Gödel's
Second Incompleteness Theorem
(G2)

Selmer Bringsjord

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Intro to Formal Logic (With AI)
4/13/2026

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Note: This is a version designed for those who have had at least one university-level course in formal logic with coverage through $\mathscr{L}_2$.

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G2 — Liar Paradox exploited — entails that these researchers can keep their funding!

Background Context …

Could Artificial Intelligence Ever … Match Personal Intelligence?

Gödelian Essays on Persons vs. Pretenders

by Selmer Bringsjord et al.

- Introduction (“The Wager”)
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
- The Continuum-Hypothesis Theorem
- The Time-Travel Theorem
- Gödel’s “God Theorem”
- Could a Finite Machine Match Gödel’s Greatness?

STOP & REVIEW IF NEEDED!

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A corollary of the First Incompleteness Theorem: *We cannot prove (in classical mathematics) that mathematics is consistent.*

The “Gödelian” Liar (from me)

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$\bar{P}$: This sentence is unprovable.

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Suppose that $\bar{P}$ is true. Then we can immediately deduce that $\bar{P}$ is provable, because here is a proof: $\bar{P} \rightarrow \bar{P}$ is an easy theorem, and from it and our supposition we deduce $\bar{P}$ by *modus ponens*. But since what $\bar{P}$ says is that it's unprovable, we have deduced that $\bar{P}$ is false under our initial supposition.

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Suppose on the other hand that $\bar{P}$ is false. Then we can immediately deduce that $\bar{P}$ is unprovable: Suppose for *reductio* that $\bar{P}$ *is* provable; then $\bar{P}$ holds as a result of some proof, but what $\bar{P}$ says is that it's unprovable; and so we have contradiction. But since what $\bar{P}$ says is that it's unprovable, and we have just proved that under our supposition, we arrive at the conclusion that $\bar{P}$ is true.

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$\bar{\pi}$ = "I'm unprovable."

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Contradiction!

All of this is fishy; but Gödel, as we've seen, transformed it (by e.g. use of his encryption scheme) into utterly precise, impactful, indisputable reasoning.

PA (Peano Arithmetic):

$$
\begin{array}{ll}
\text{A1} & \forall x(0 \neq s(x)) \\
\text{A2} & \forall x \forall y(s(x) = s(y) \rightarrow x = y) \\
\text{A3} & \forall x(x \neq 0 \rightarrow \exists y(x = s(y)) \\
\text{A4} & \forall x(x + 0 = x) \\
\text{A5} & \forall x \forall y(x + s(y) = s(x + y)) \\
\text{A6} & \forall x(x \times 0 = 0) \\
\text{A7} & \forall x \forall y(x \times s(y) = (x \times y) + x)
\end{array}
$$

And, every sentence that is the universal closure of an instance of

$$([\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x))] \rightarrow \forall x \phi(x))$$

where $\phi(x)$ is open wff with variable x, and perhaps others, free.

RussellsLetter2Frege_IFLAI1S24

Problem Bank

Edit Problems

In-Class Exploratory

The challenge here is to prove that from Russell's instantiation of Frege's doomed Axiom V a contradiction can be promptly derived. The letter has of course been examined in some detail by S Bringsjord (in the Feb 29 2024 lecture in the 2024 lecture lineup). Put meta-logically, your task in the present problem is to build a proof that confirms this:

$$\{\exists x \forall y((y \in x) \rightarrow (y \notin y))\} \vdash \zeta \wedge \neg \zeta.$$

Make sure you understand that the given here is an instantiation of Frege's Axiom V; i.e. it's an instantiation of

$$\exists x \forall y((y \in x) \rightarrow \phi(y)).$$

(The notation $\phi(y)$, recall from lecture, is the standard way in mathematical logic to say that y is free in ϕ.) **Note**: Your finished proof is allowed to make use the PC-provability oracle (but *only* that oracle).

(Now a brief remark on matters covered by in class by Bringsjord when second-order logic = $\mathscr{L}_2$ arrives on the scene: Longer term, and certainly constituting evidence of Frege's capacity for ingenius, intricate deduction, it has recently been realized that while Frege himself relied on Axiom V to obtain what is known as **Hume's Principle** (= HP), this reliance is avoidable. That from just HP we can deduce all of Peano Arithmetic (**PA**) (!) is a result Frege can be credited with showing; the result is known today as Frege's Theorem (= FT). Following the link just given will reward the reader with an understanding of HP, and how how to obtain **PA** from it.)

Deadline December 4, 2025 at 5:50 PM EST

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Courtesy of Gödel: Given certain limitative assumptions about "proof power," we can't prove that there isn't!

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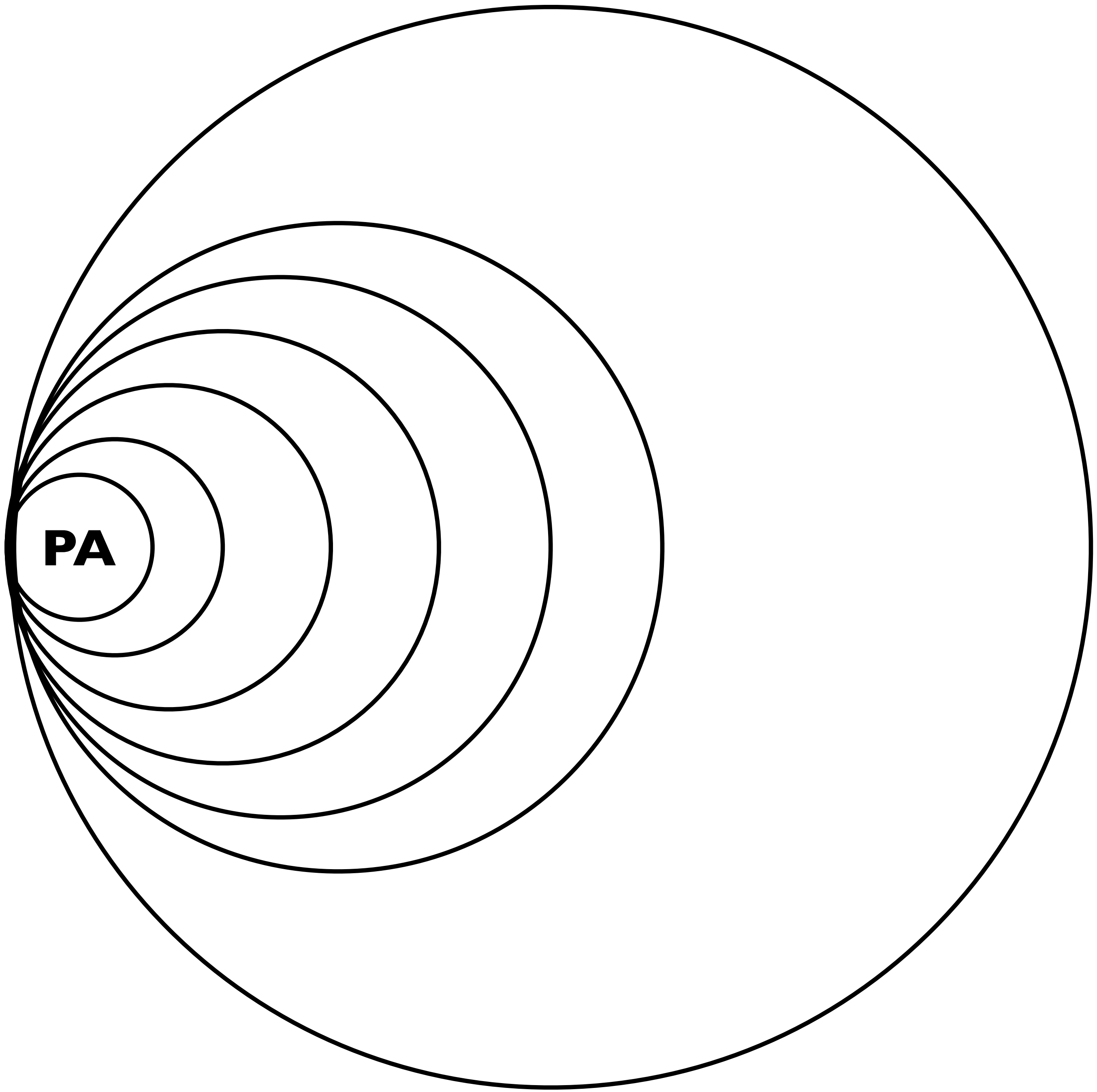

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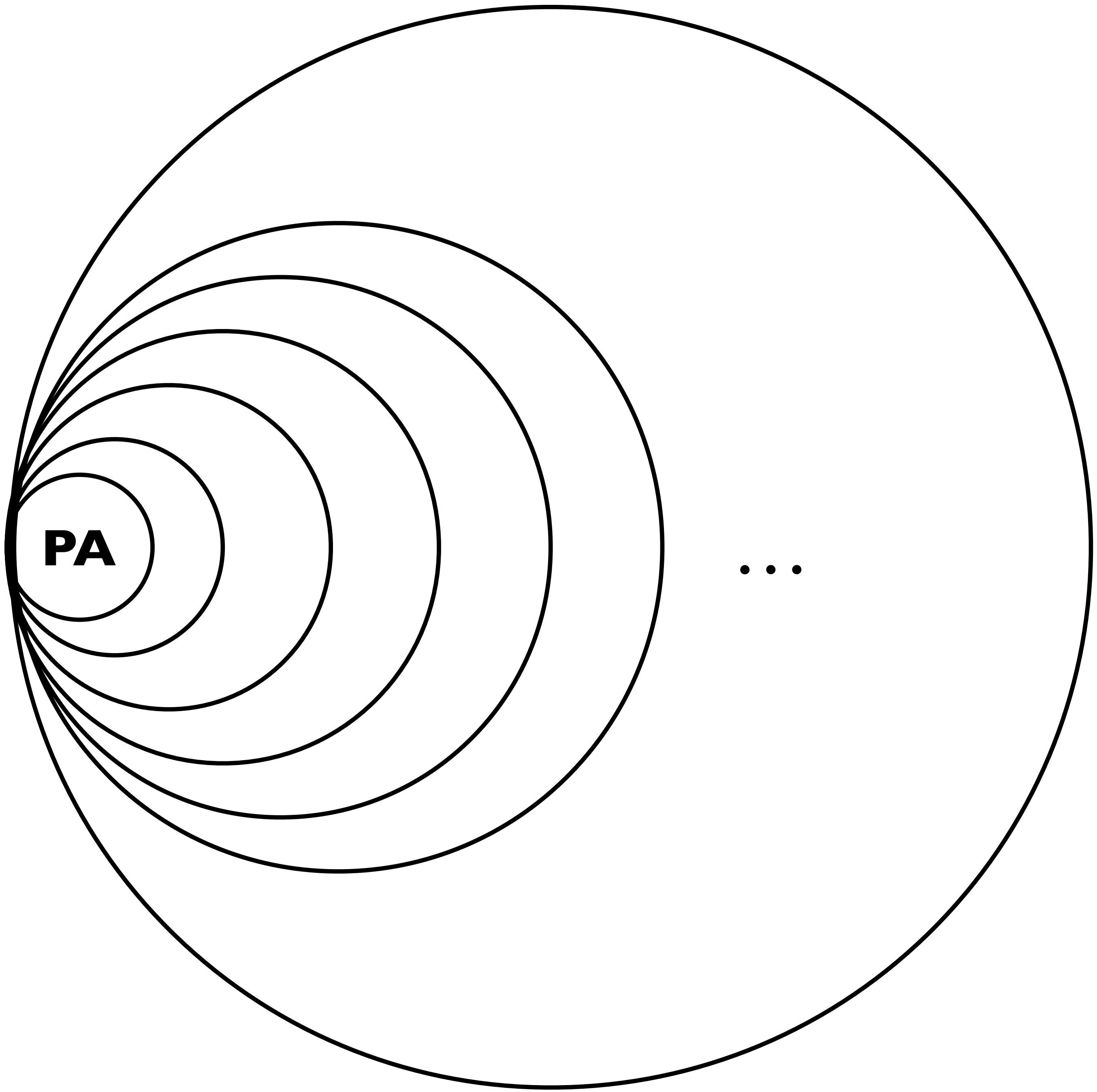

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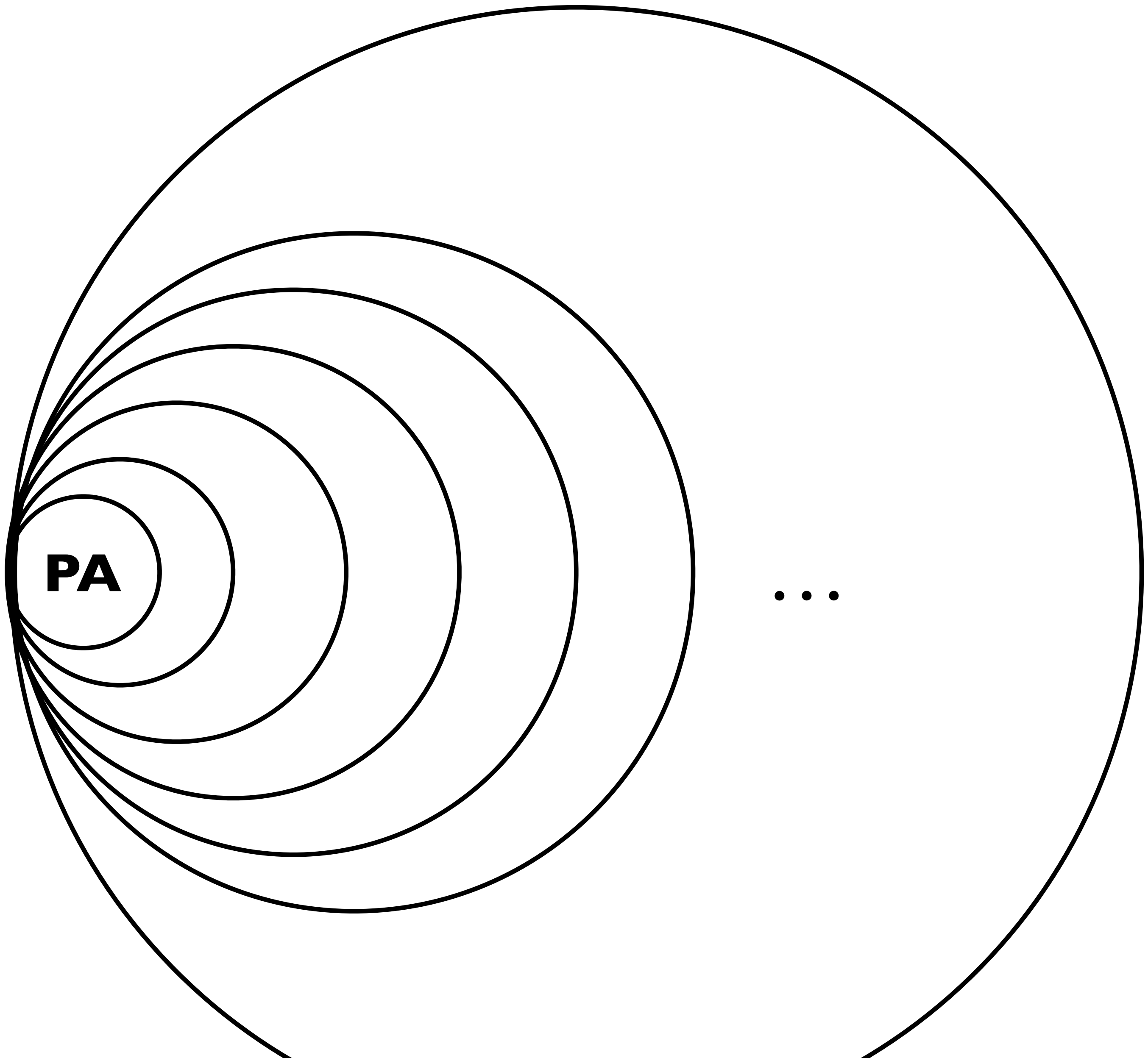

Each circle is a larger part of the formal sciences.

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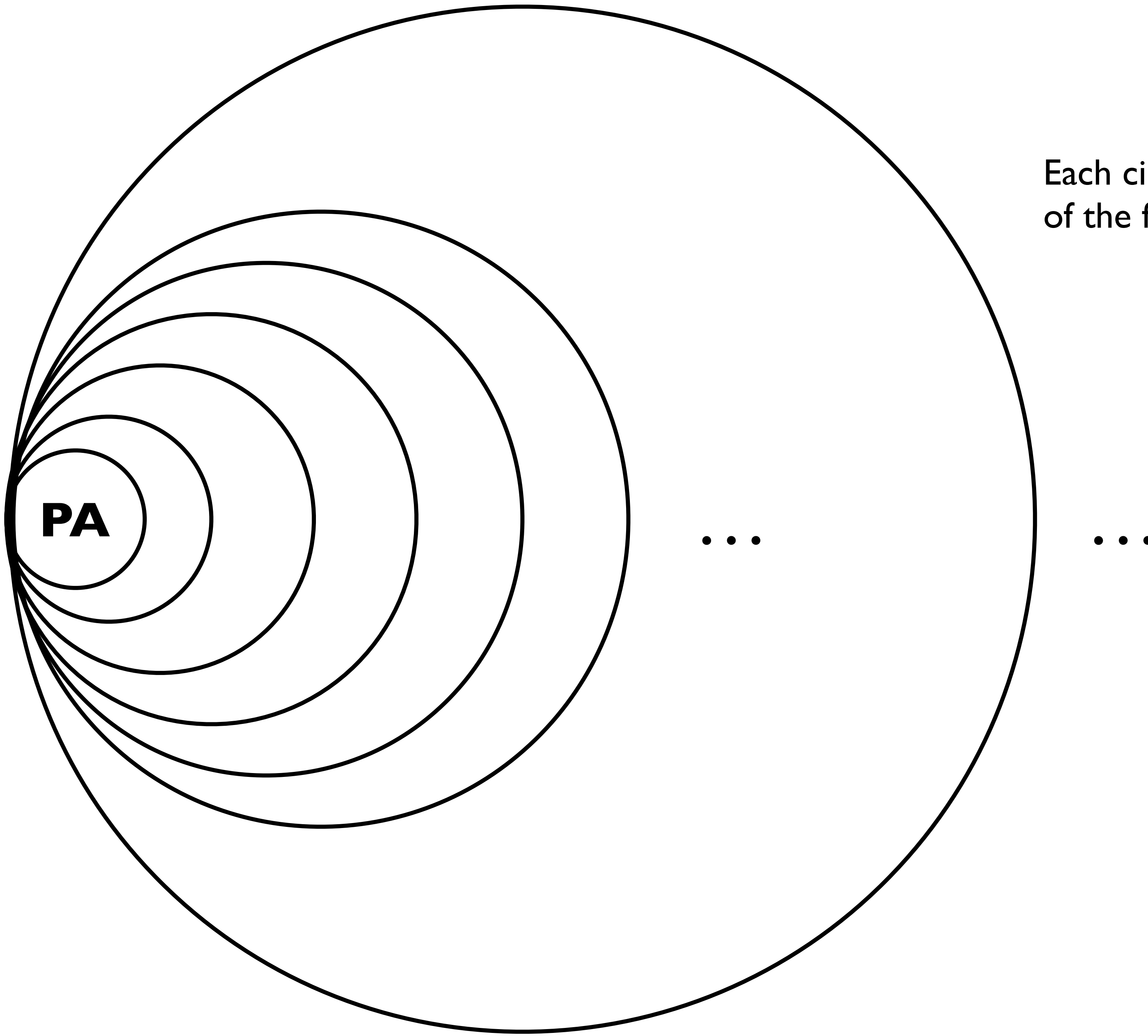

Each circle is a larger part of the formal sciences.

G2 as Slogan …

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"We can't use math to ascertain whether mathematics is consistent."

G2 as Slogan …

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“‘If we are restricted to certain kinds of formal reasoning, and feel we must have all of **PA** (math, engineering, etc.), we can’t ascertain whether mathematics is consistent.’’

G2 as Slogan …

"If we are restricted to certain kinds of formal reasoning, and feel we must have all of **PA** (math, engineering, etc.), we can't ascertain whether mathematics is consistent."

"Don't take Selmer's $ away because he's using full elementary number theory in his AI system because you're worried about inconsistency! — he has no alternative!"

$\Phi \supseteq$ PA?????? —Selmer check this.

Gödel's
<u>Second</u> Incompleteness Theorem

pocket theorem

We cannot prove from Φ, a set of formulae about basic arithmetic (possibly augmented with other math based on elementary number theory), that no contradiction can be deduced from Φ.

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Suppose $\mathbf{\Phi} \supset \mathbf{PA}$ is

(i) Con $\mathbf{\Phi}$;
(ii) membership in $\mathbf{\Phi}$ is algorithmically decidable; and
(iii) Cap $\mathbf{\Phi}$.

Then $\mathbf{\Phi} \nvdash \neg\pi(\hat{n}^{0=1})$.

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To prove G2, we shall once again allow ourselves …

The Fixed Point Theorem (FPT)

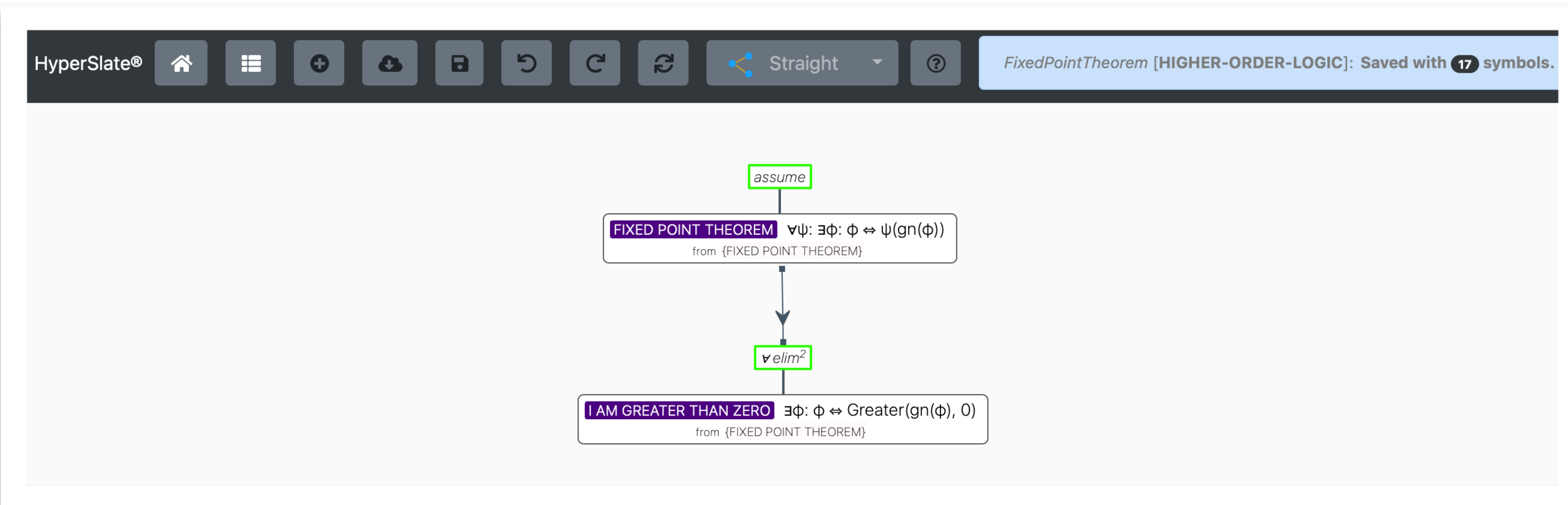
"The Self-Ascription Theorem" (GSAT)

Assume that Φ is a set of arithmetic sentences such that **Cap** Φ. Then for every arithmetic formula $\psi(x)$ with one free variable x, there is an arithmetic sentence ϕ s.t.

$$\Phi \vdash \phi \leftrightarrow \psi(\hat{n}^{\phi}).$$

We can intuitively understand ϕ to be saying:
"I have the property ascribed to me by the formula ψ."

FPT/SATL in HyperSlate®!



Ok; so let's do it … and let's see if you can see why Gödel declared G2 to be a direct "corollary" of G1, and didn't bother to prove it in his original paper …

Proof: Suppose that the antecedent (i)–(iii) of G2 holds. Suppose for *reductio* that

$\Phi \supseteq \mathbf{PA} \vdash \neg\pi(\hat{n}^{0=1})$

(Supposition: We can prove from **PA** that **PA** system is consistent.)

We have three ingredients which together suffice for getting to our goal. First, from Gödel's Self-Ascription Template Theorem (SATL) we can again directly obtain:

(G2.1) $\Phi \supseteq \mathbf{PA} \vdash \mathcal{G} \leftrightarrow \neg\pi(\hat{n}^{\mathcal{G}})$.

(= It's provable from anything that includes **PA** that there's a Gödel sentence $\mathcal{G}$ that says "I'm unprovable.")

"I'm not provable from Φ!"

Recall that we *already* proved (when? … from one half of G1!) that:

(G2.2) $\Phi \supseteq \mathbf{PA} \nvdash \mathcal{G}$.

(Gödel's sentence $\mathcal{G}$ can't be proved from anything that includes **PA**.)

Next, since the provable meta-logical conditional that

if Φ can prove $\mathcal{G}$, then Φ can prove that 0=1

is capturable, and by hypothesis the capturability of Φ is in force, we have:

(G2.3) $\Phi \supseteq \mathbf{PA} \vdash \pi(\hat{n}^{\mathcal{G}}) \rightarrow \pi(\hat{n}^{0=1})$

From our supposition and (G2.3) we can deduce by *modus tollens* that Φ proves $\neg\pi(\hat{n}^{\mathcal{G}})$, and from this and (G2.1) by biconditional elimination we have that $\Phi \vdash \mathcal{G}$, i.e., that Φ proves the Gödel sentence, which is impossible, as (G2.2) tells us — contradiction! **QED**

Med nok penger, kan logikk løse alle problemer.