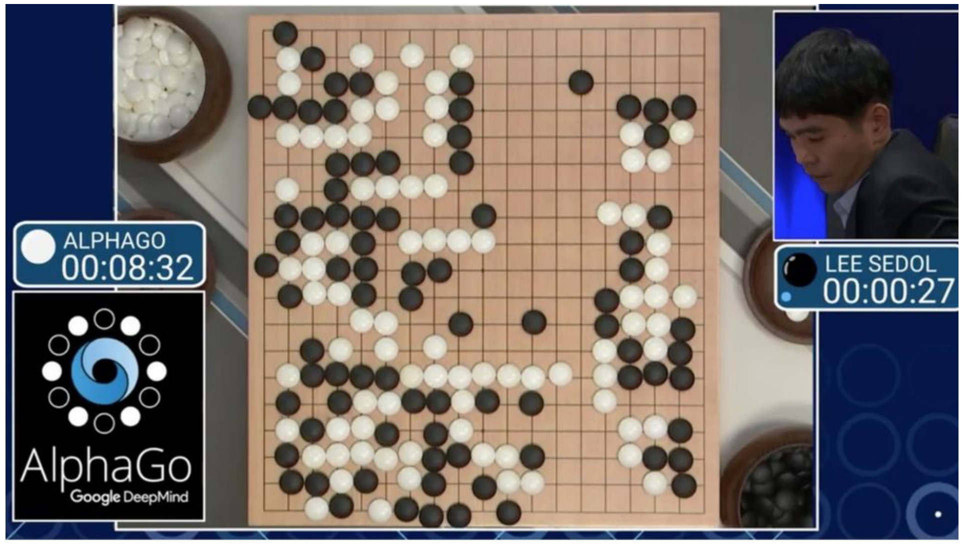
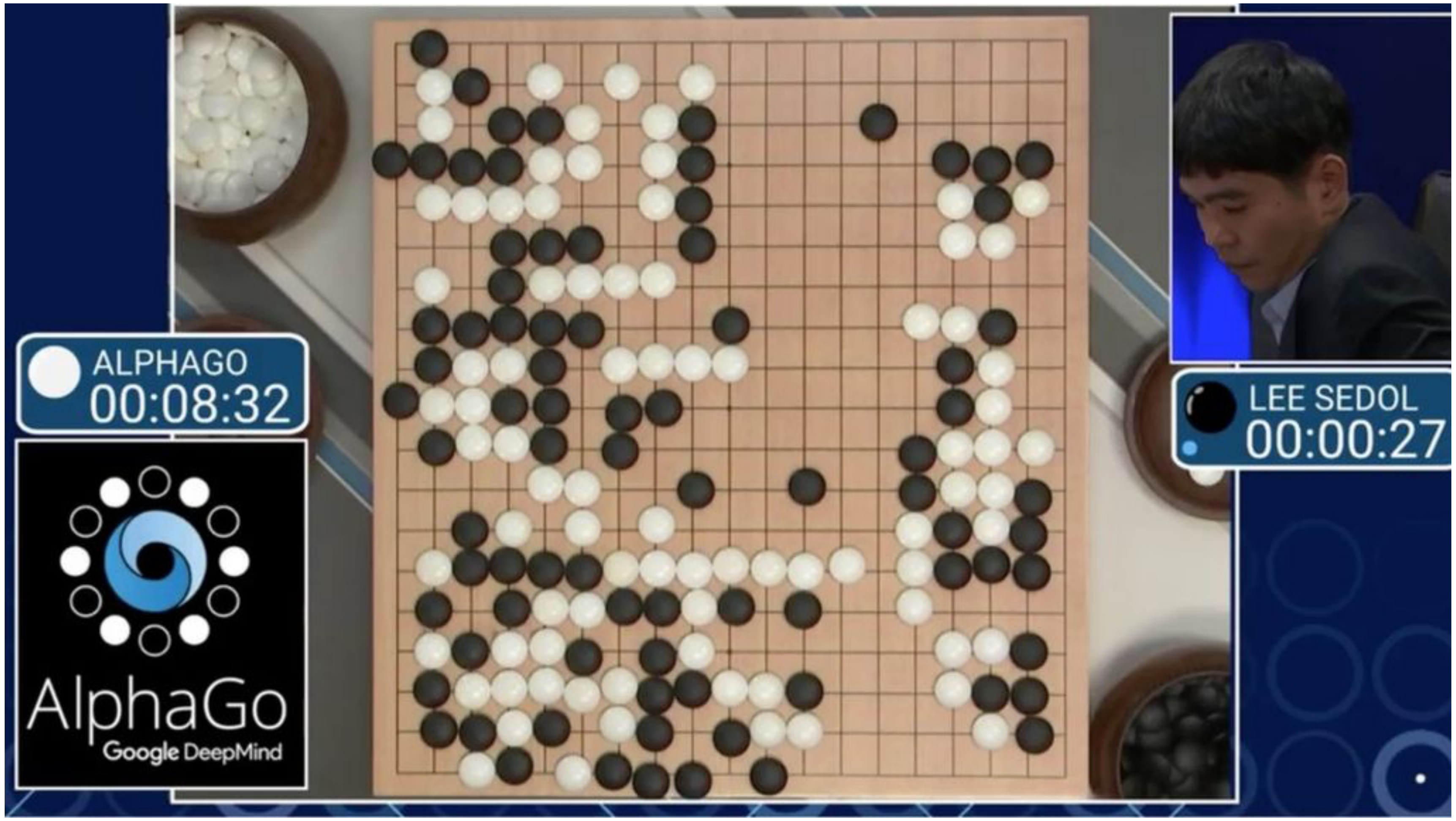



ALPHAGO
00:08:32
AlphaGo
Google DeepMind
LEE SEDOL
00:00:27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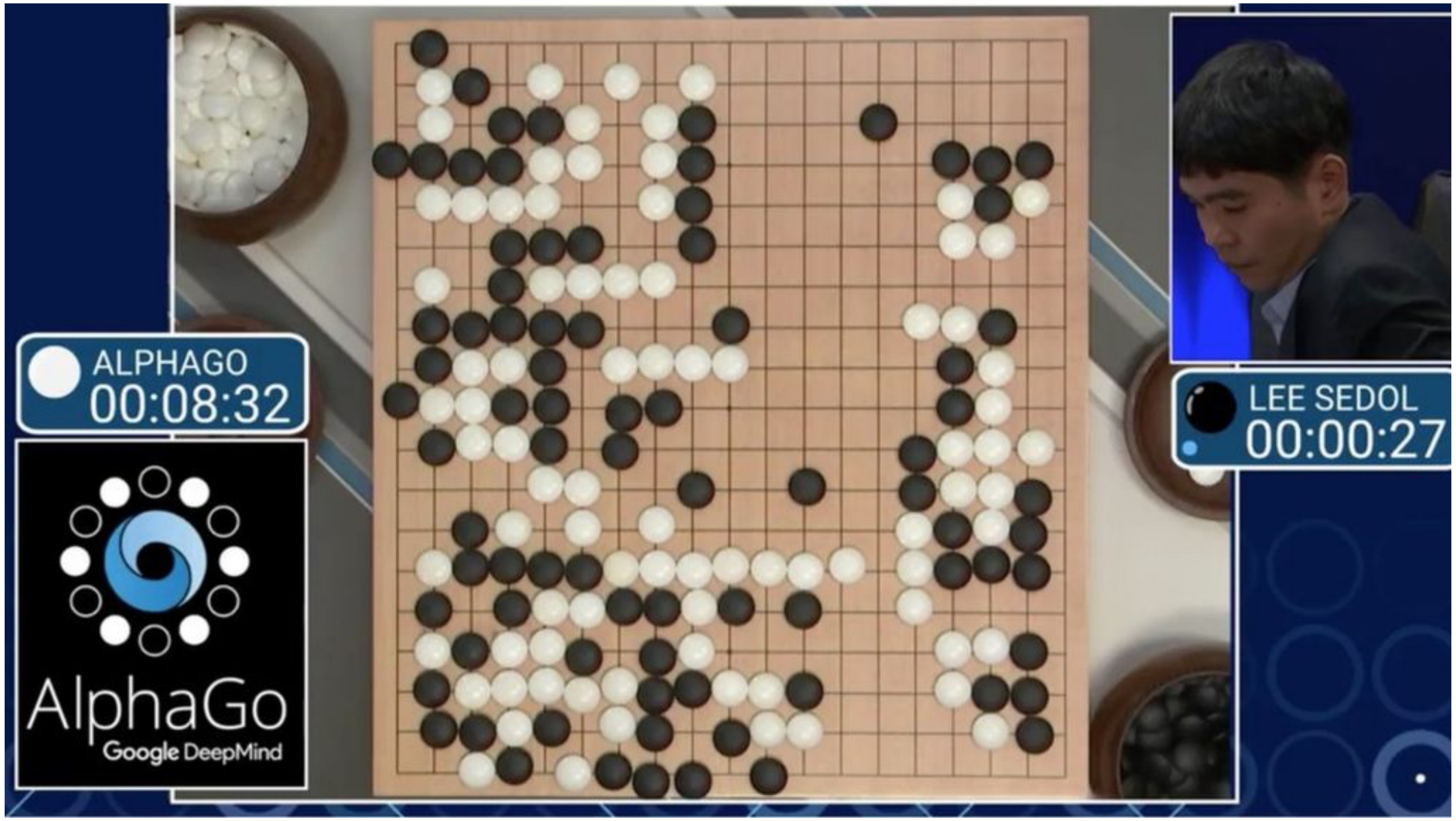
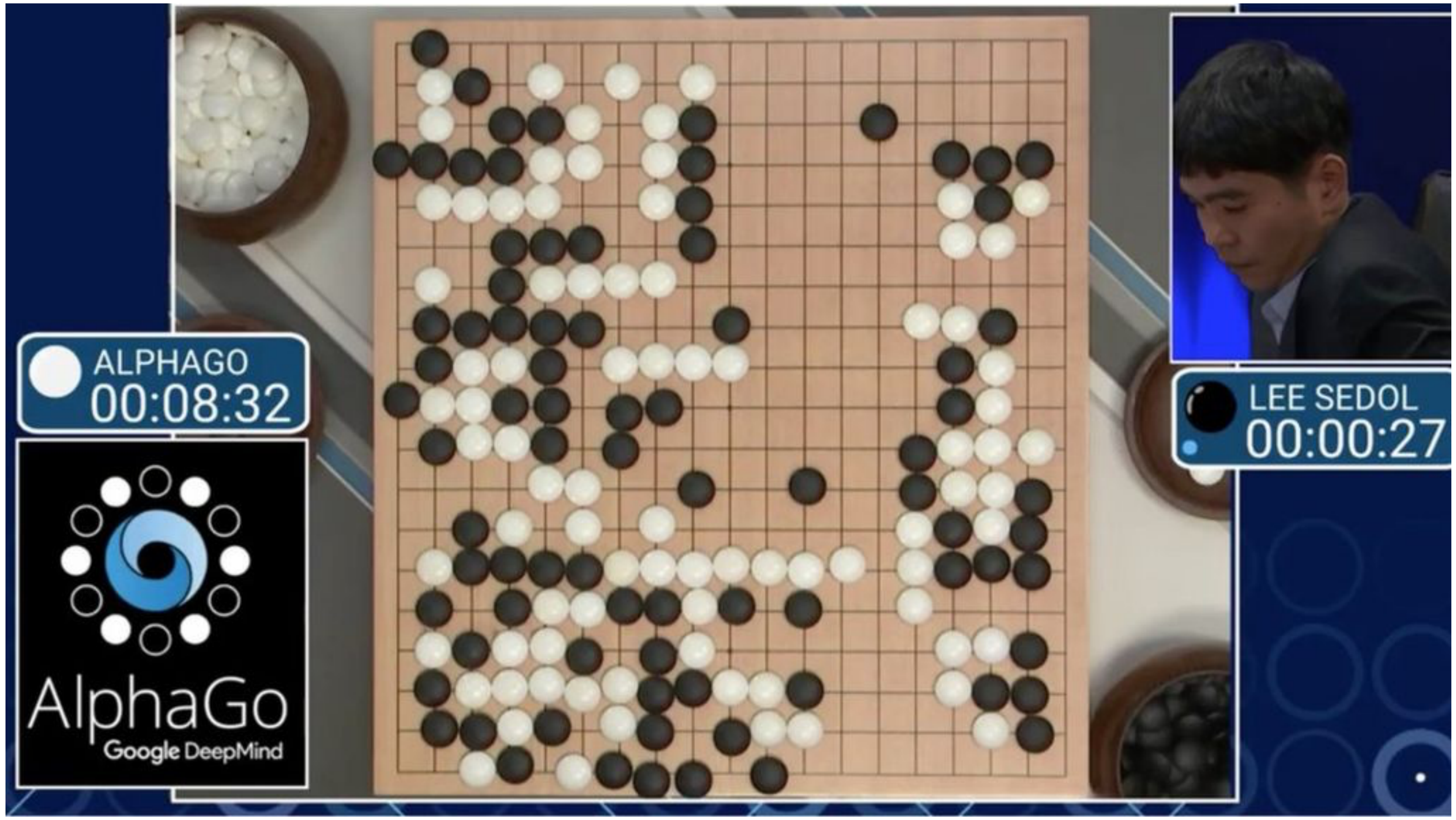
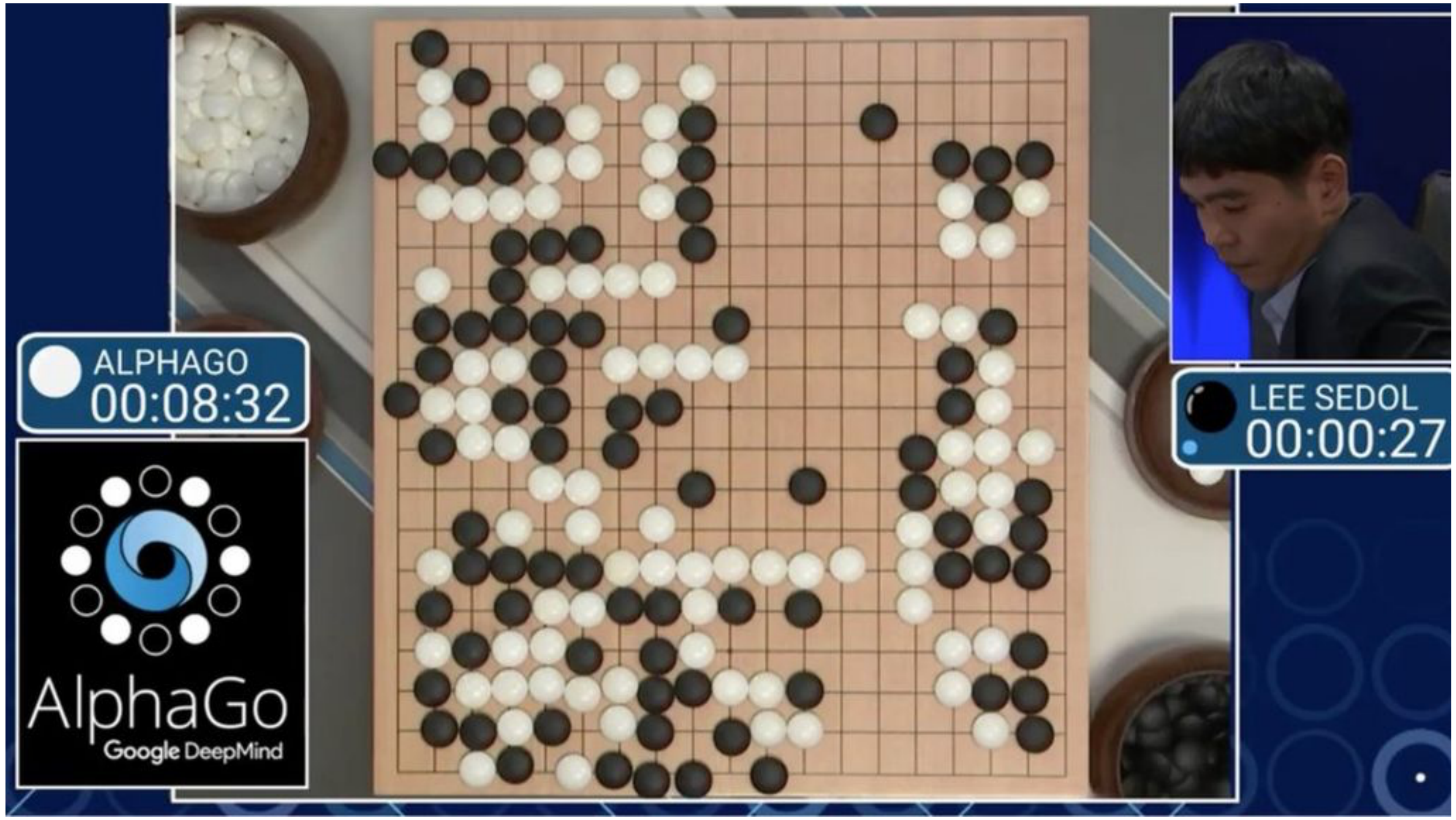


Praiseworthy AI *simplicter*, perhaps.



Praiseworthy AI *simplicter*, perhaps.

But certainly *not* AI = HI!



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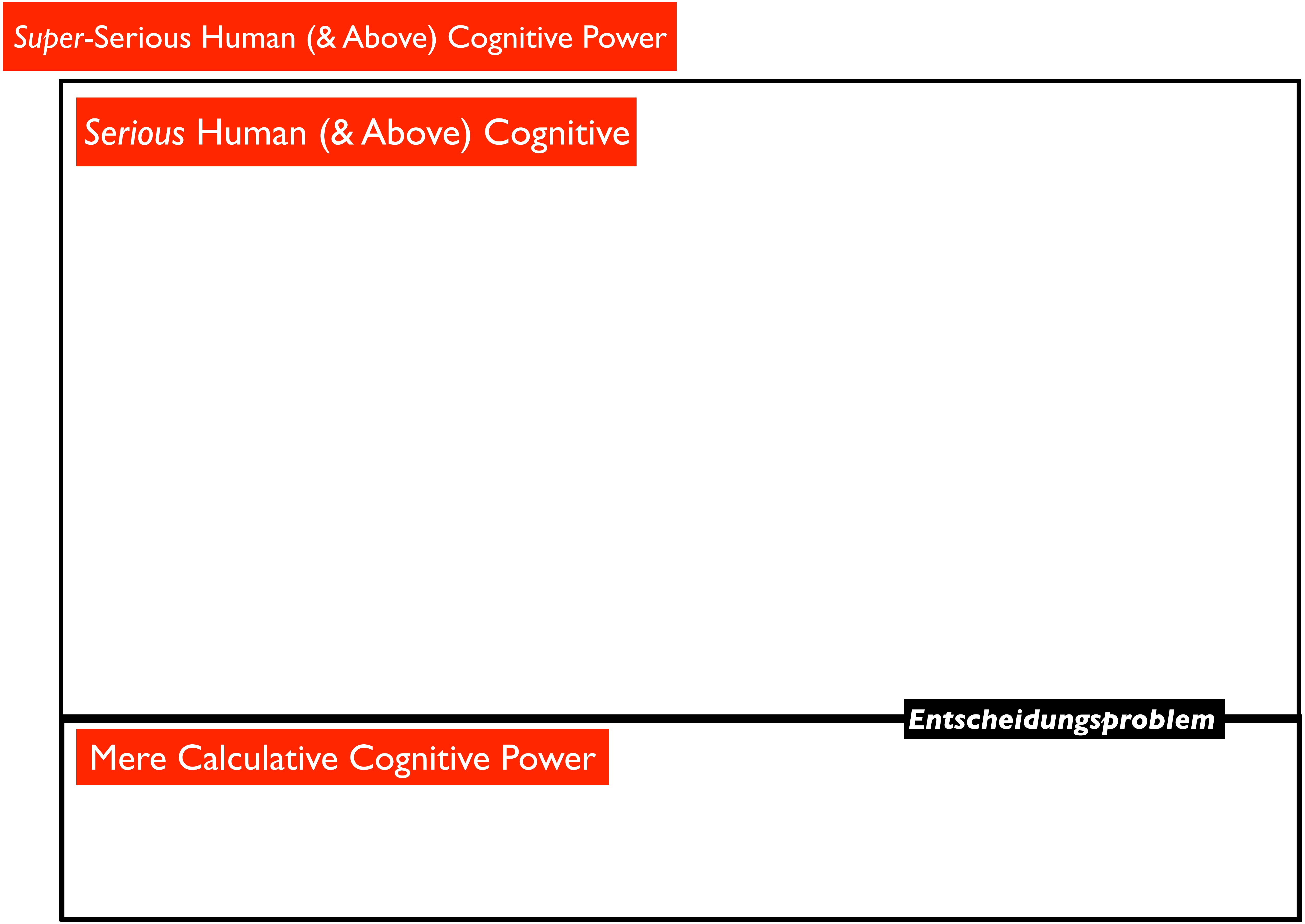
"AlphaGo, from the perspective of South, how many majuscule Roman letters are in black? Why do you say that?"

Logico-Mathematical Landscape that Has Them Turning in Their Graves



Logico-Mathematical Landscape that Has Them Turning in Their Graves

Super-Serious Human (& Above) Cognitive Power

Serious Human (& Above) Cognitive

Descartes

Entscheidungsproblem

Mere Calculative Cognitive Power

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Polynomial Hierarchy

$$\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} = \mathbf{NPSPACE} \subseteq \mathbf{EXPTIME} \subseteq \mathbf{NEXPTIME} \subseteq \mathbf{EXPSPACE}$$

Logico-Mathematical Landscape that Has Them Turning in Their Graves

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Arithmetical Hierarchy

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Descartes

Leibniz

Church

Gödel

$\vdots$

Π_2

Σ_2

Π_1

Σ_1

Σ_0

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$\mathscr{L}_1$

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Π_1

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Go: AlphaGo

$\vdots$

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Σ_2

Π_1

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Arithmetical Hierarchy

The first (procedural) programmer!

Jeopardy!: ●

Go: AlphaGo ●

$\vdots$

Π_2

Σ_2

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Chess: Deep Blue

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Checkers: Chinook

Chess: Deep Blue

Jeopardy!:

Go: AlphaGo

$\vdots$

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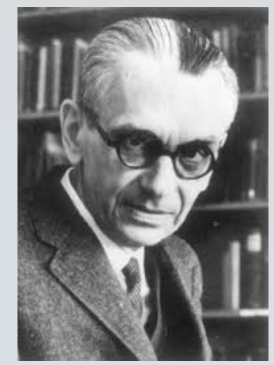

Gödel

Turing

$\vdots$

Π_2

Σ_2

Π_1

Σ_1

Σ_0

Entscheidungsproblem

Polynomial Hierarchy

Chess: Deep Blue

Checkers: Chinook

Jeopardy!: **Watson**

Go: AlphaGo

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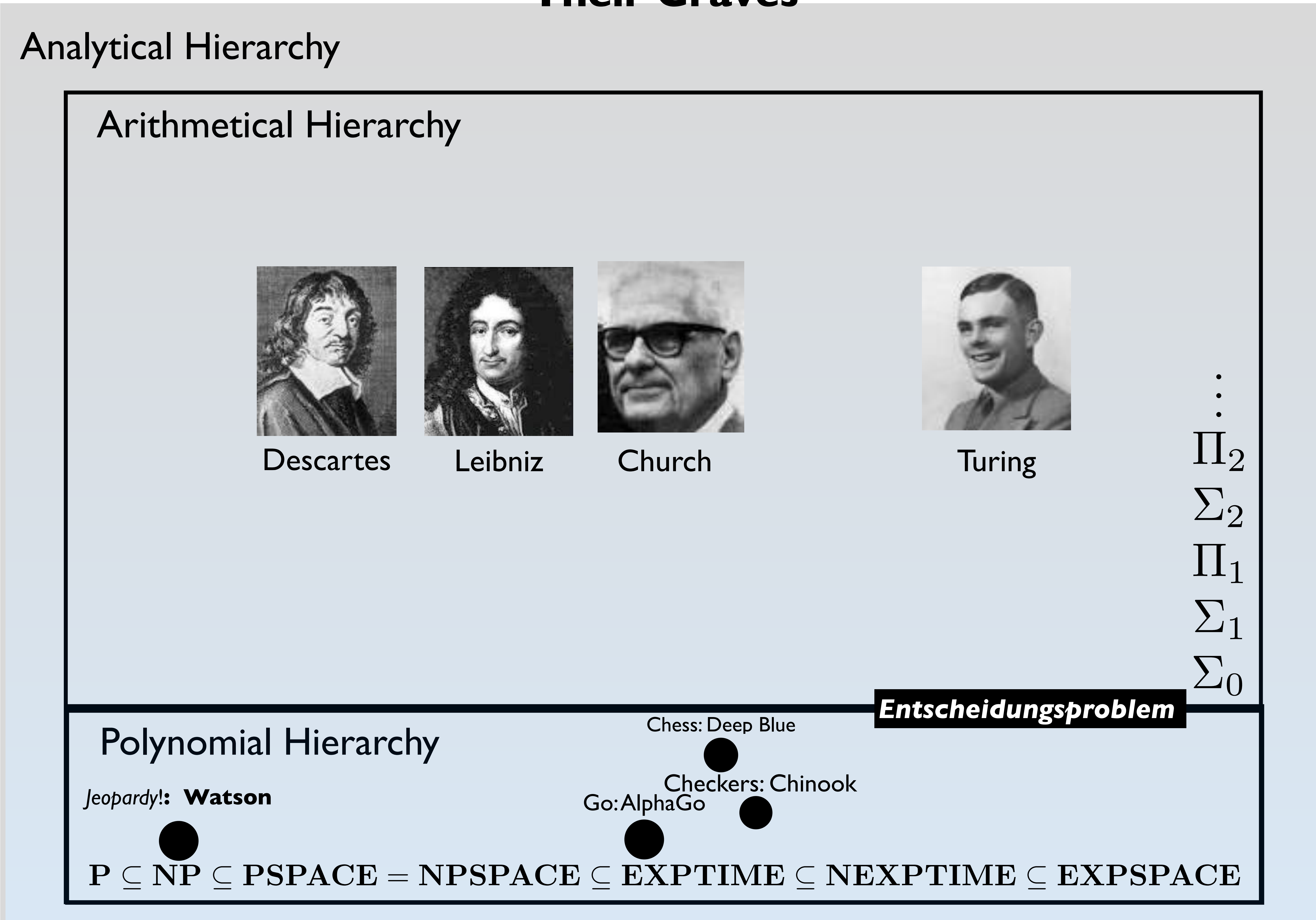

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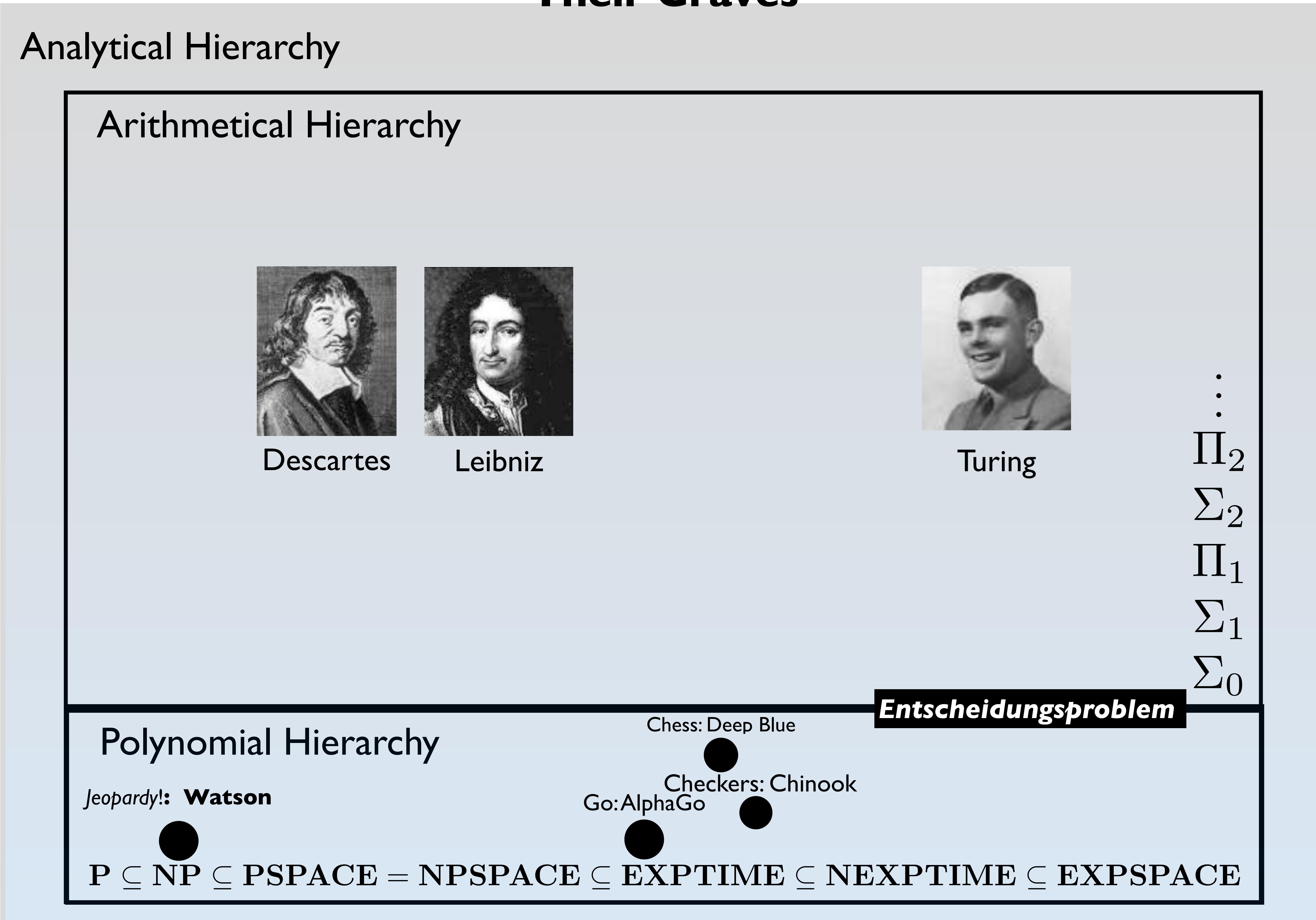

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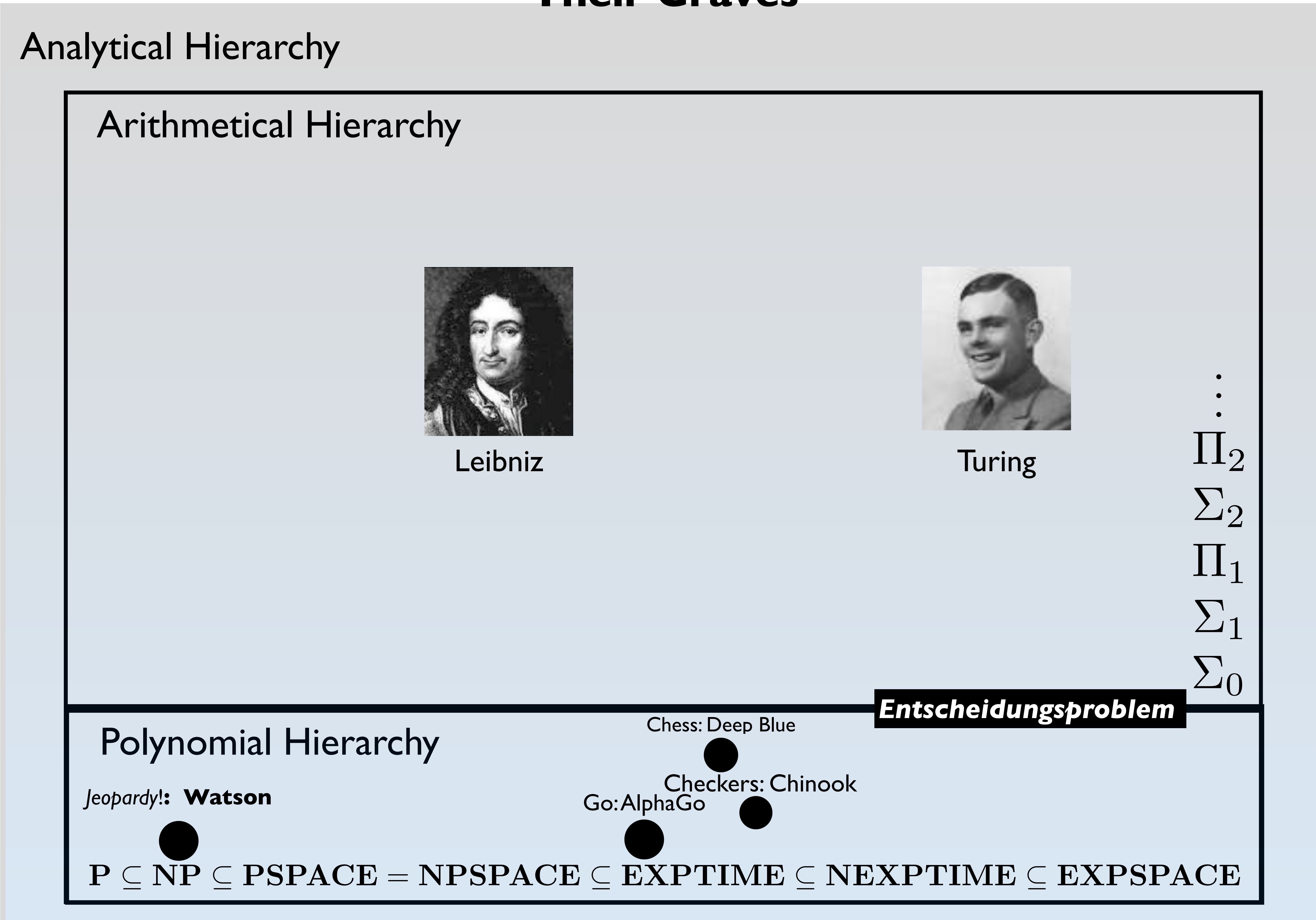

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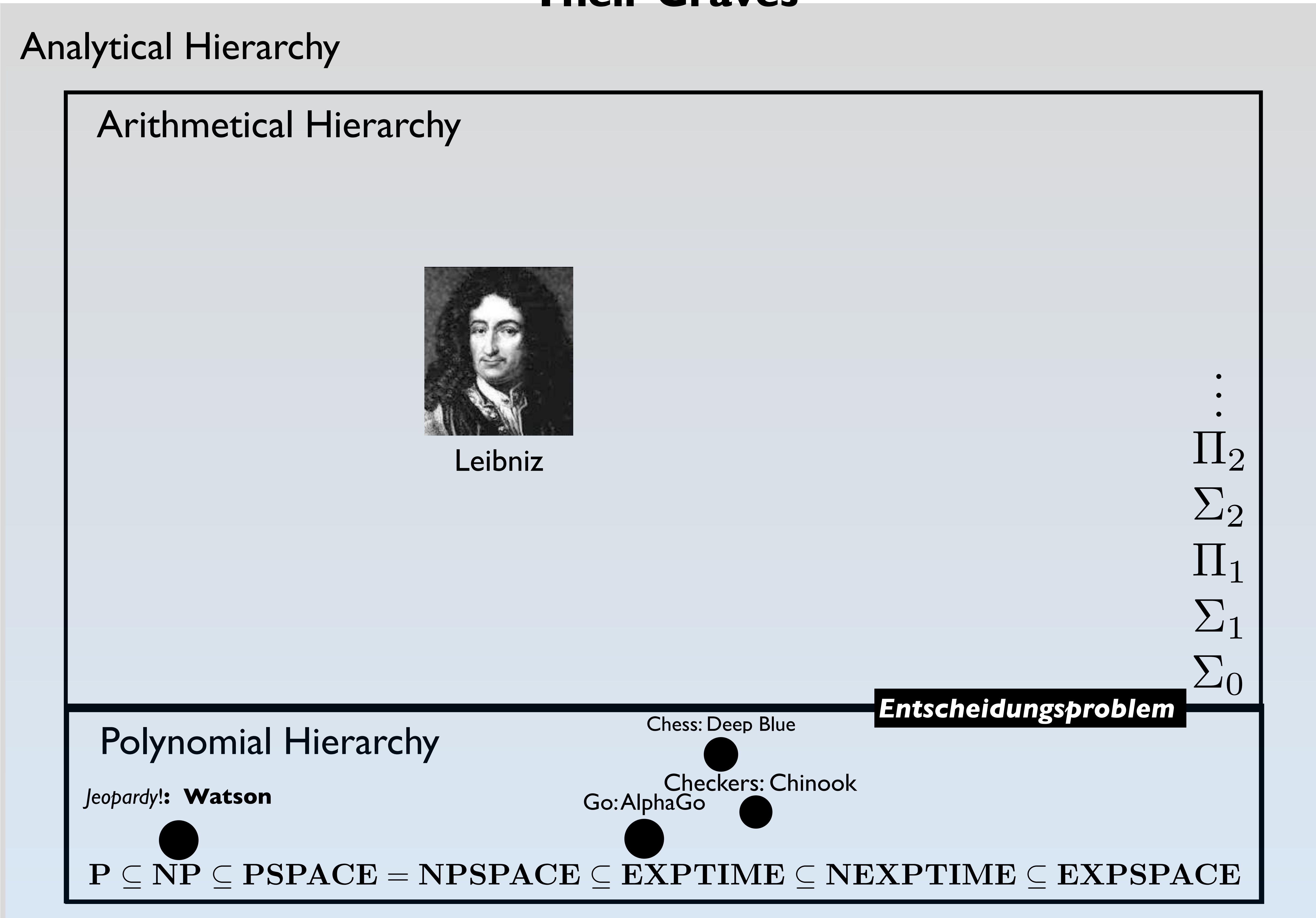

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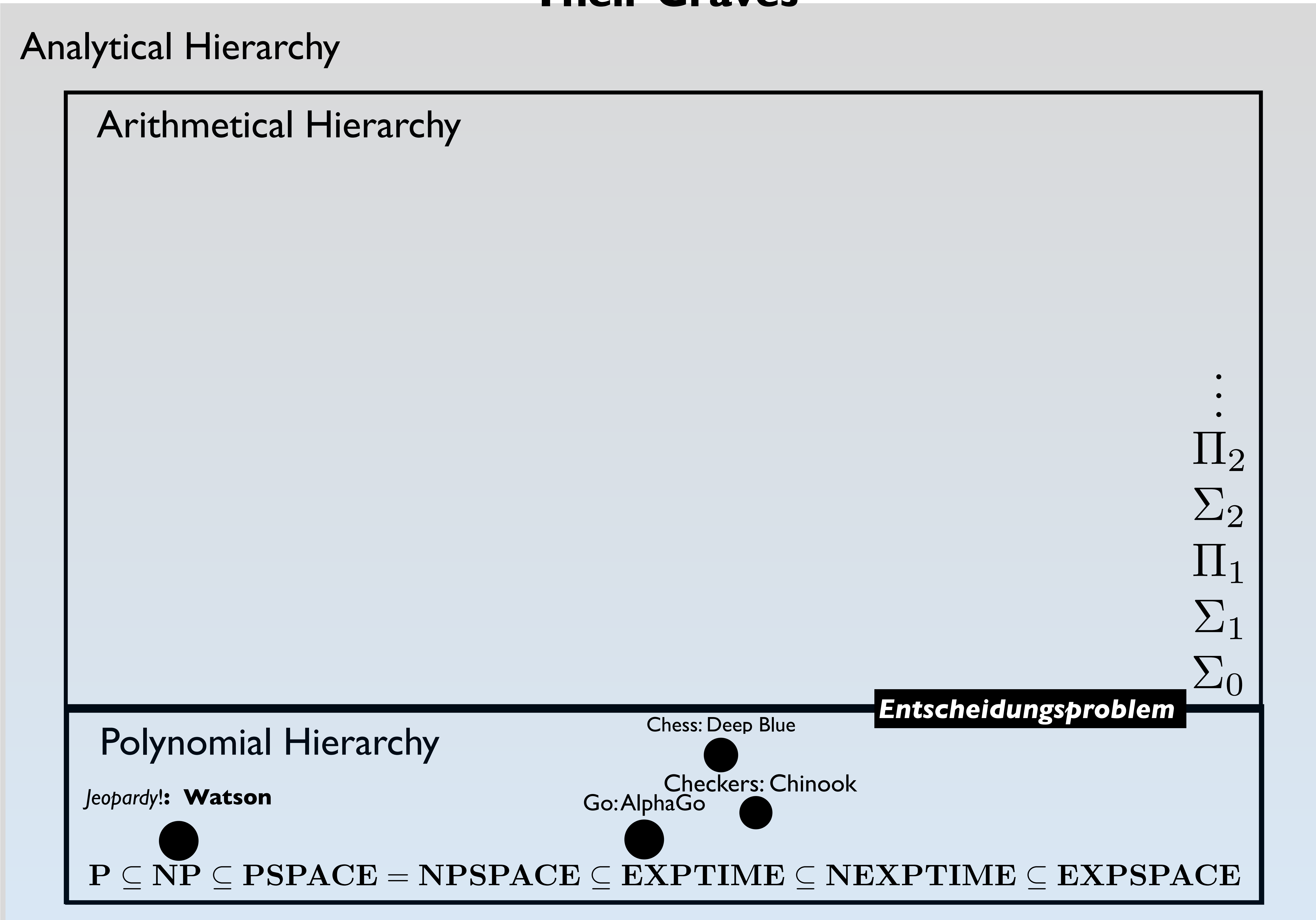

Rengo Kriegspiel

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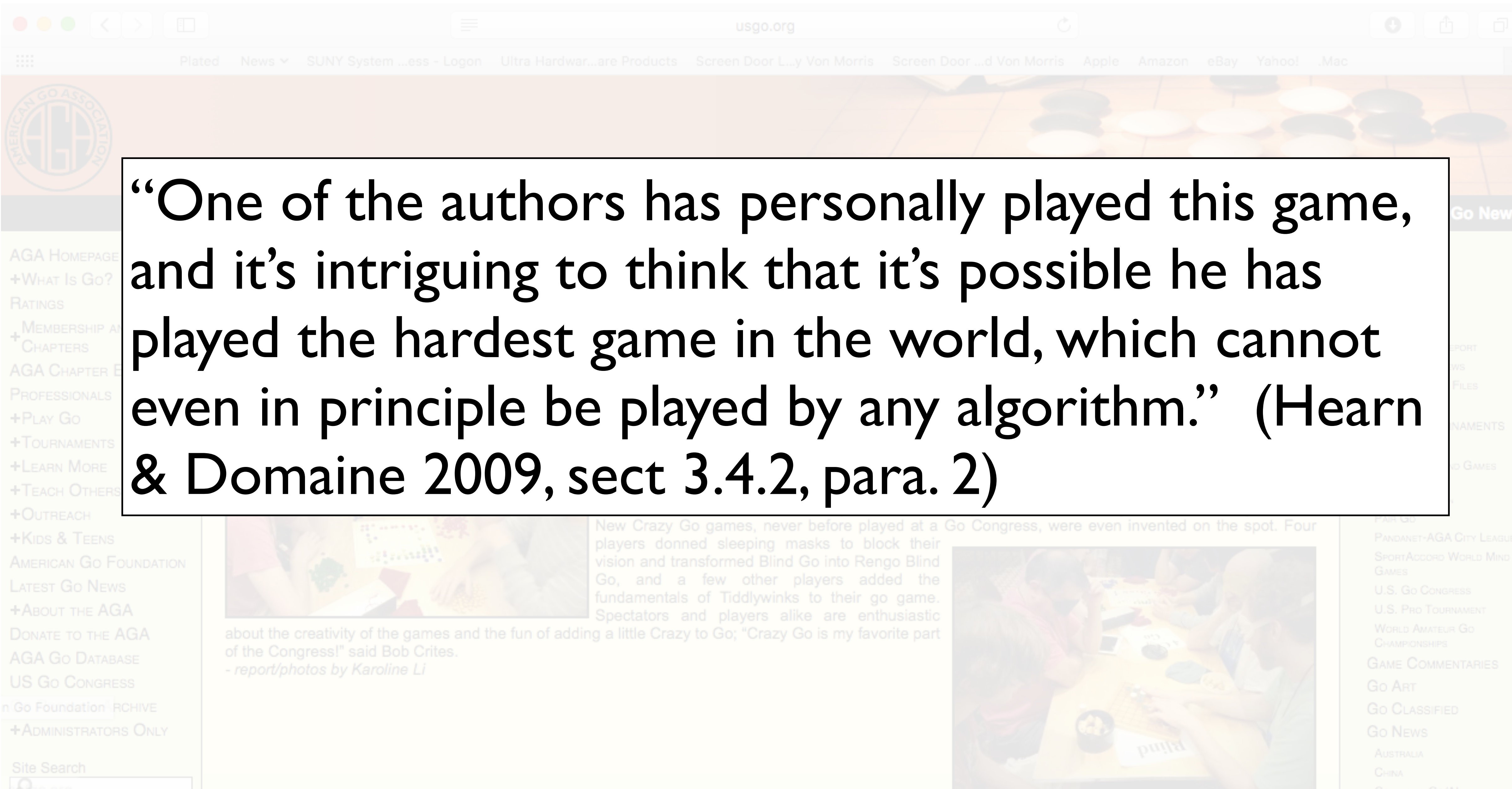
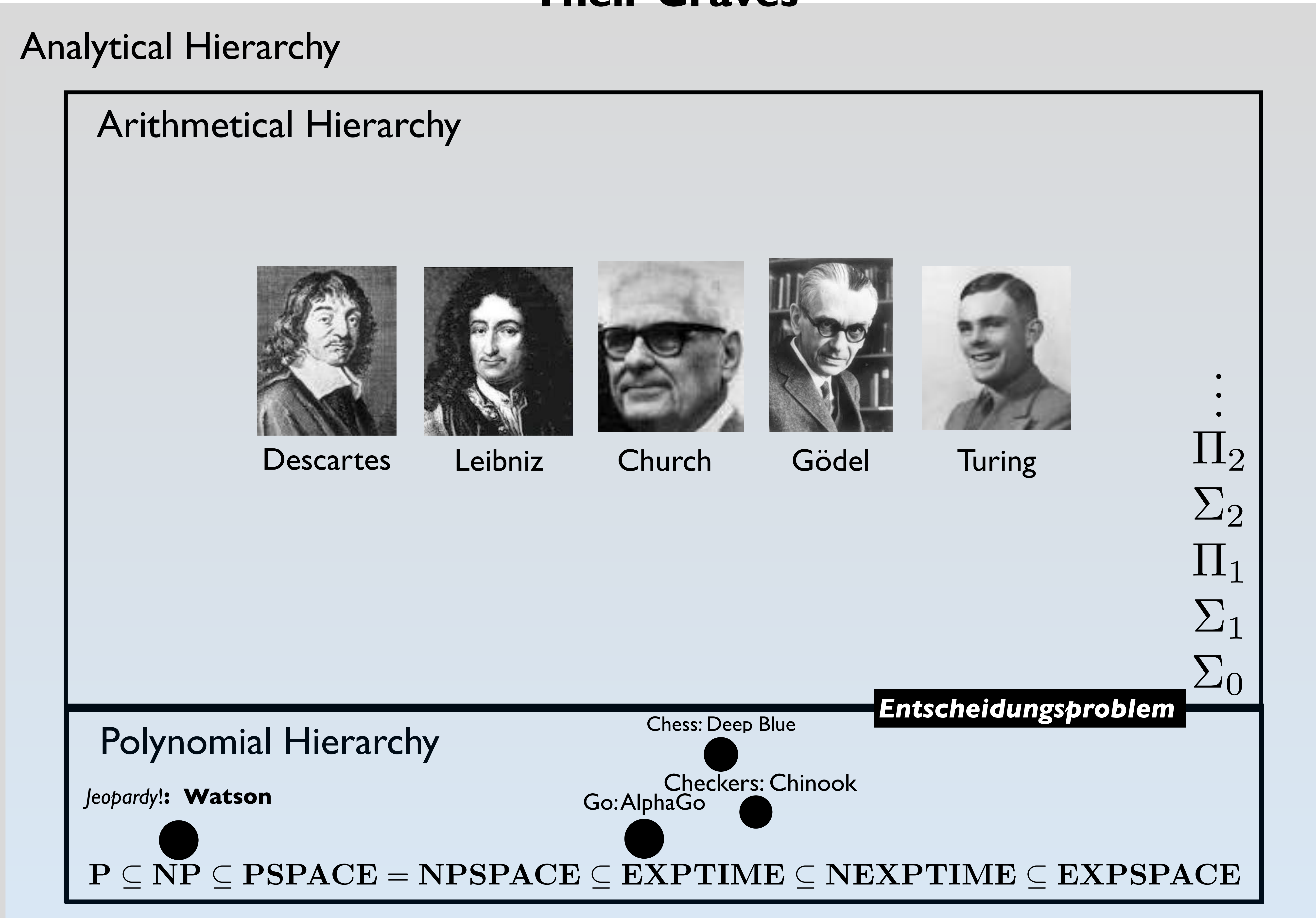


"One of the authors has personally played this game, and it's intriguing to think that it's possible he has played the hardest game in the world, which cannot even in principle be played by any algorithm." (Hearn & Domaine 2009, sect 3.4.2, para. 2)

Logico-Mathematical Landscape that Has Them Turning in Their Graves



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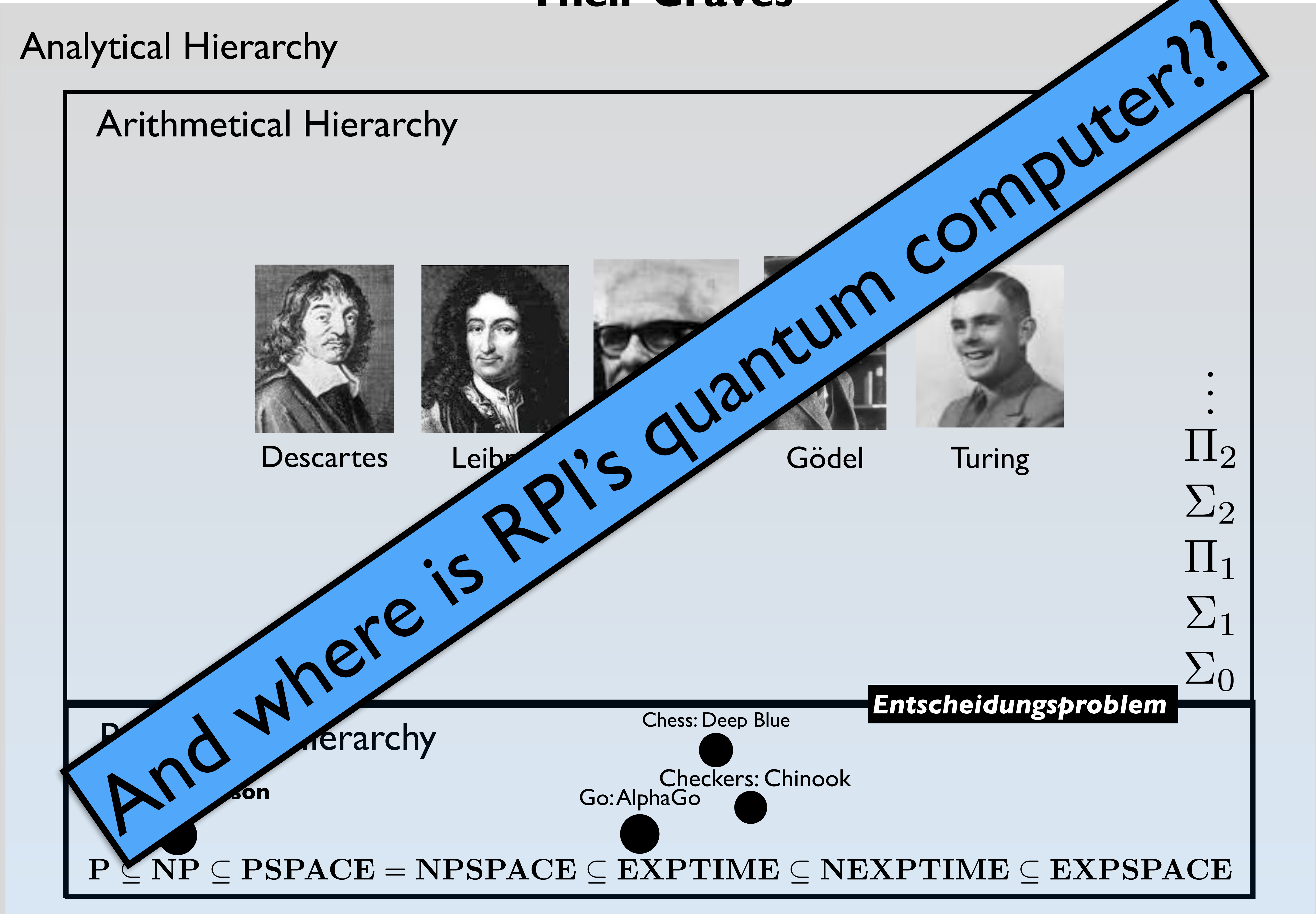
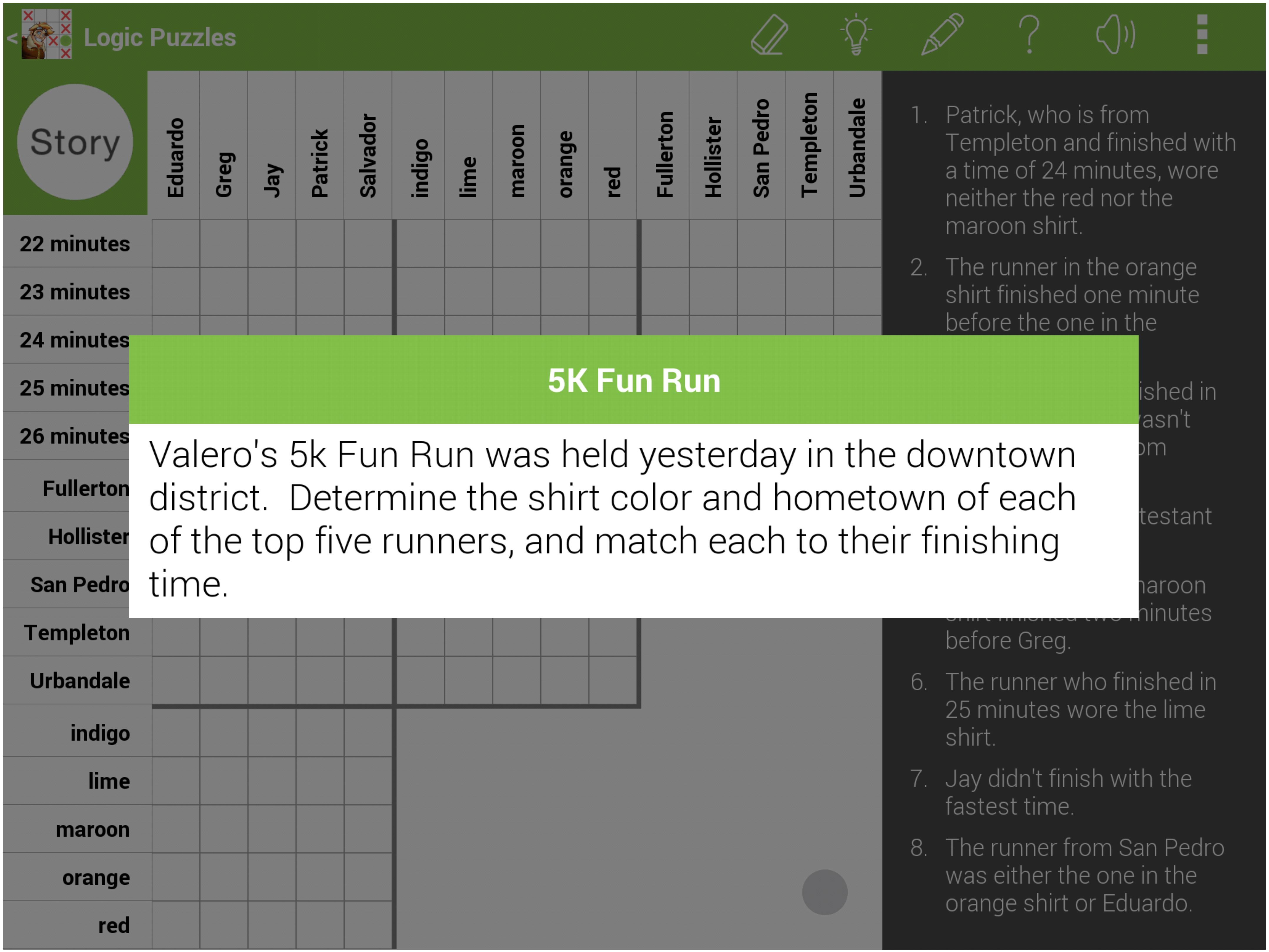


But starting simpler …

Tabular "Deduction" (Example)



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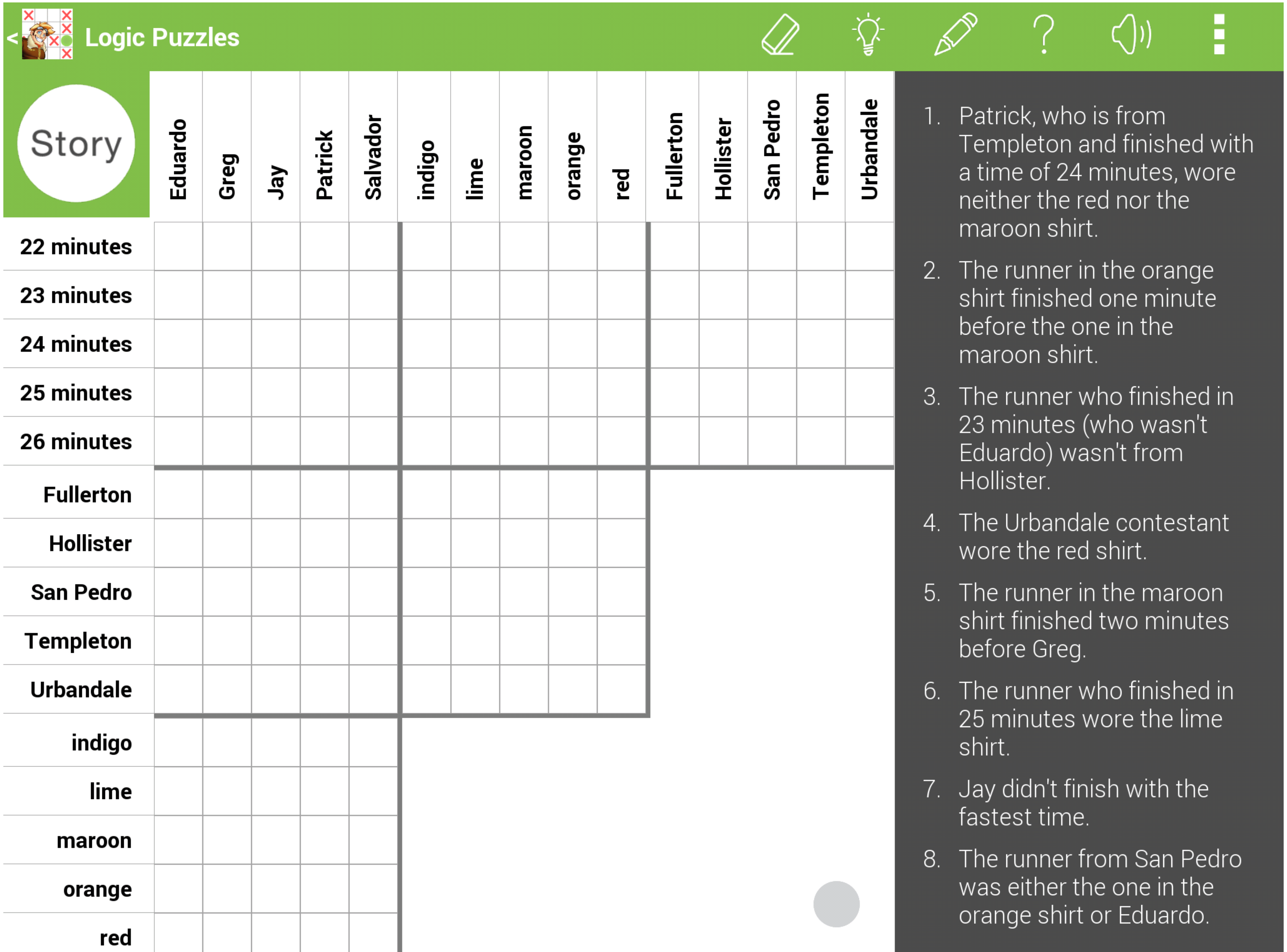

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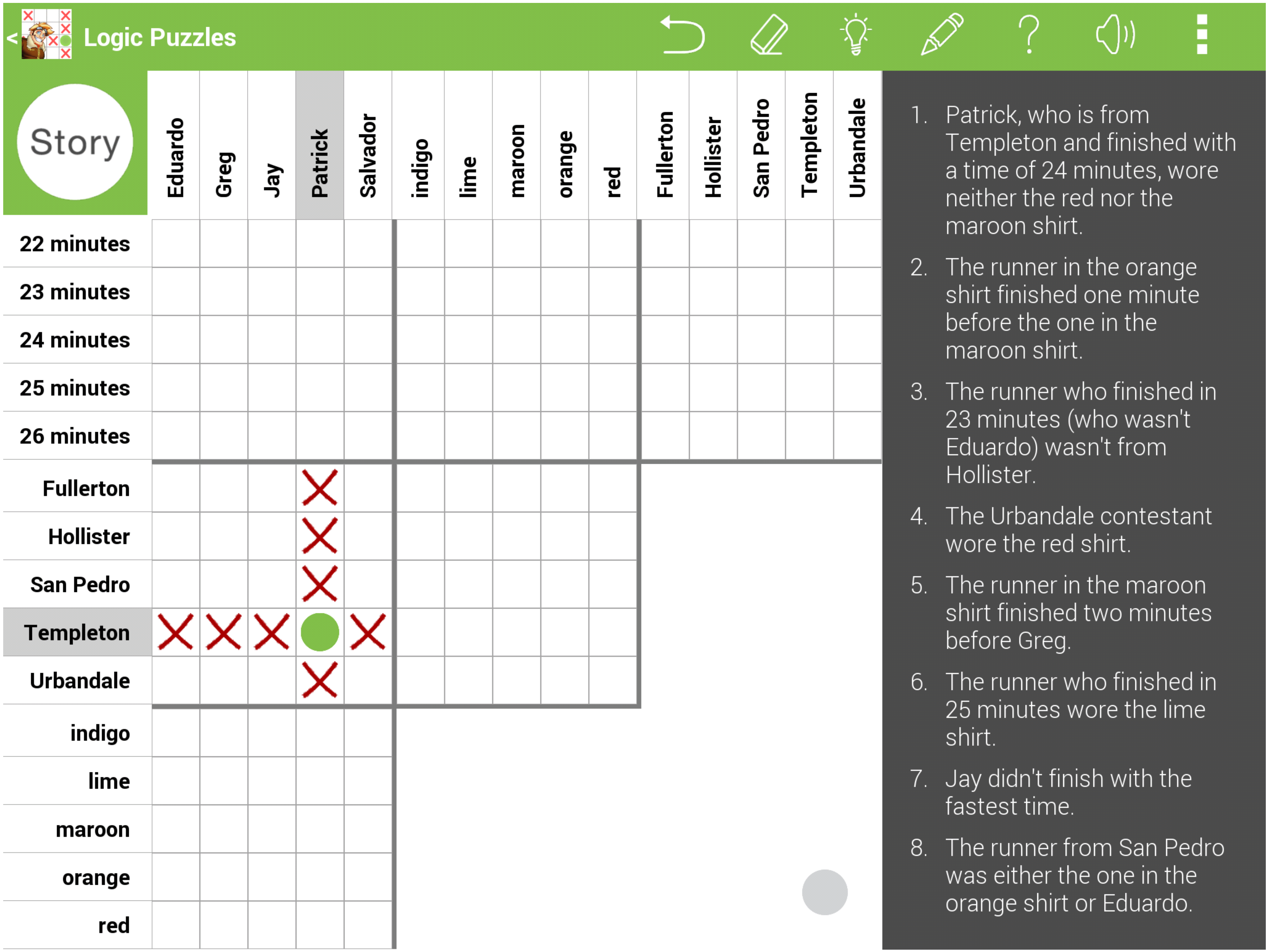

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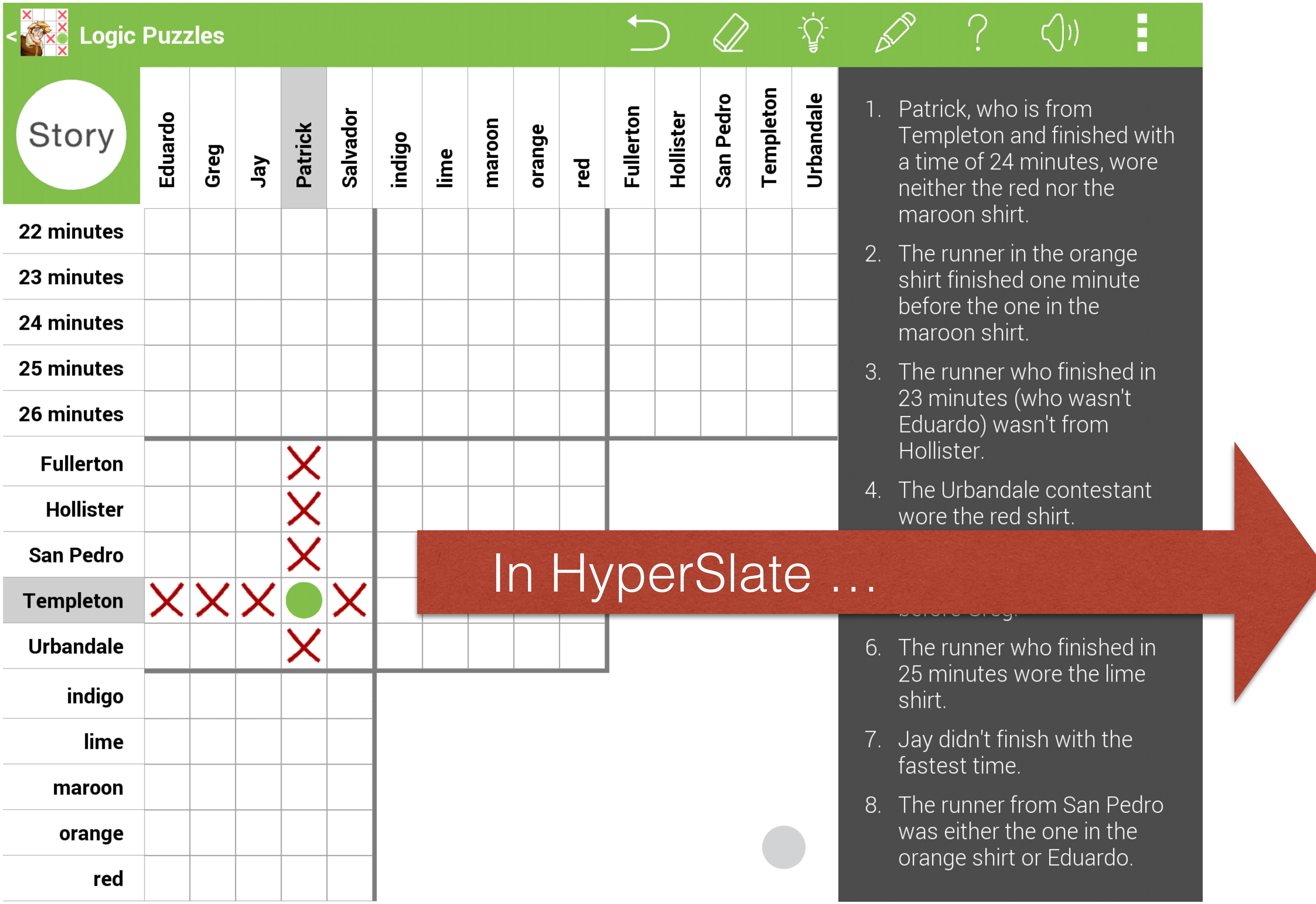

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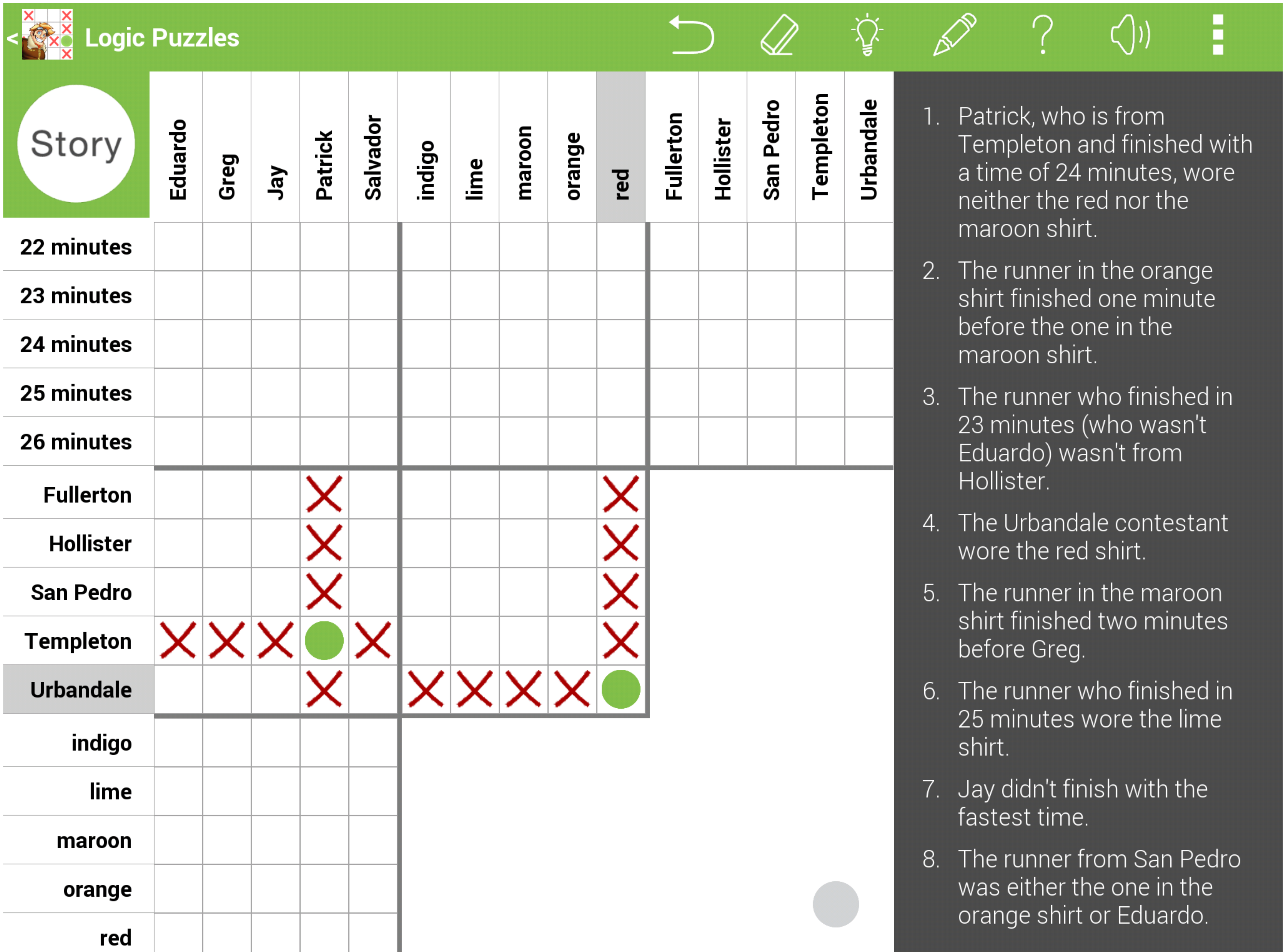

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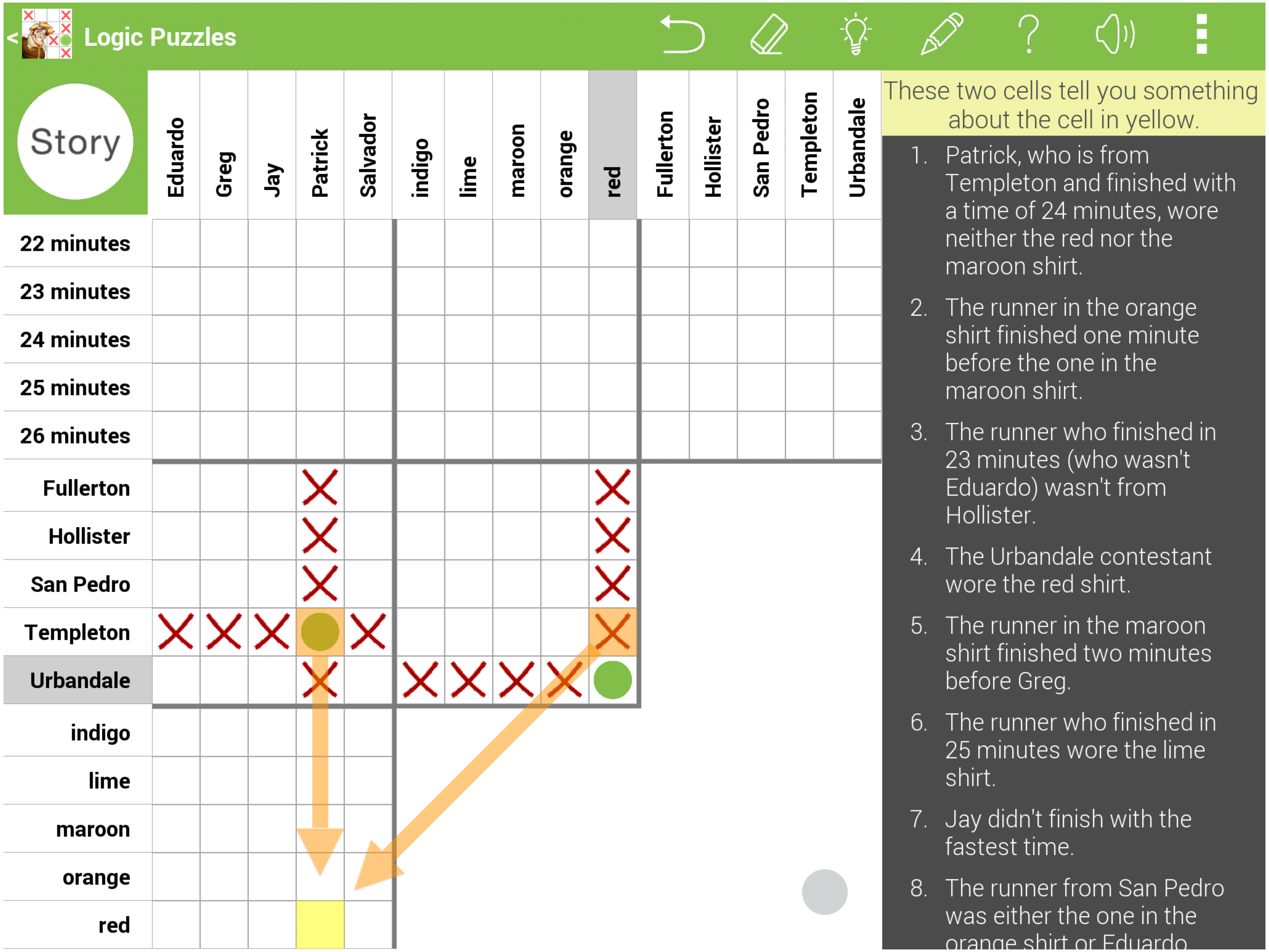


Tabular "Deduction": It's Taught!

Example

Grace, Dylan, Kira, and Diego are each wearing different colored shirts. Grace's shirt is red. Dylan's shirt is not white. Kira's shirt is not green. Diego's shirt is not yellow or white. What color shirt is each person wearing?

First, make a chart to show what you know.

- Each shirt is a different color.
- Grace's shirt is red.
- Dylan's shirt is not white.
- Kira's shirt is not green.
- Diego's shirt is not yellow or white.

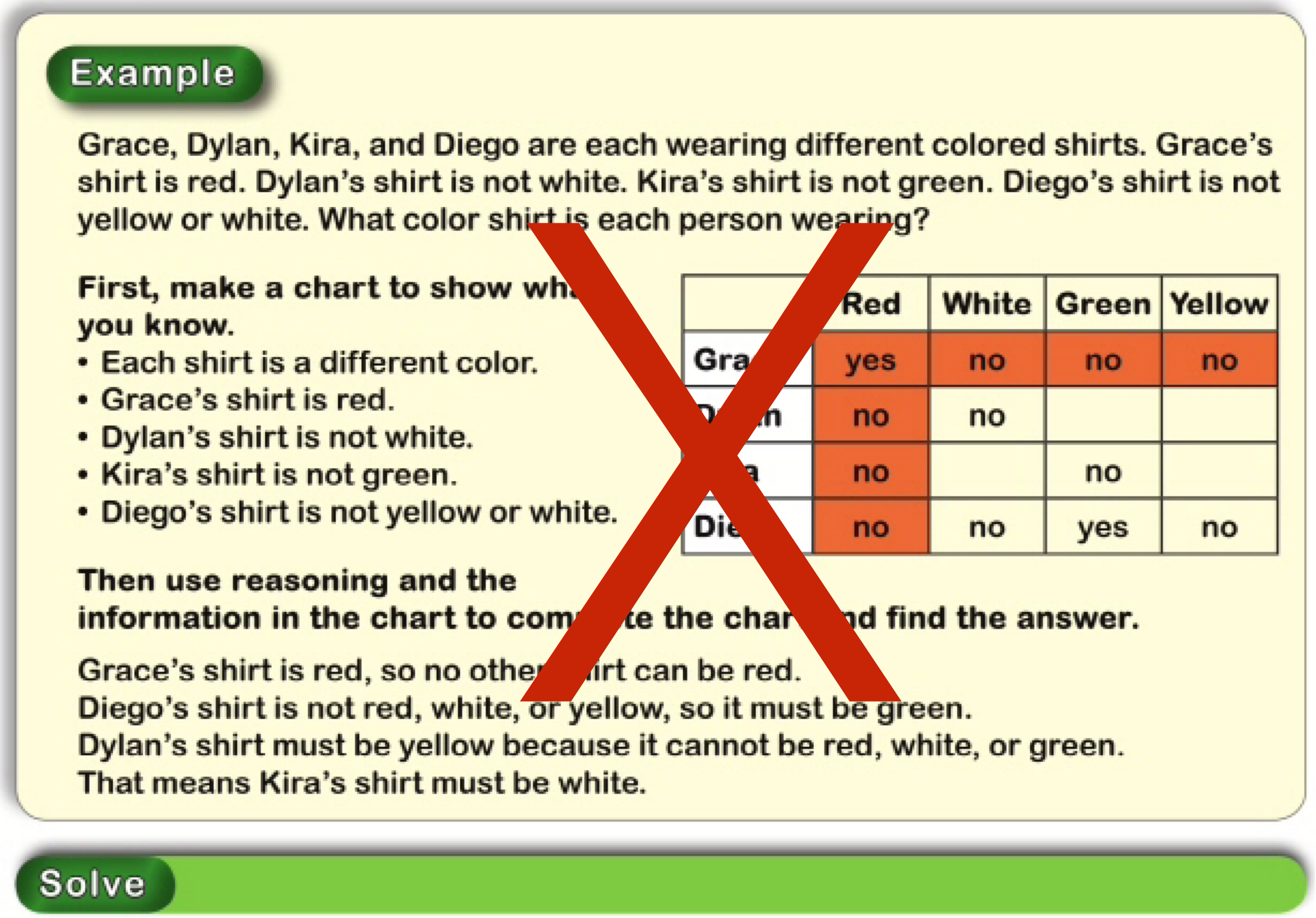
	Red	White	Green	Yellow
Grace	yes	no	no	no
Dylan	no	no		
Kira	no		no	
Diego	no	no	yes	no

Then use reasoning and the information in the chart to complete the chart and find the answer.

Grace's shirt is red, so no other shirt can be red.
Diego's shirt is not red, white, or yellow, so it must be green.
Dylan's shirt must be yellow because it cannot be red, white, or green.
That means Kira's shirt must be white.

Solve

Tabular "Deduction": It's Taught!



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	Red	White	Green	Yellow
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[illegible]n	no	no		
[illegible]a	no		no	
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Solve

IMHO very bad idea—if *before* real learning of deduction to answer "Why, exactly? Prove it!"

Tabular "deduction" *not* the skill that's valuable.

8:29 AM iPad 8:29 AM 91

Recall from Lesson 4-8 that the complex numbers $a + bi$ and $a - bi$ are conjugates. Similarly, the irrational numbers $a + \sqrt{b}$ and $a - \sqrt{b}$ are conjugates. If a complex number or an irrational number is a root of a polynomial equation with rational coefficients, so is its conjugate.

TAKE NOTE Theorem

Conjugate Root Theorem

If $P(x)$ is a polynomial with *rational* coefficients, then irrational roots of $P(x) = 0$ that have the form $a + \sqrt{b}$ occur in conjugate pairs. That is, if $a + \sqrt{b}$ is an irrational root with a and b rational, then $a - \sqrt{b}$ is also a root.

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Every quadratic polynomial equation has two roots, every cubic polynomial equation has three roots, and so on.

This result is related to the *Fundamental Theorem of Algebra*. The German mathematician Carl Friedrich Gauss (1777–1855) is credited with proving this theorem.

TAKE NOTE Theorem

The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x) = 0$ has exactly n roots, including multiple and complex roots.

From Algebra 2

iPad

Theorems About Roots of Polynomial Equations

Practice and Problem-Solving Exercises - Contir

Determine whether each of the following statements is *always*, *sometimes*, or *never* true.

41. A polynomial function with real coefficients has real zeros.

42. Polynomial functions with complex coefficients have one complex zero.

43. A polynomial function that does not intercept the x-axis has complex roots only.

44. **Reasoning** A 4th-degree polynomial function has zeros at 3 and $5 - i$. Can $4 + i$ also be a zero of the function? Explain your reasoning.

45. **Open-Ended** Write a polynomial function that has four possible rational zeros but no actual rational zeros.

46. **Reasoning** Show that the Fundamental Theorem of Algebra must be true for all quadratic polynomial functions.

C • Challenge

47. Use the Fundamental Theorem of Algebra and the Conjugate Root Theorem to show that any odd degree polynomial equation with real coefficients has at least one real root.

48. **Reasoning** What is the maximum number of points of intersection between the graphs of a quartic and a quintic polynomial function?

49. **Reasoning** What is the least possible degree of a polynomial with rational coefficients, leading coefficient 1, constant term 5, and zeros at $\sqrt{2}$ and $\sqrt{3}$? Show that such a polynomial has a rational zero and indicate this zero.

Tabular "deduction" *not* the skill that's valuable.

8:29 AM iPad 8:29 AM

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The Game of LogiNim$_{\mathscr{L}}$

In HyperSlate®

The Game of LogiNim$_{\mathscr{L}}$ in HyperSlate®

Selmer Bringsjord
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A *logicist directed acyclic hypergraph*, hereafter simply a *hypergraph*, is a pair

$$\mathscr{H} \coloneqq \langle N, A \rangle$$

where

- each *node* $\nu \in N$ contains some formula $\phi \in \mathcal{L}$, where $\mathcal{L}$ is a background formal language, possibly including a label l_ϕ for the node, and the set of all assumptions on which inference of ϕ may rely;
- each *arc* $a \in A$ is a pair composed of a label l_σ for some $\sigma \in \mathcal{I}$, a collection of inference schemata, and one of $\{r, g\}$;
- arcs are directed; and
- no *cycles* are permitted.

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Back to FOL …

Our Final New Inference Rule in FOL

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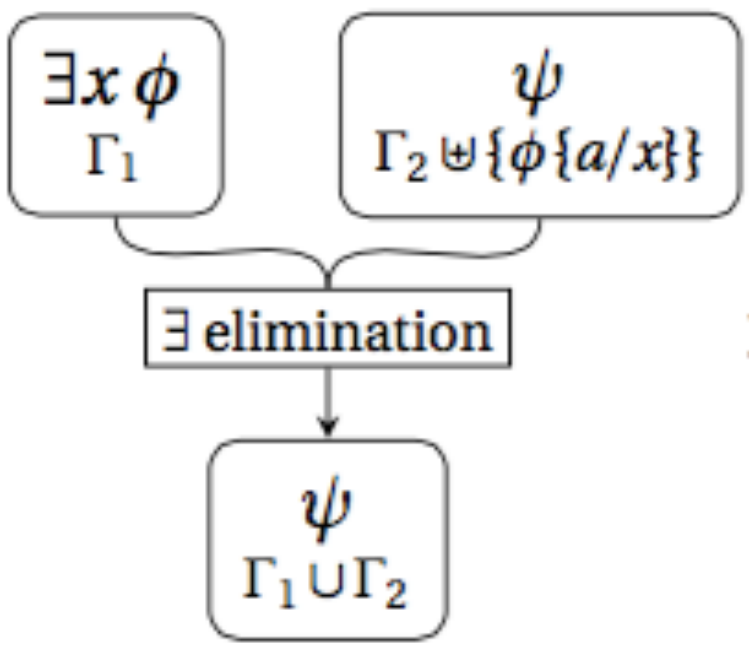
- `existential elimination` (intuitively put):

Our Final New Inference Rule in FOL

- `existential elimination` (intuitively put):
    - If we know that (i) there's something *x* which is an *R*, and (ii) on the supposition that *a* is an arbitrary representative (a "witness") of such an *x* we can prove *P*, then we are permitted to deduce *P* from (i) alone.

existential elimination, precise version:



provided that a does not appear free in Γ_1, Γ_2, or ψ

Starting State of Workspace

(Assumes a domain of e.g. players on a March-madness basketball court.)

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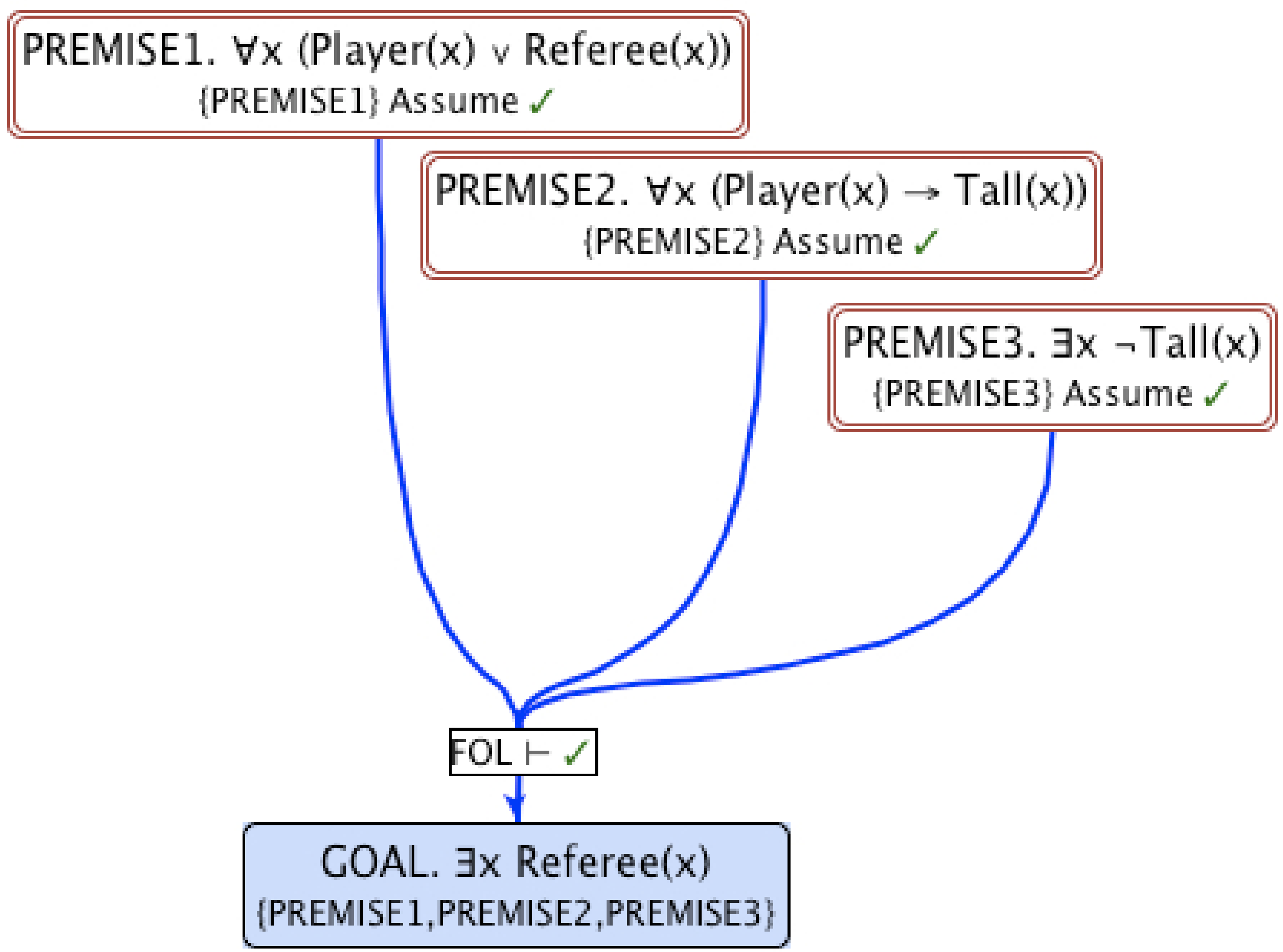

Starting State of Workspace

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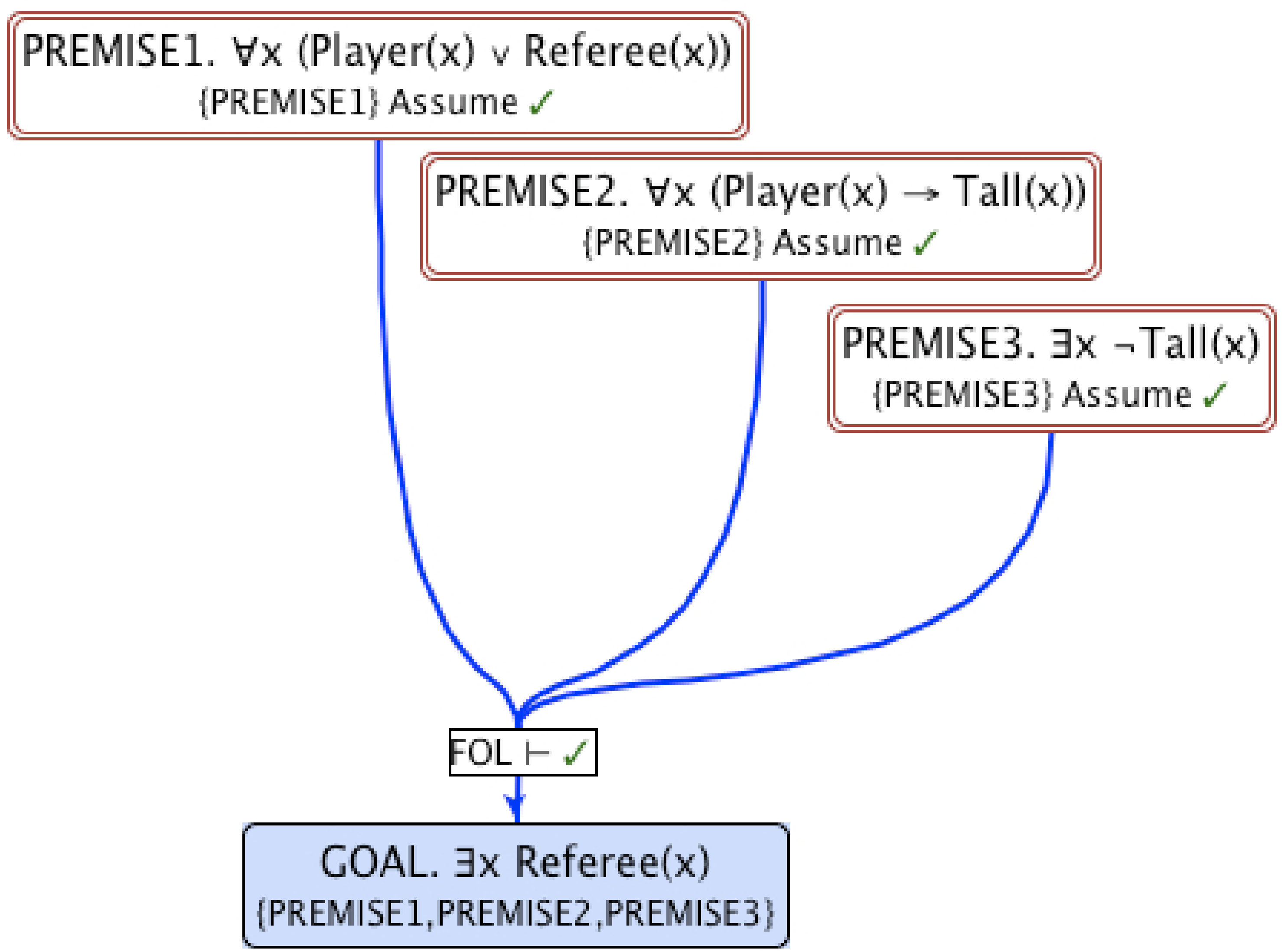
"Each and every thing is either a player or a referee."



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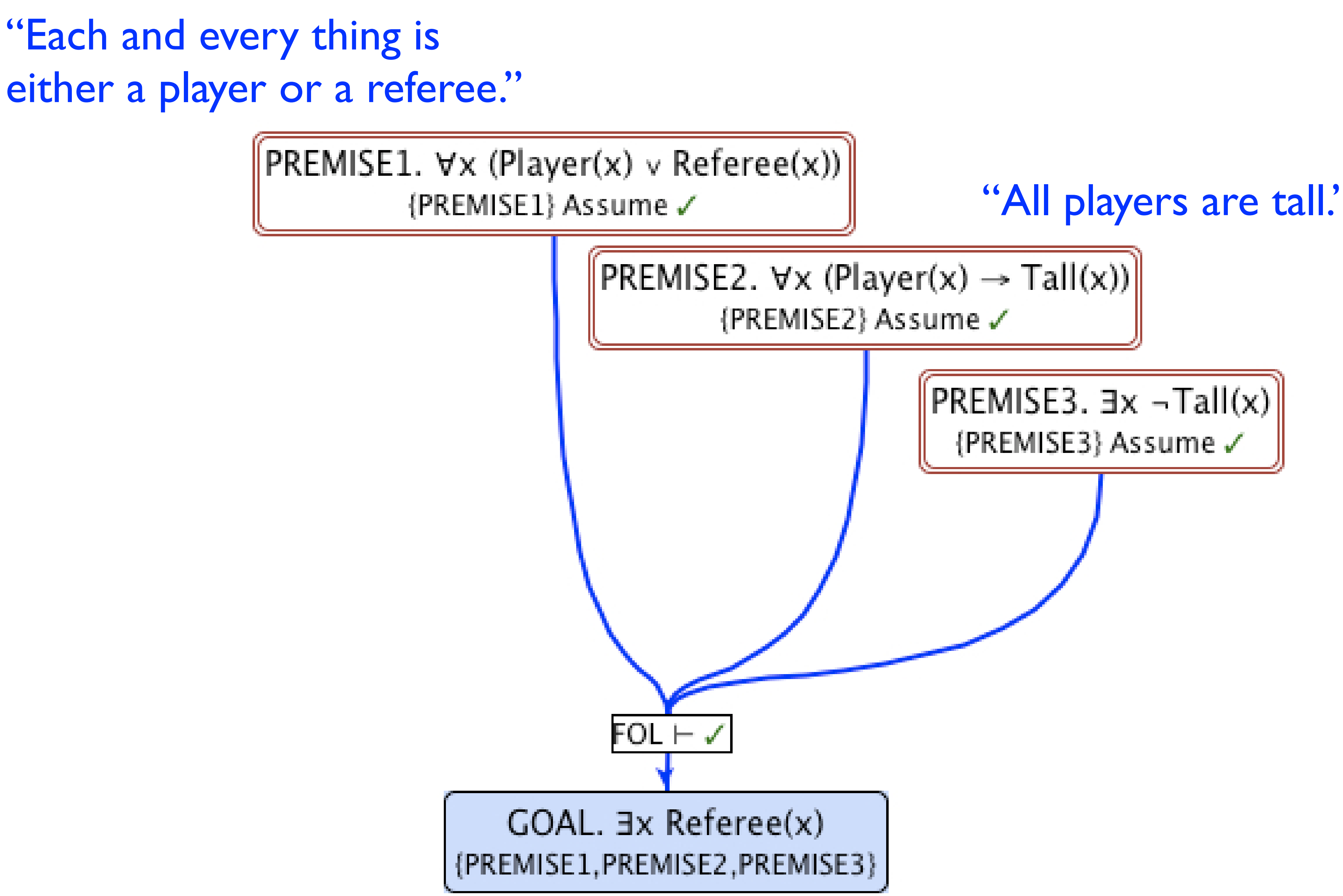

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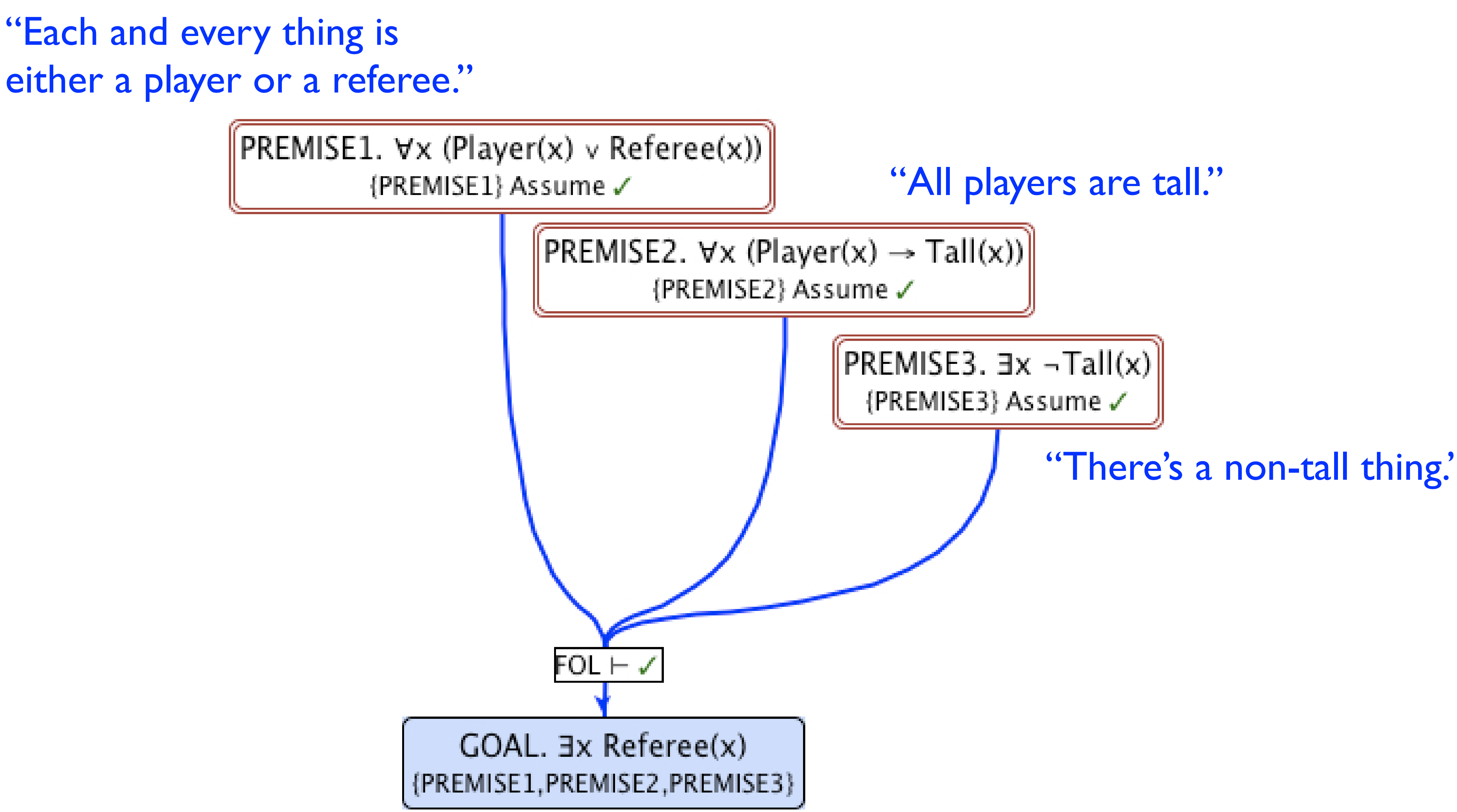


Step 1

PREMISE1. ∀x (Player(x) ∨ Referee(x))
{PREMISE1} Assume ✓

PREMISE2. ∀x (Player(x) → Tall(x))
{PREMISE2} Assume ✓

PREMISE3. ∃x ¬Tall(x)
{PREMISE3} Assume ✓

5. ¬Tall(a)
{5} Assume ✓

GOAL. ∃x Referee(x)
FOL ⊢ ✗

Step 1

PREMISE1. $\forall x\ (Player(x) \lor Referee(x))$
{PREMISE1} Assume ✓

PREMISE2. $\forall x\ (Player(x) \rightarrow Tall(x))$
{PREMISE2} Assume ✓

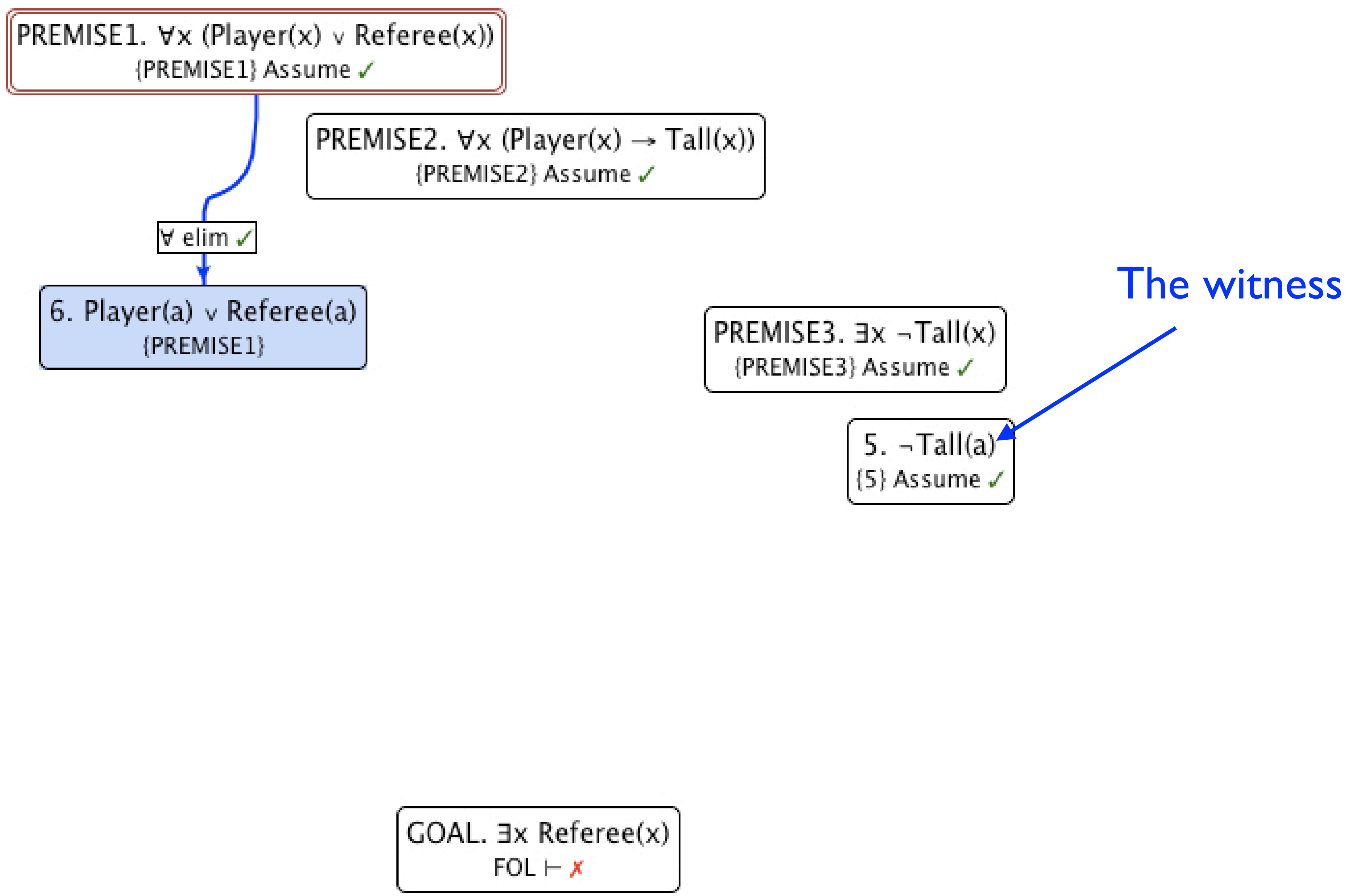
The witness!

PREMISE3. $\exists x\ \neg Tall(x)$
{PREMISE3} Assume ✓

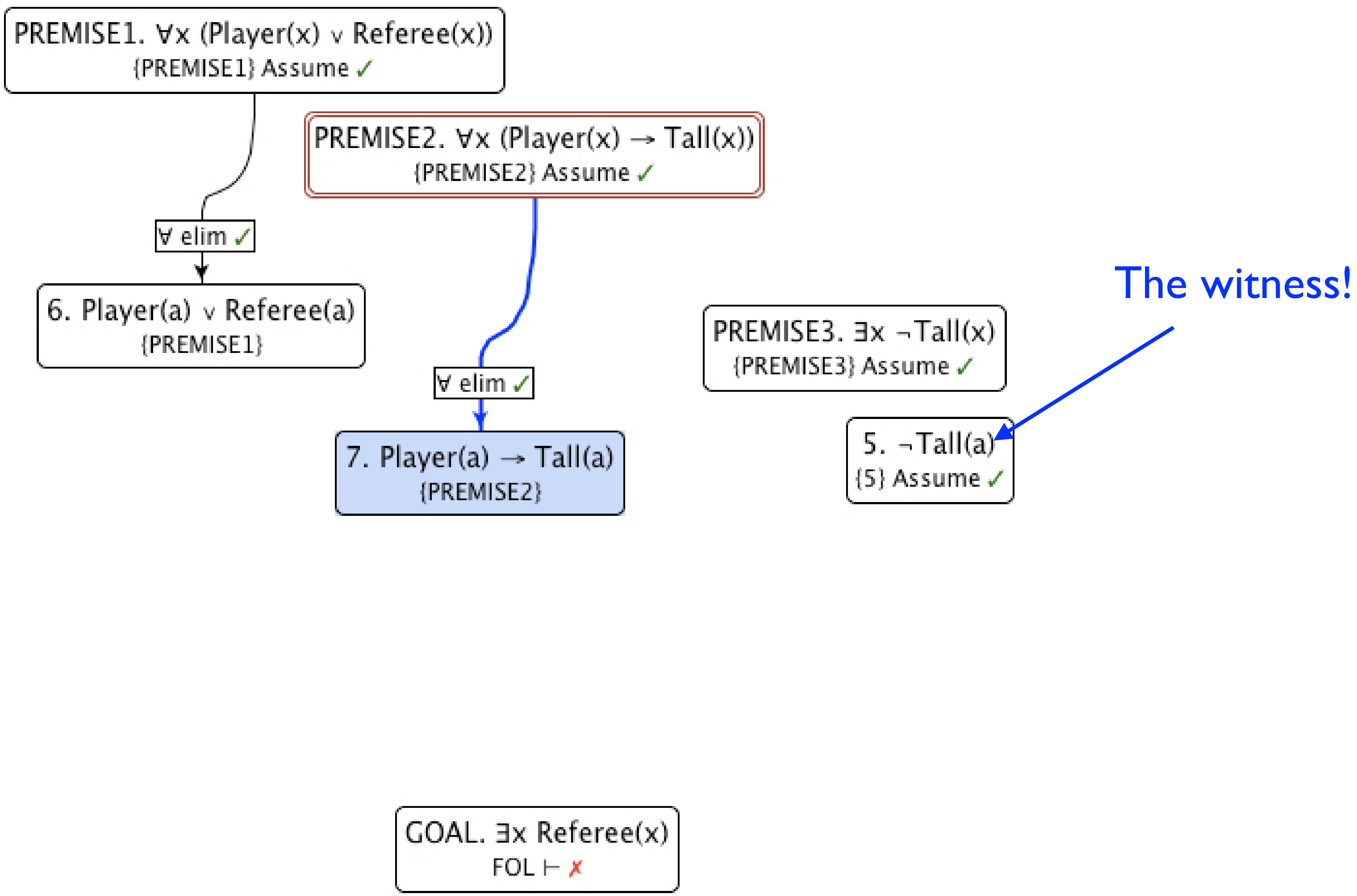
5. $\neg Tall(a)$
{5} Assume ✓

GOAL. $\exists x\ Referee(x)$
FOL $\vdash$ ✗

Step 2


PREMISE1. ∀x (Player(x) ∨ Referee(x))
{PREMISE1} Assume ✓
PREMISE2. ∀x (Player(x) → Tall(x))
{PREMISE2} Assume ✓
∀ elim ✓
6. Player(a) ∨ Referee(a)
{PREMISE1}
The witness!
PREMISE3. ∃x ¬Tall(x)
{PREMISE3} Assume ✓
5. ¬Tall(a)
{5} Assume ✓
GOAL. ∃x Referee(x)
FOL ⊢ ✗

Step 3



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PREMISE2. ∀x (Player(x) → Tall(x))
{PREMISE2} Assume ✓
∀ elim ✓
6. Player(a) ∨ Referee(a)
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∀ elim ✓
7. Player(a) → Tall(a)
{PREMISE2}
PREMISE3. ∃x ¬Tall(x)
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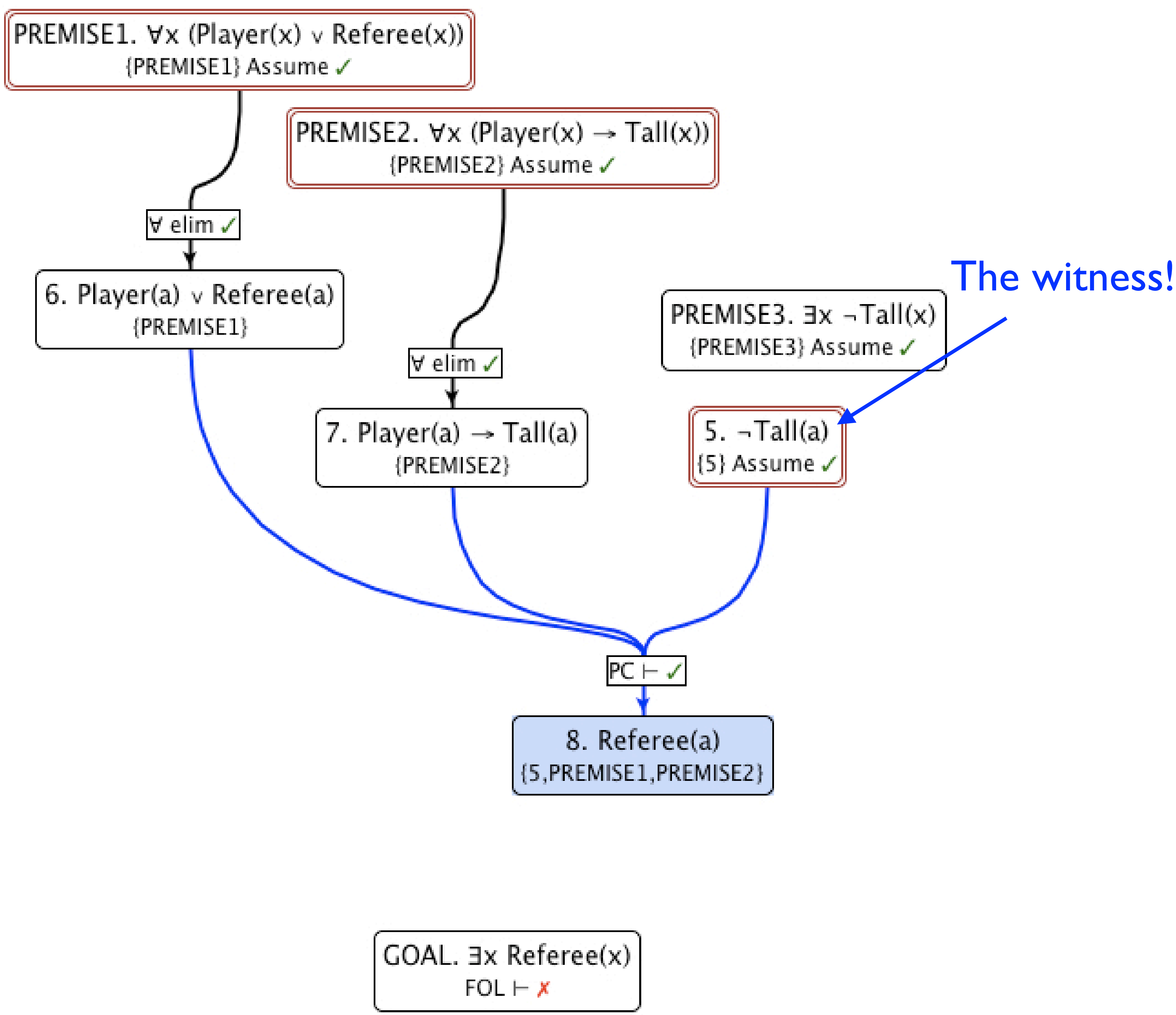
5. ¬Tall(a)
{5} Assume ✓
The witness!
GOAL. ∃x Referee(x)
FOL ⊢ ✗

Step 4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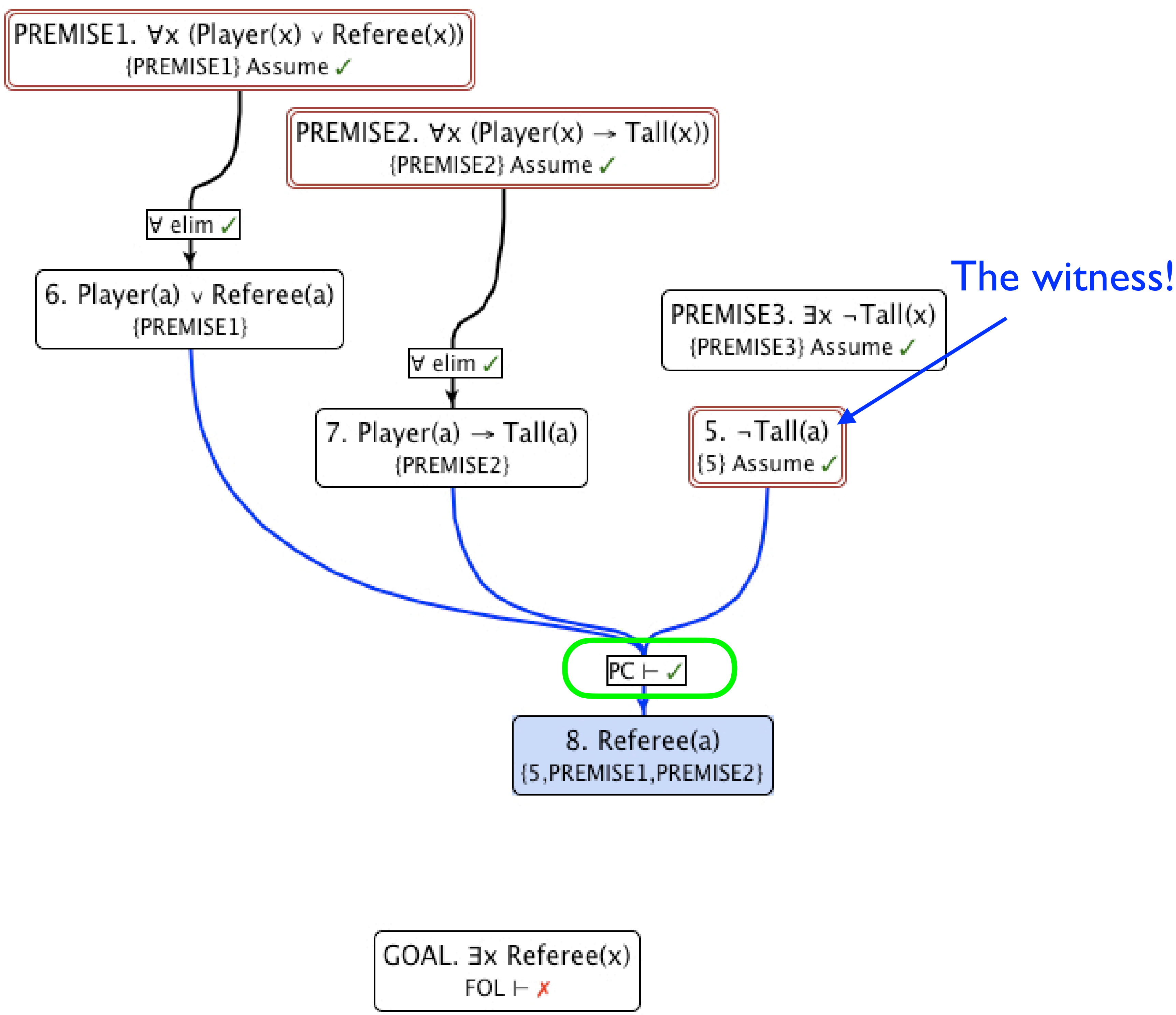
{PREMISE1} Assume ✓
PREMISE2. ∀x (Player(x) → Tall(x))
{PREMISE2} Assume ✓
∀ elim ✓
6. Player(a) ∨ Referee(a)
{PREMISE1}
∀ elim ✓
The witness!
PREMISE3. ∃x ¬Tall(x)
{PREMISE3} Assume ✓
7. Player(a) → Tall(a)
{PREMISE2}
5. ¬Tall(a)
{5} Assume ✓
PC ⊢ ✓
8. Referee(a)
{5,PREMISE1,PREMISE2}
GOAL. ∃x Referee(x)
FOL ⊢ ✗

Step 4



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PREMISE2. ∀x (Player(x) → Tall(x))
{PREMISE2} Assume ✓
∀ elim ✓
6. Player(a) ∨ Referee(a)
{PREMISE1}
∀ elim ✓
The witness!
PREMISE3. ∃x ¬Tall(x)
{PREMISE3} Assume ✓
7. Player(a) → Tall(a)
{PREMISE2}
5. ¬Tall(a)
{5} Assume ✓
PC ⊢ ✓
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{5,PREMISE1,PREMISE2}
GOAL. ∃x Referee(x)
FOL ⊢ ✗

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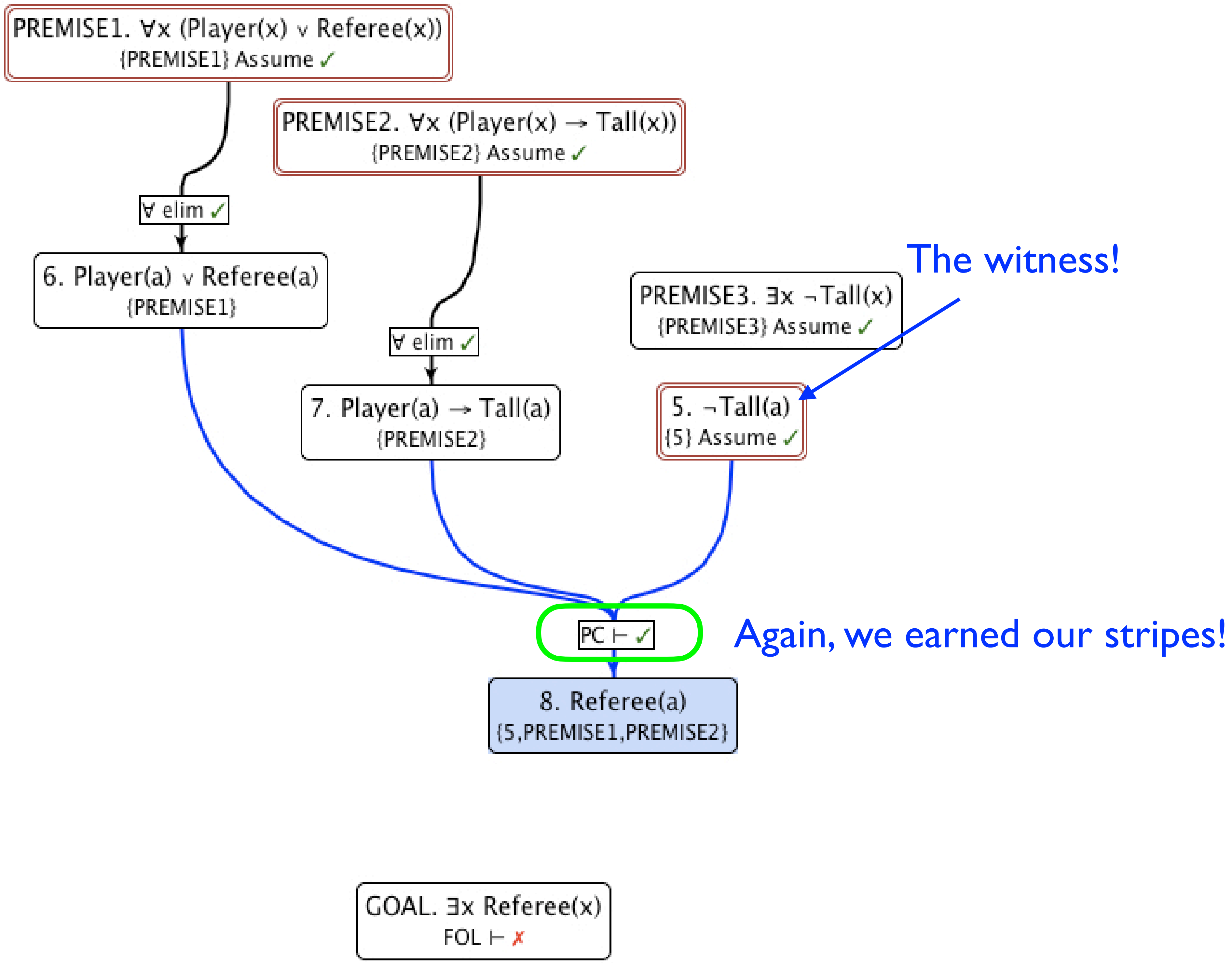
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PREMISE2. ∀x (Player(x) → Tall(x))
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∀ elim ✓
6. Player(a) ∨ Referee(a)
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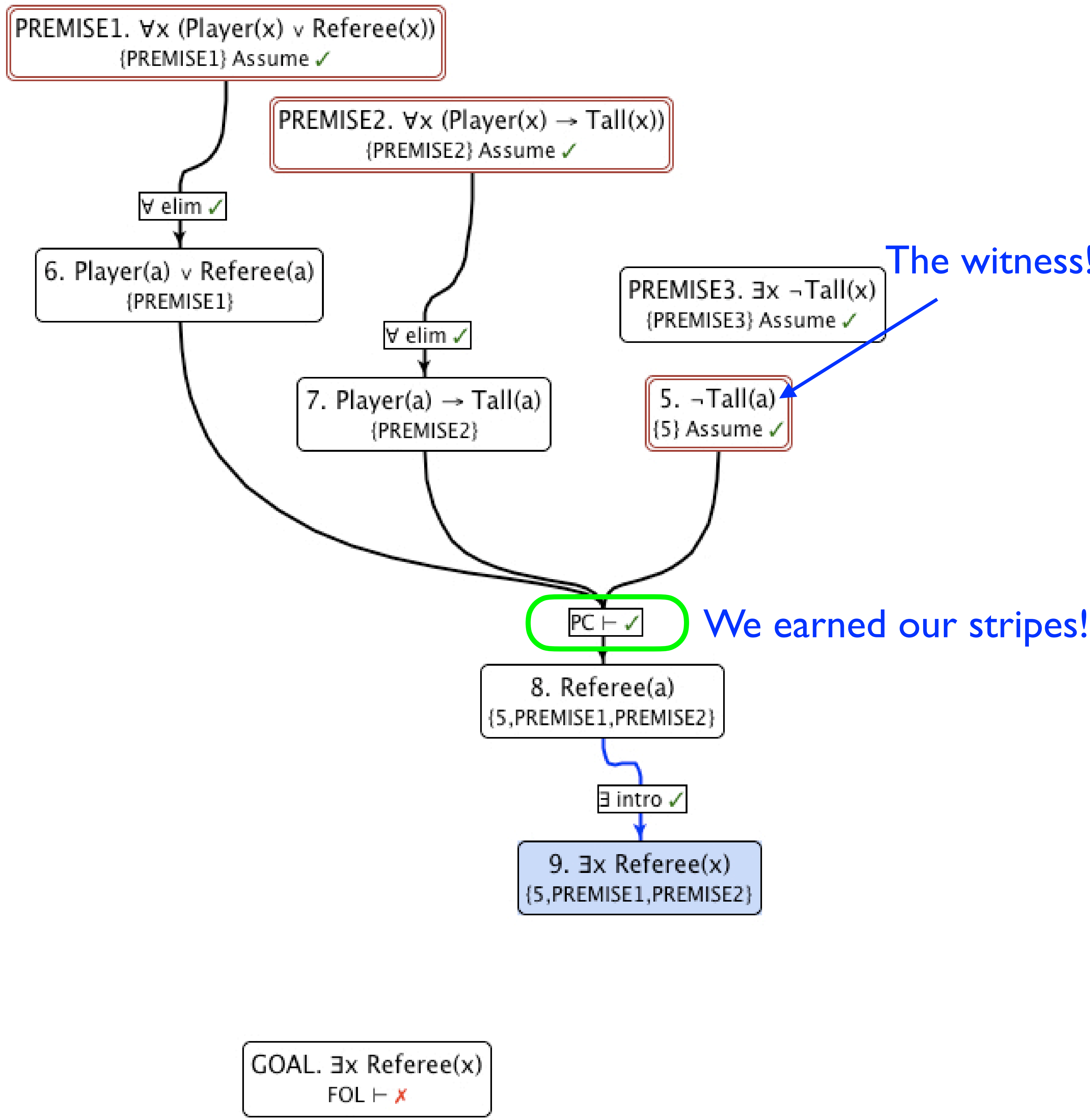
∀ elim ✓
The witness!
PREMISE3. ∃x ¬Tall(x)
{PREMISE3} Assume ✓
7. Player(a) → Tall(a)
{PREMISE2}
5. ¬Tall(a)
{5} Assume ✓
PC ⊢ ✓
Again, we earned our stripes!
8. Referee(a)
{5,PREMISE1,PREMISE2}
GOAL. ∃x Referee(x)
FOL ⊢ ✗

Step 5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PREMISE2. ∀x (Player(x) → Tall(x))
{PREMISE2} Assume ✓
∀ elim ✓
6. Player(a) ∨ Referee(a)
{PREMISE1}
∀ elim ✓
The witness!
PREMISE3. ∃x ¬Tall(x)
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7. Player(a) → Tall(a)
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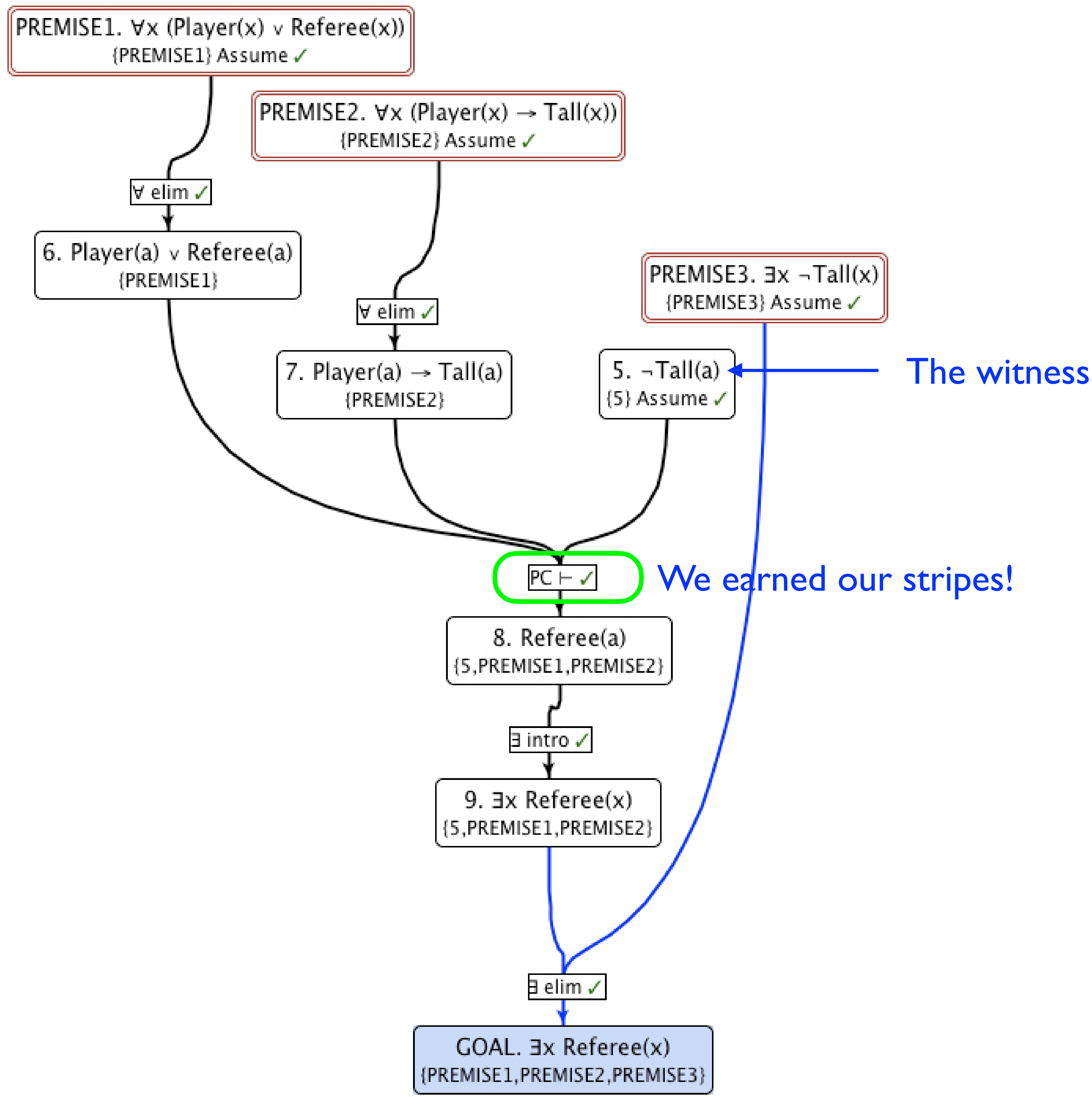
5. ¬Tall(a)
{5} Assume ✓
PC ⊢ ✓
We earned our stripes!
8. Referee(a)
{5,PREMISE1,PREMISE2}
∃ intro ✓
9. ∃x Referee(x)
{5,PREMISE1,PREMISE2}
GOAL. ∃x Referee(x)
FOL ⊢ ✗

Step 6



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PREMISE2. ∀x (Player(x) → Tall(x))
{PREMISE2} Assume ✓
∀ elim ✓
6. Player(a) ∨ Referee(a)
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PREMISE3. ∃x ¬Tall(x)
{PREMISE3} Assume ✓
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7. Player(a) → Tall(a)
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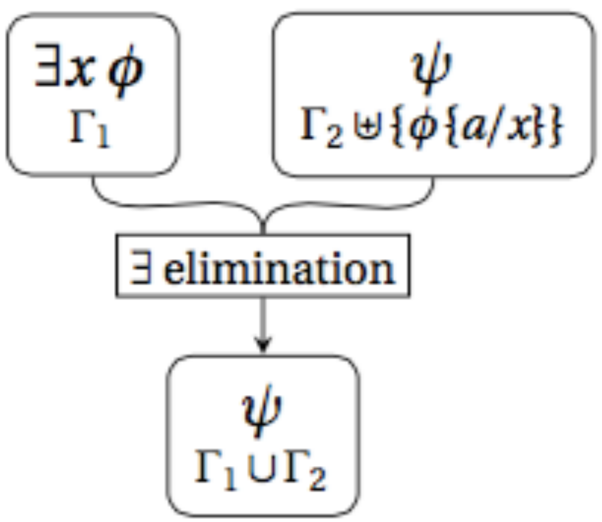
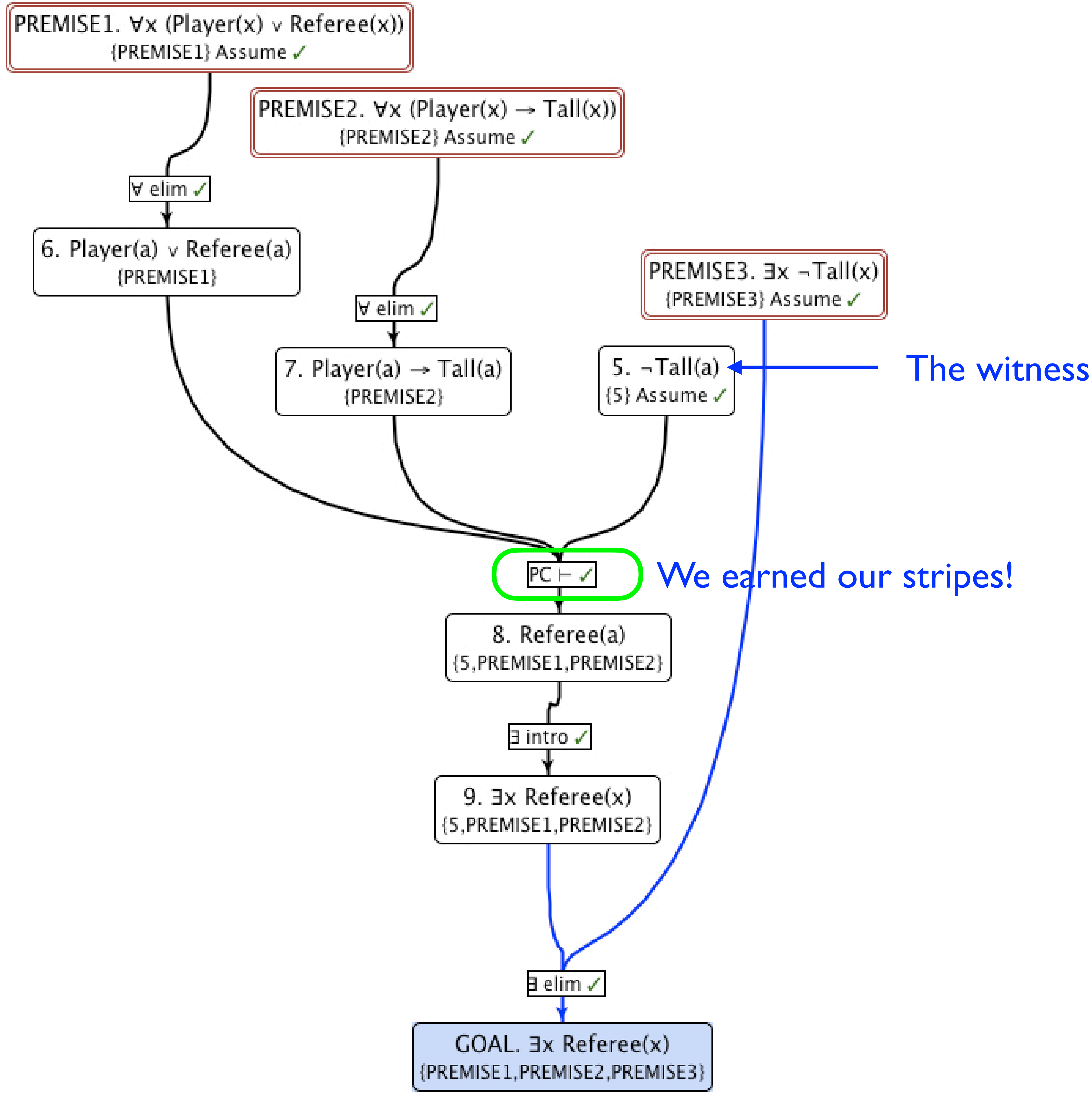
The witness!
PC ⊢ ✓
We earned our stripes!
8. Referee(a)
{5,PREMISE1,PREMISE2}
∃ intro ✓
9. ∃x Referee(x)
{5,PREMISE1,PREMISE2}
∃ elim ✓
GOAL. ∃x Referee(x)
{PREMISE1,PREMISE2,PREMISE3}

Step 6



∃x φ
Γ1
ψ
Γ2 ⊎ {φ{a/x}}
∃ elimination
ψ
Γ1 ∪ Γ2



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{5,PREMISE1,PREMISE2}
∃ elim ✓
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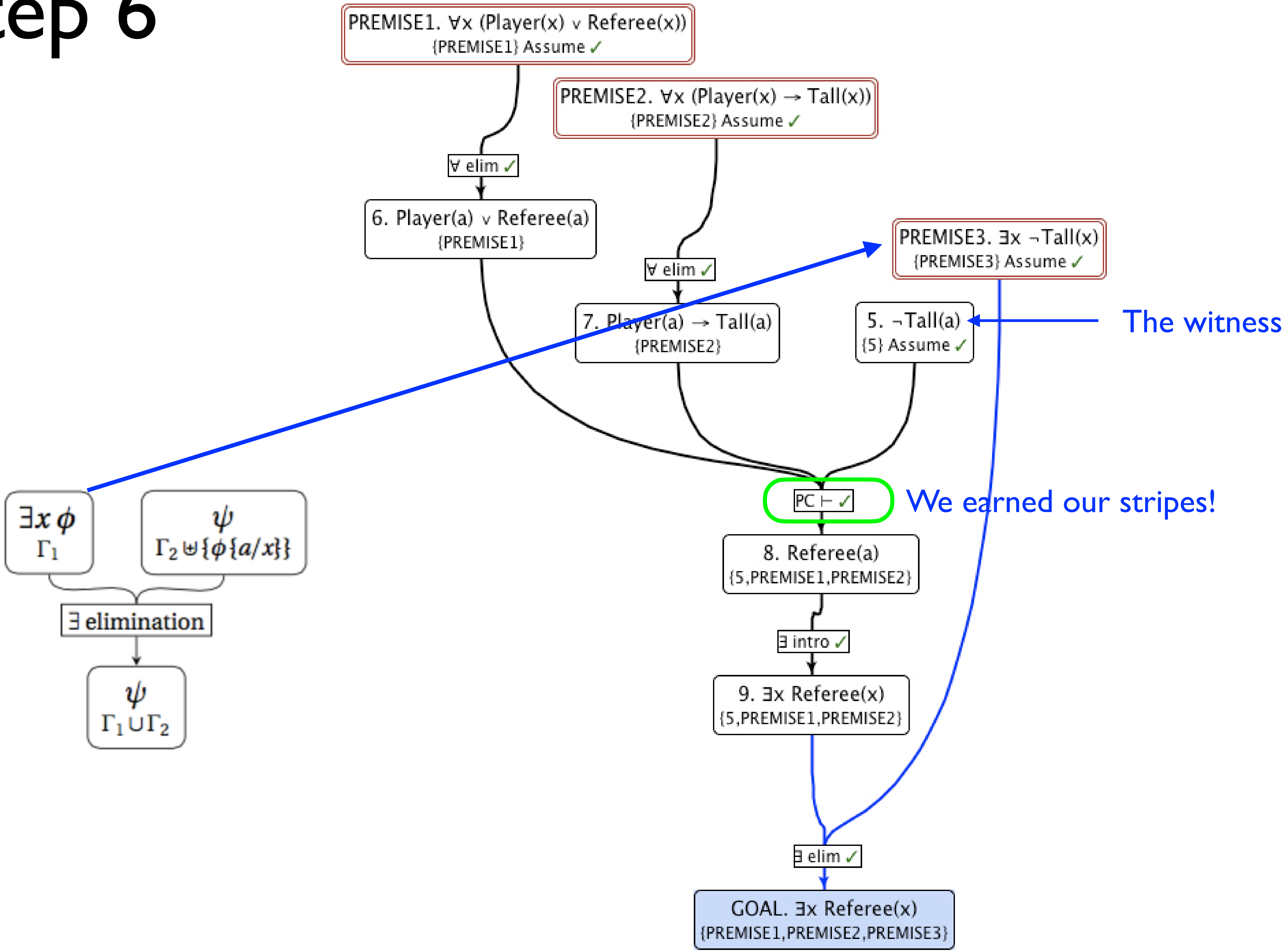
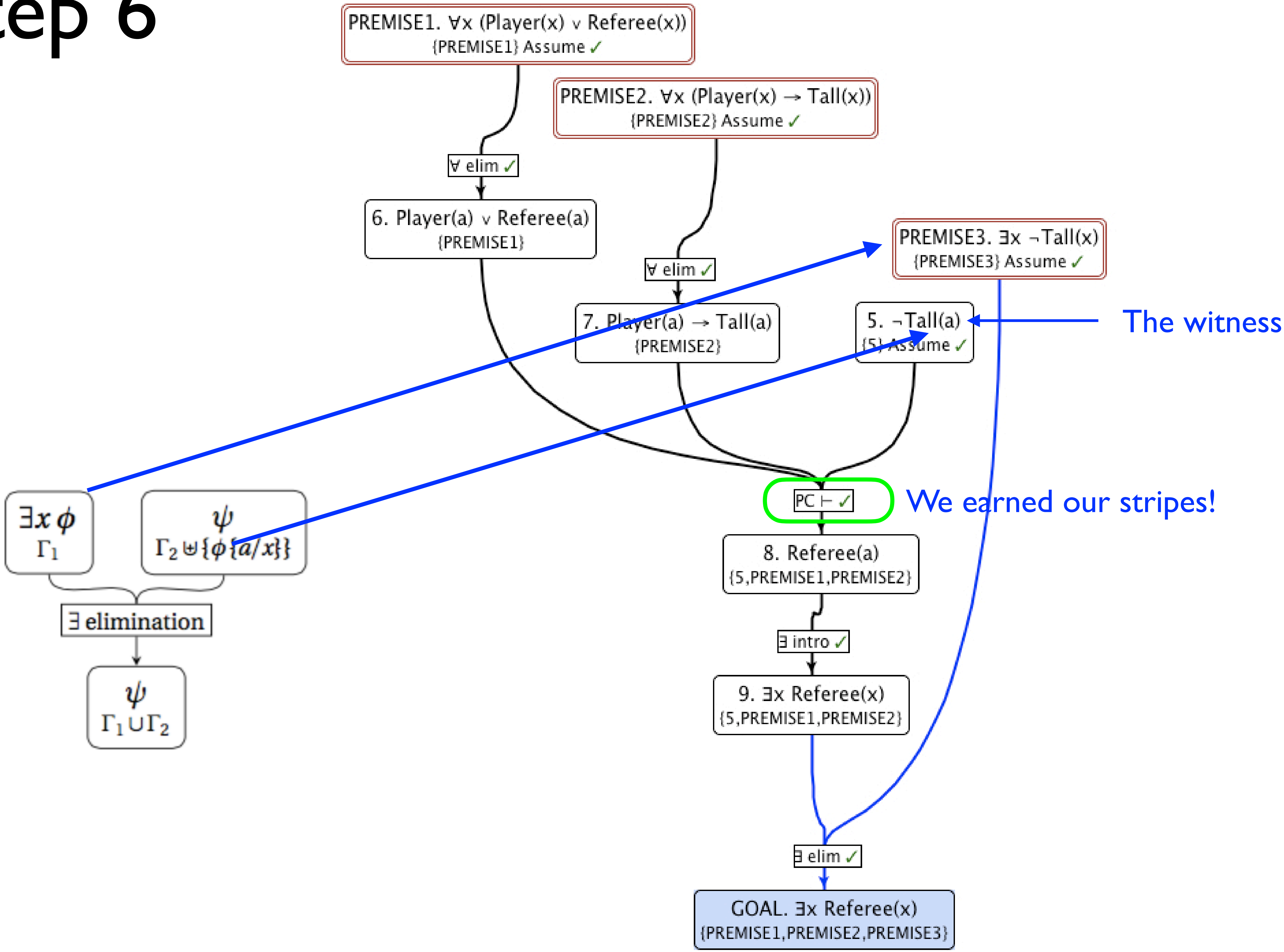
{PREMISE1,PREMISE2,PREMISE3}

Step 6


PREMISE1. ∀x (Player(x) ∨ Referee(x))
{PREMISE1} Assume ✓
PREMISE2. ∀x (Player(x) → Tall(x))
{PREMISE2} Assume ✓
∀ elim ✓
6. Player(a) ∨ Referee(a)
{PREMISE1}
∀ elim ✓
PREMISE3. ∃x ¬Tall(x)
{PREMISE3} Assume ✓
7. Player(a) → Tall(a)
{PREMISE2}
5. ¬Tall(a)
{5} Assume ✓
The witness!
PC ⊢ ✓
We earned our stripes!
8. Referee(a)
{5,PREMISE1,PREMISE2}
∃ intro ✓
9. ∃x Referee(x)
{5,PREMISE1,PREMISE2}
∃ elim ✓
GOAL. ∃x Referee(x)
{PREMISE1,PREMISE2,PREMISE3}
∃x φ
Γ1
ψ
Γ2 ⊎ {φ{a/x}}
∃ elimination
ψ
Γ1 ∪ Γ2

Step 6



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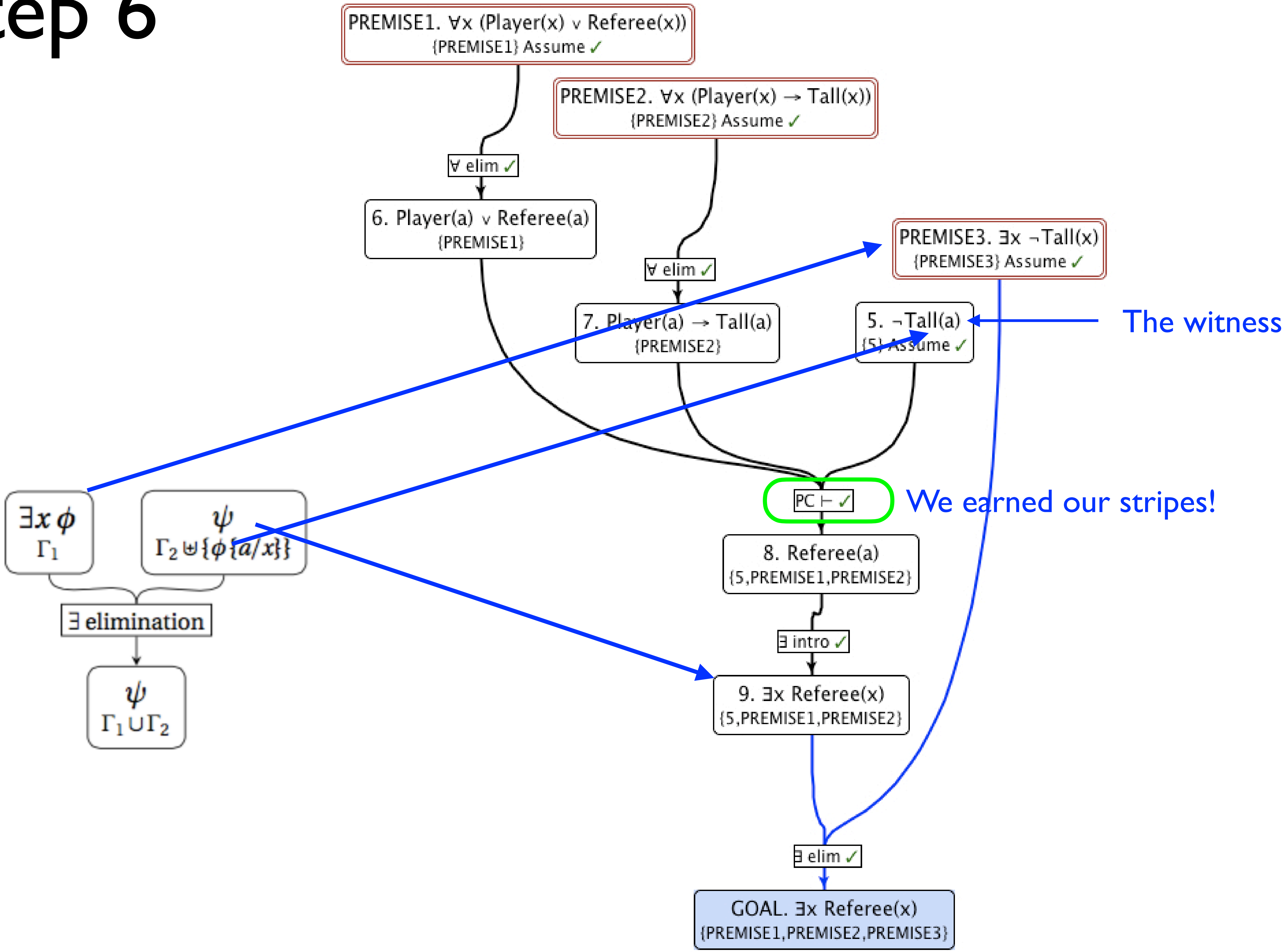
Step 6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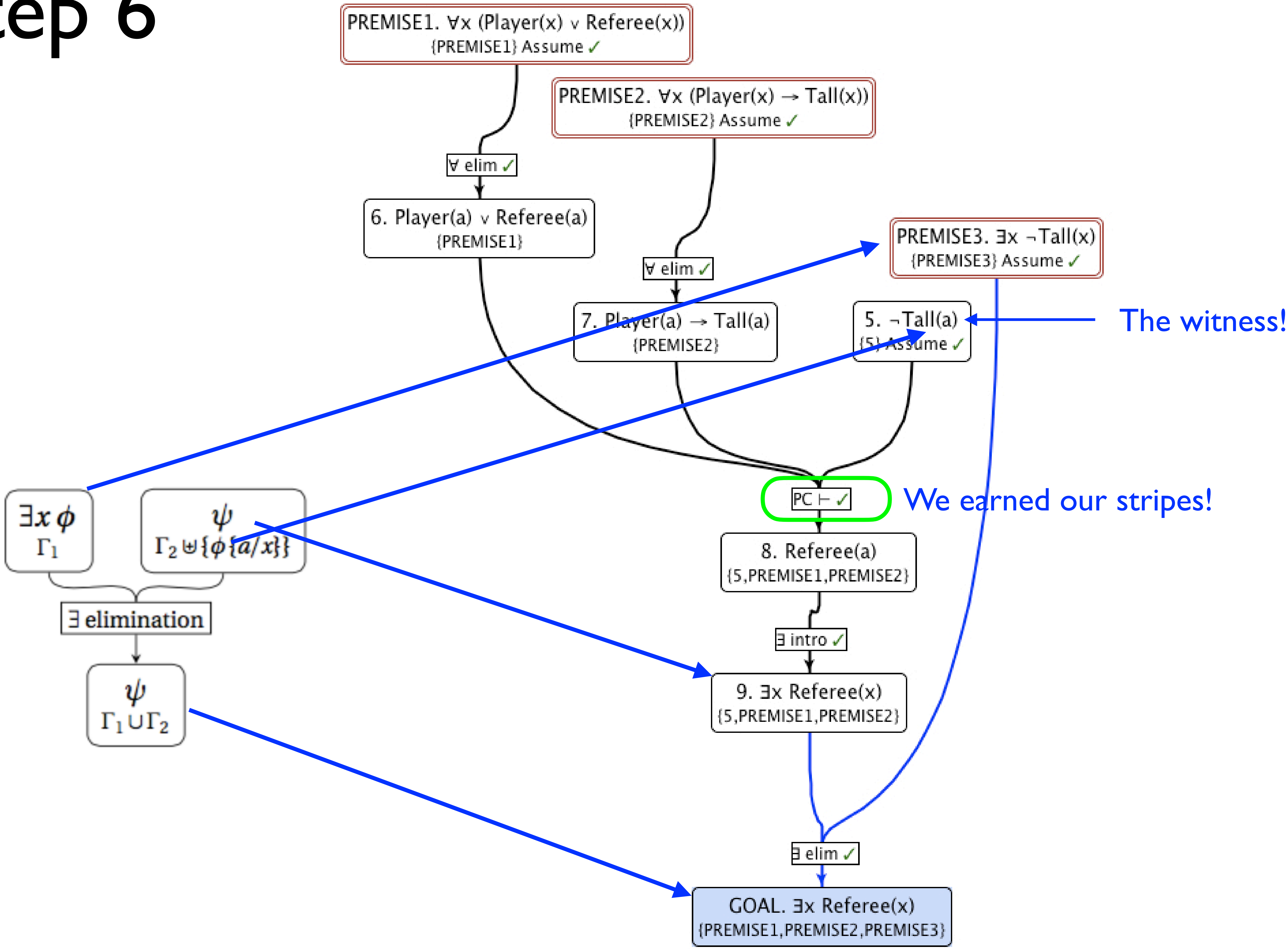
Step 6



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PREMISE3. ∃x ¬Tall(x)
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∃x φ
Γ1
ψ
Γ2 ⊎ {φ{a/x}}
∃ elimination
ψ
Γ1 ∪ Γ2

Practice:

$$\{\forall \mathtt{x}(\mathtt{Scared}(\mathtt{x}) \leftrightarrow \mathtt{Small}(\mathtt{x})), \exists \mathtt{x} \neg \mathtt{Scared}(x)\} \vdash \exists \mathtt{x} \neg \mathtt{Small}(x)$$

$$\{\exists \mathtt{x}, \mathtt{y} \mathtt{Contiguous}(\mathtt{x}, \mathtt{y}), \forall \mathtt{x}, \mathtt{y}(\mathtt{Contiguous}(\mathtt{x}, \mathtt{y}) \rightarrow \neg \mathtt{SameCountry}(\mathtt{x}, \mathtt{y}))\} \vdash \exists \mathtt{x}, \mathtt{y} \neg \mathtt{SameCountry}(\mathtt{x}, \mathtt{y})$$

Hvis du forstår det, kan du bevise det.