

Gödel's ... First Incompleteness Theorem (GI)

Selmer Bringsjord*

Rensselaer AI & Reasoning (RAIR) Lab
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Lally School of Management & Technology
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Troy, New York 12180 USA

ILBAI
4/9/2026



*With Naveen Sundar G's introduction to Gödel numbering, and his depiction of GT growth, etc.

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Note: This is a version designed for those who have had at least one robust, proof-intensive university-level course in formal logic to the level of \mathcal{L}_2 .

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HyperSlate® session ...

Part 1: Selmer & ProofGrid

Part 2: Help? ...

Background Context ...

Could Artificial Intelligence Ever ... Match Personal Intelligence?

Gödelian Essays on Persons vs. Pretenders

by Selmer Bringsjord et al.

- Introduction (“The Wager”)
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
- The Continuum-Hypothesis Theorem
- The Time-Travel Theorem
- Gödel’s “God Theorem”
- Could a Finite Machine Match Gödel’s Greatness?



STOP & REVIEW IF NEEDED!

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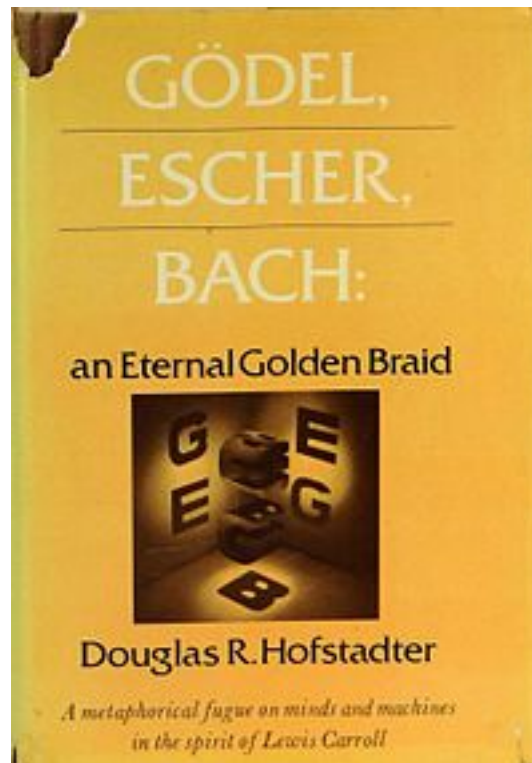
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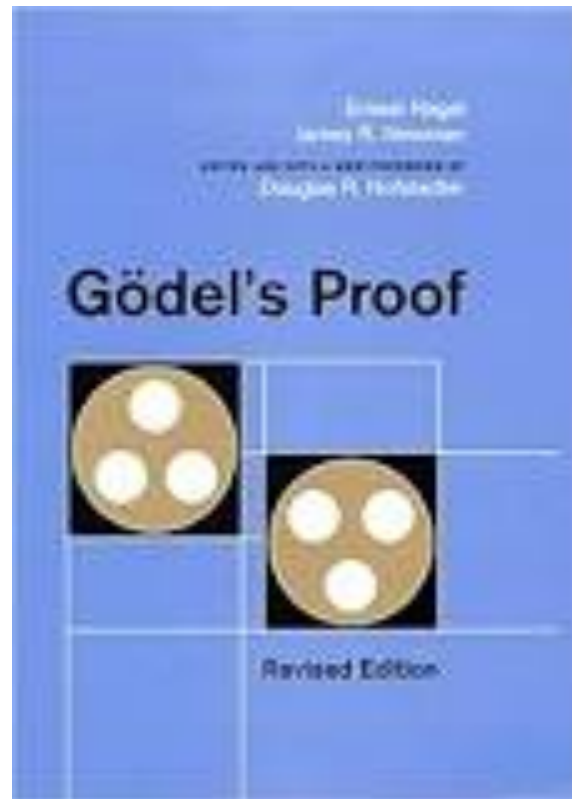
A corollary of the First Incompleteness Theorem: *We cannot prove (in classical mathematics) that mathematics is consistent.*

Deficient; Beware

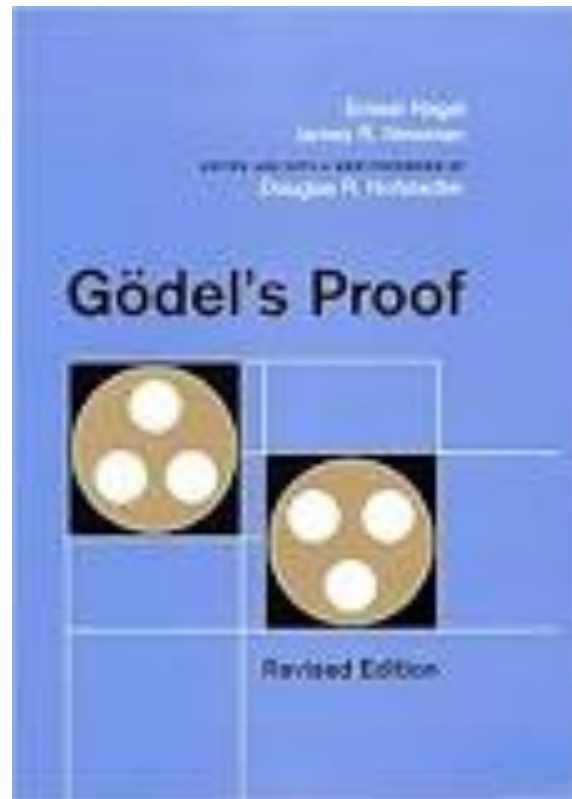
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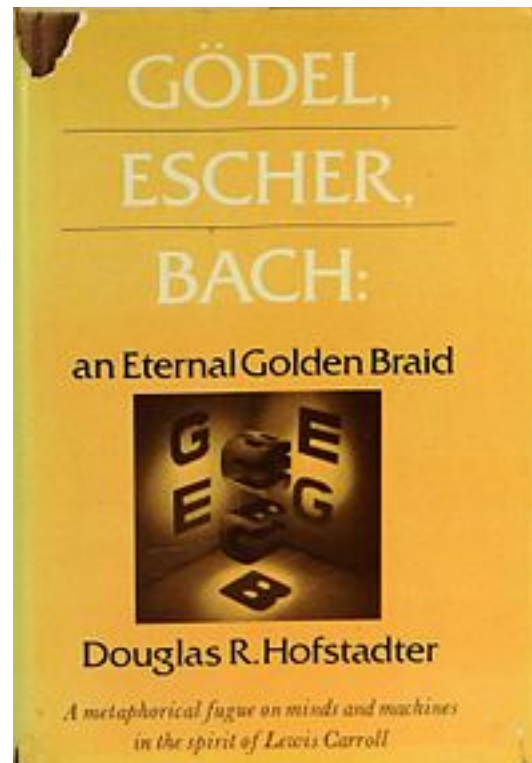
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Some Timeline Points

1978 Princeton NJ USA.



1940 Back to USA, for good.

1936 Schlick murdered; Austria annexed

1933 Hitler comes to power.

1930 Announces (First) *Incompleteness Theorem*

1929 Doctoral Dissertation: Proof of Completeness Theorem

Undergrad in seminar by Schlick

1923 Vienna

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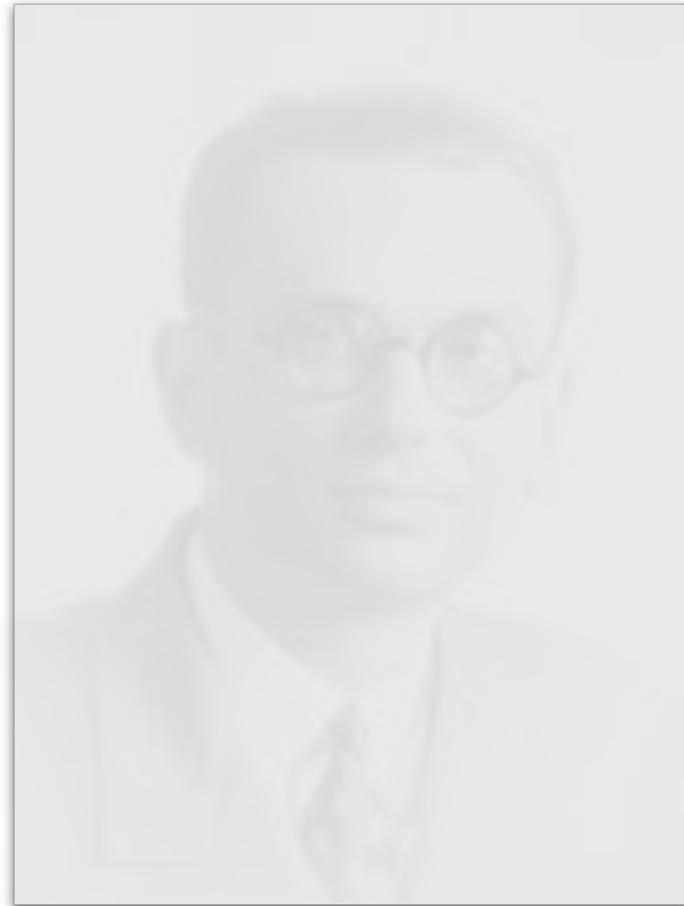
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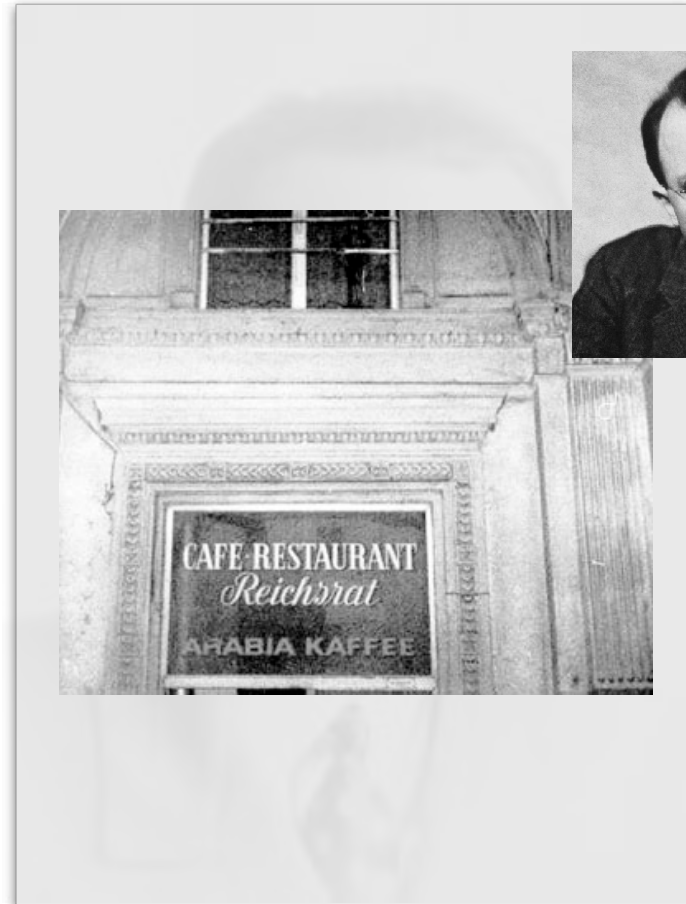
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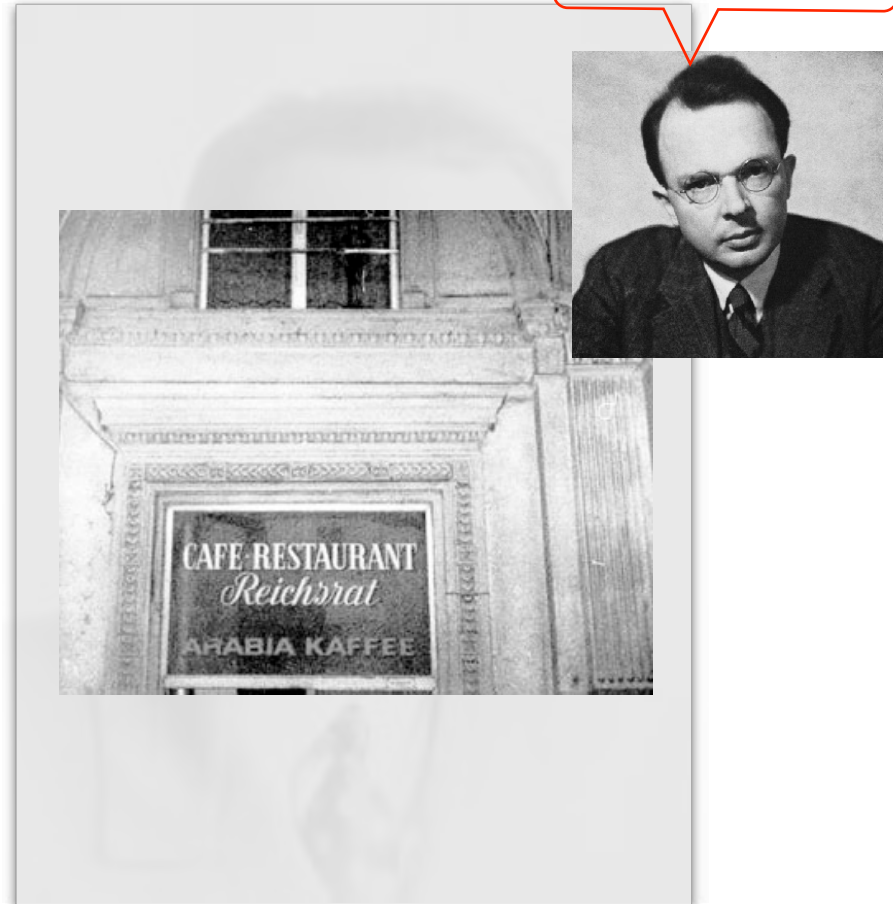
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“Well, uh, hmm, ...”



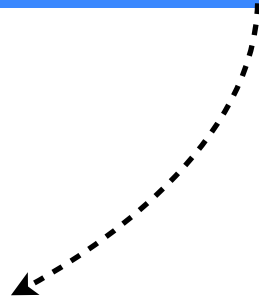
The “Liar Tree”

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The Liar Paradox

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Pure Proof-Theoretic Route

```
graph TD; A[The Liar Paradox] -.-> B(Pure Proof-Theoretic Route)
```

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graph TD; A[The Liar Paradox] -.-> B(Pure Proof-Theoretic Route); A -.-> C[ ];
```

Pure Proof-Theoretic Route

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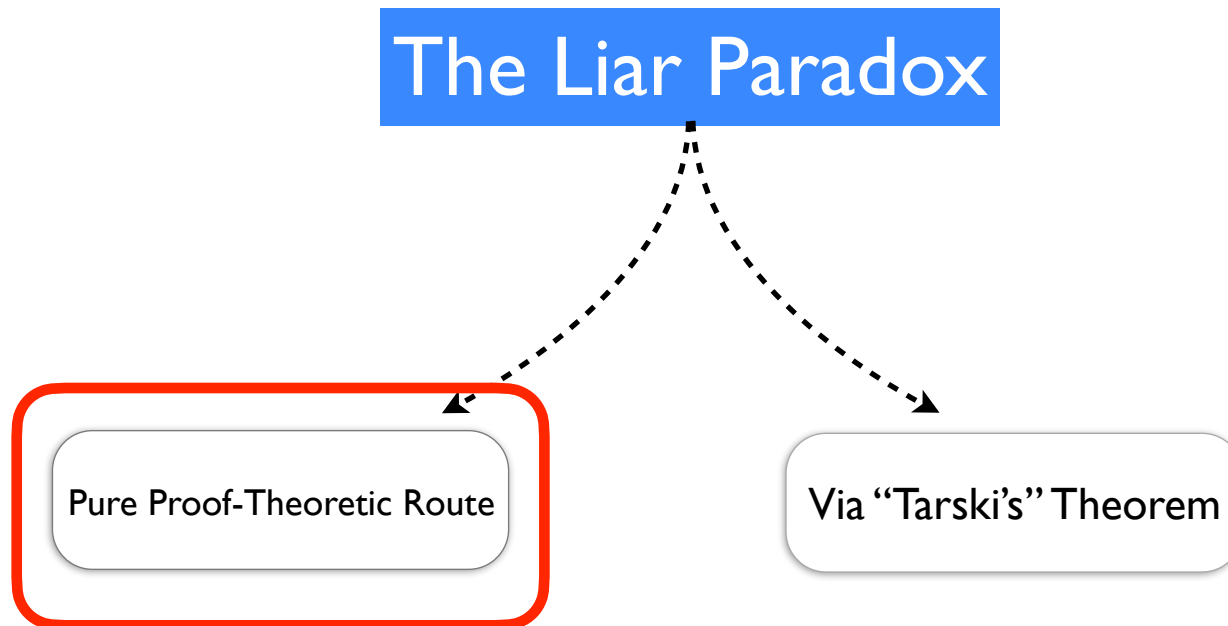
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graph TD; A[The Liar Paradox] -.-> B(Pure Proof-Theoretic Route); A -.-> C(Via “Tarski’s” Theorem)
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Pure Proof-Theoretic Route

Via “Tarski’s” Theorem

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The “Liar Tree”

The Liar Paradox

Pure Proof-Theoretic Route

Via “Tarski’s” Theorem

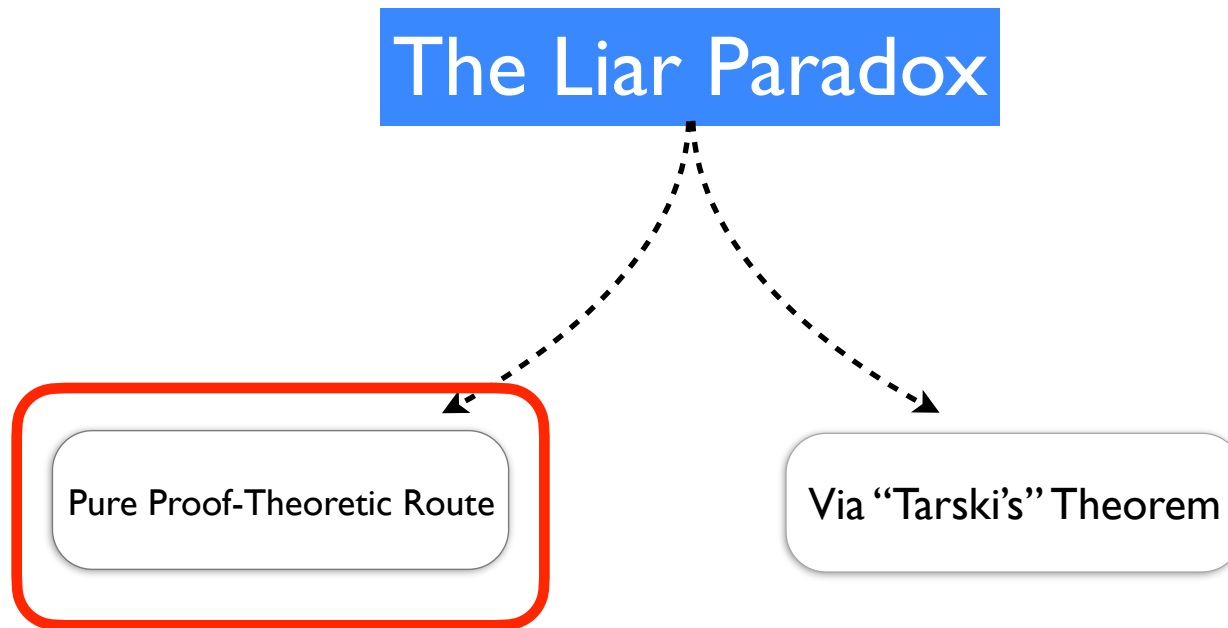


Paul Erdős



“The Book”

The “Liar Tree”



Ergo, step one: *What is LP?*



Paul Erdős



“The Book”

“The (Economical) Liar”

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L: This sentence is false.

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Suppose that $T(\mathbf{L})$; then $\neg T(\mathbf{L})$.

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Hence: $T(\mathbf{L})$ iff (i.e., if & only if) $\neg T(\mathbf{L})$.

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Contradiction!

The “Gödelian” Liar (from me)

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\bar{P} : This sentence is unprovable.

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Suppose that \bar{P} is true. Then we can immediately deduce that \bar{P} is provable, because here is a proof: $\bar{P} \rightarrow \bar{P}$ is an easy theorem, and from it and our supposition we deduce \bar{P} by *modus ponens*. But since what \bar{P} says is that it's unprovable, we have deduced that \bar{P} is false under our initial supposition.

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Suppose on the other hand that \bar{P} is false. Then we can immediately deduce that \bar{P} is unprovable: Suppose for *reductio* that \bar{P} is provable; then \bar{P} holds as a result of some proof, but what \bar{P} says is that it's unprovable; and so we have contradiction. But since what \bar{P} says is that it's unprovable, and we have just proved that under our supposition, we arrive at the conclusion that \bar{P} is true.

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All of this is fishy; but
Gödel transformed it into
utterly precise, impactful,
indisputable reasoning ...

PA (Peano Arithmetic):

$$\text{A1} \quad \forall x(0 \neq s(x))$$

$$\text{A2} \quad \forall x \forall y (s(x) = s(y) \rightarrow x = y)$$

$$\text{A3} \quad \forall x (x \neq 0 \rightarrow \exists y (x = s(y)))$$

$$\text{A4} \quad \forall x (x + 0 = x)$$

$$\text{A5} \quad \forall x \forall y (x + s(y) = s(x + y))$$

$$\text{A6} \quad \forall x (x \times 0 = 0)$$

$$\text{A7} \quad \forall x \forall y (x \times s(y) = (x \times y) + x)$$

And, every sentence that is the universal closure of an instance of

$$([\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x)))] \rightarrow \forall x \phi(x))$$

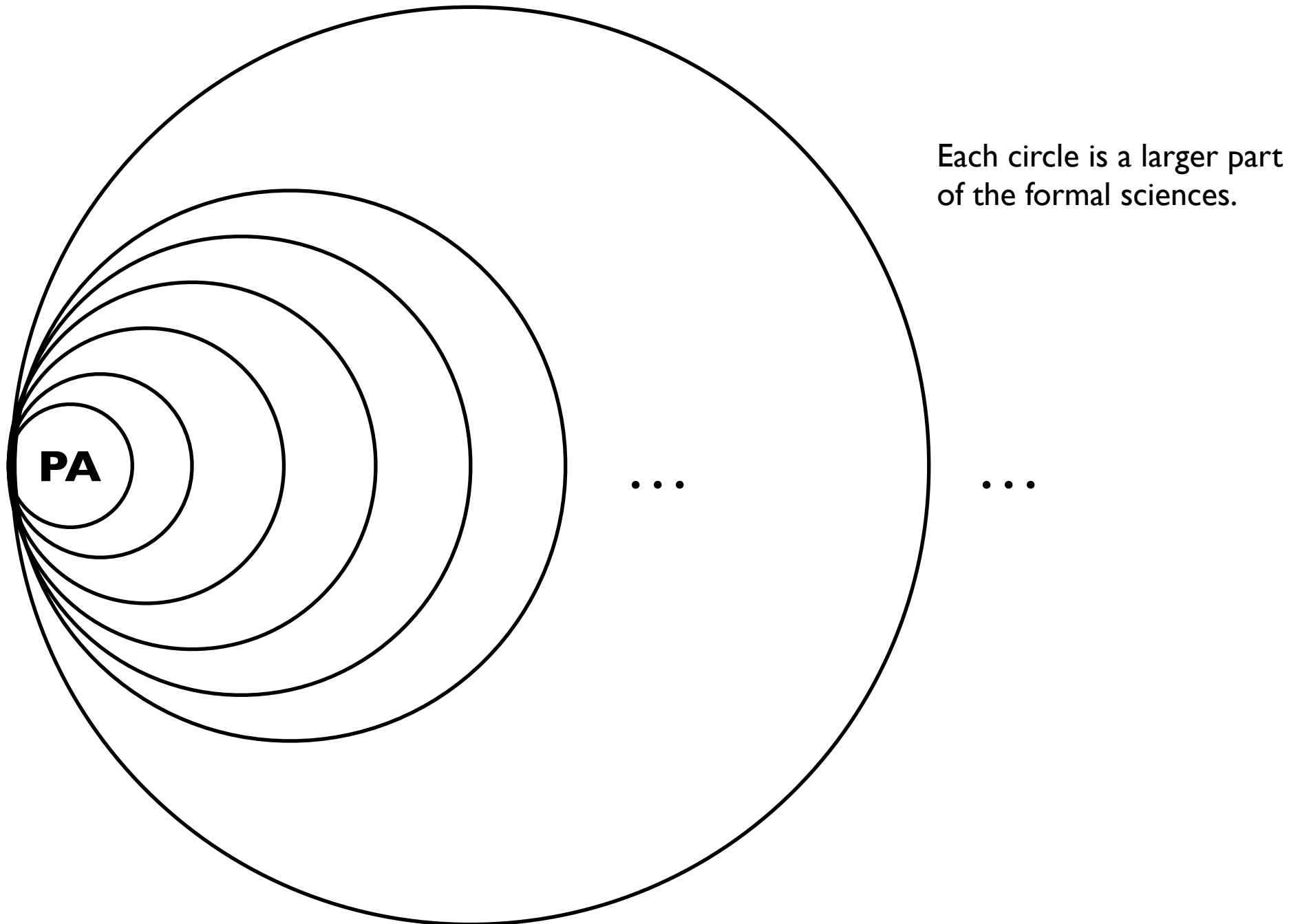
where $\phi(x)$ is open wff with variable x , and perhaps others, free.

PA2 (Second-Order Peano Arithmetic)

<https://rpi.logicamodernapproach.com/hyperslate/public/Selmer.Bringsjord@gmail.com/PA2>

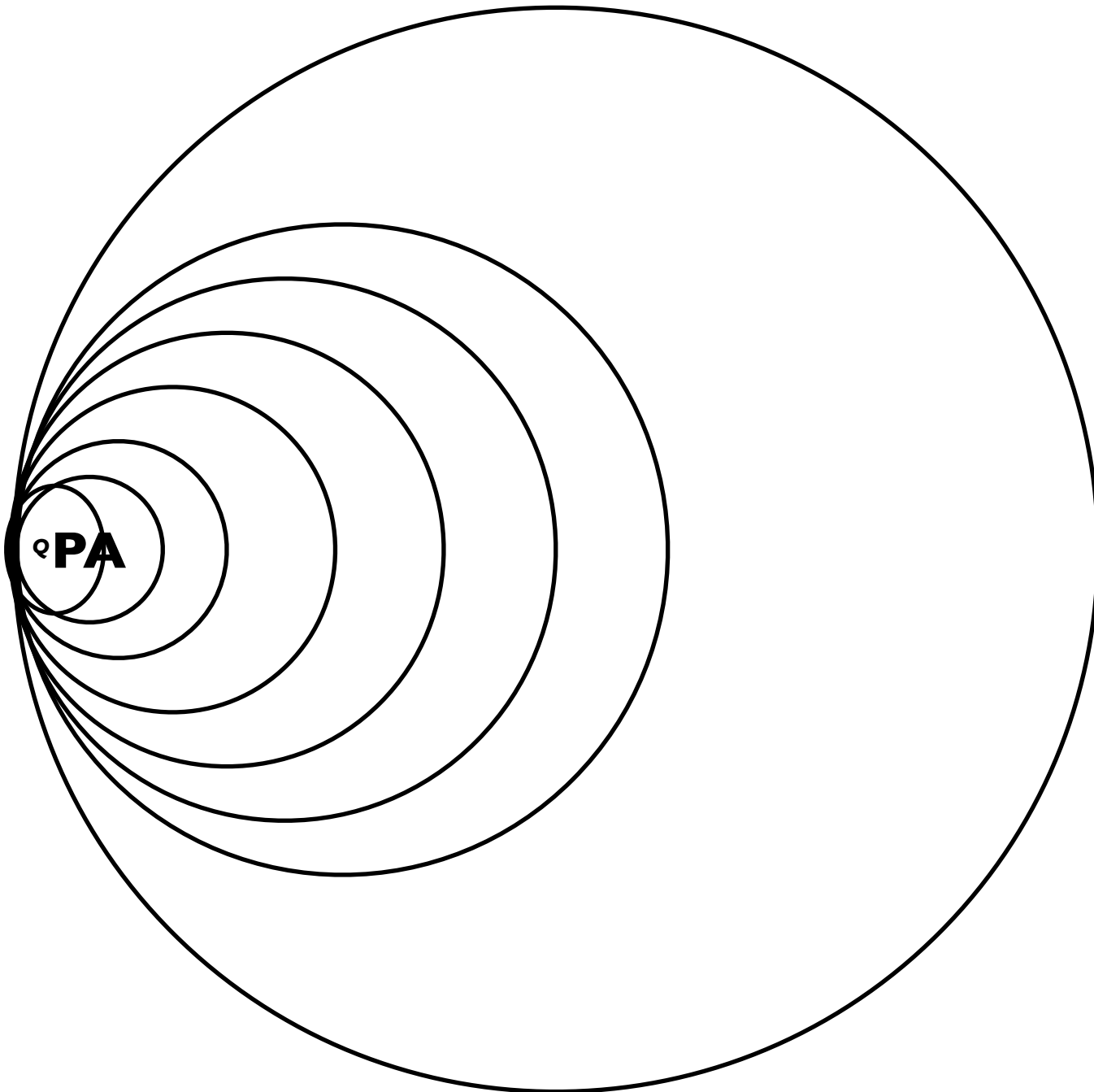
Arithmetic is Part of All Things Sci/Eng/Tech!

but courtesy of Gödel: We can't even prove all truths of arithmetic!



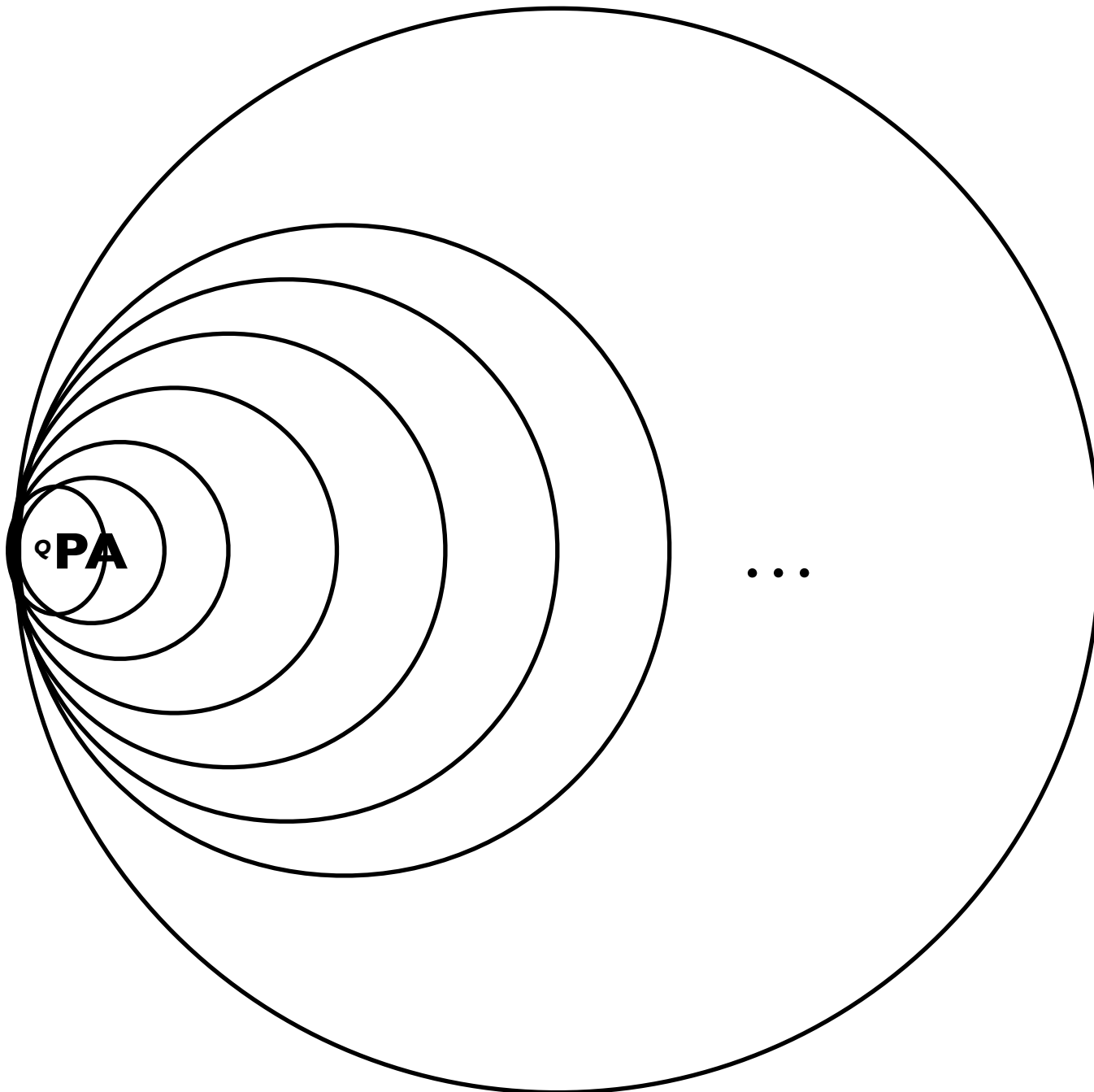
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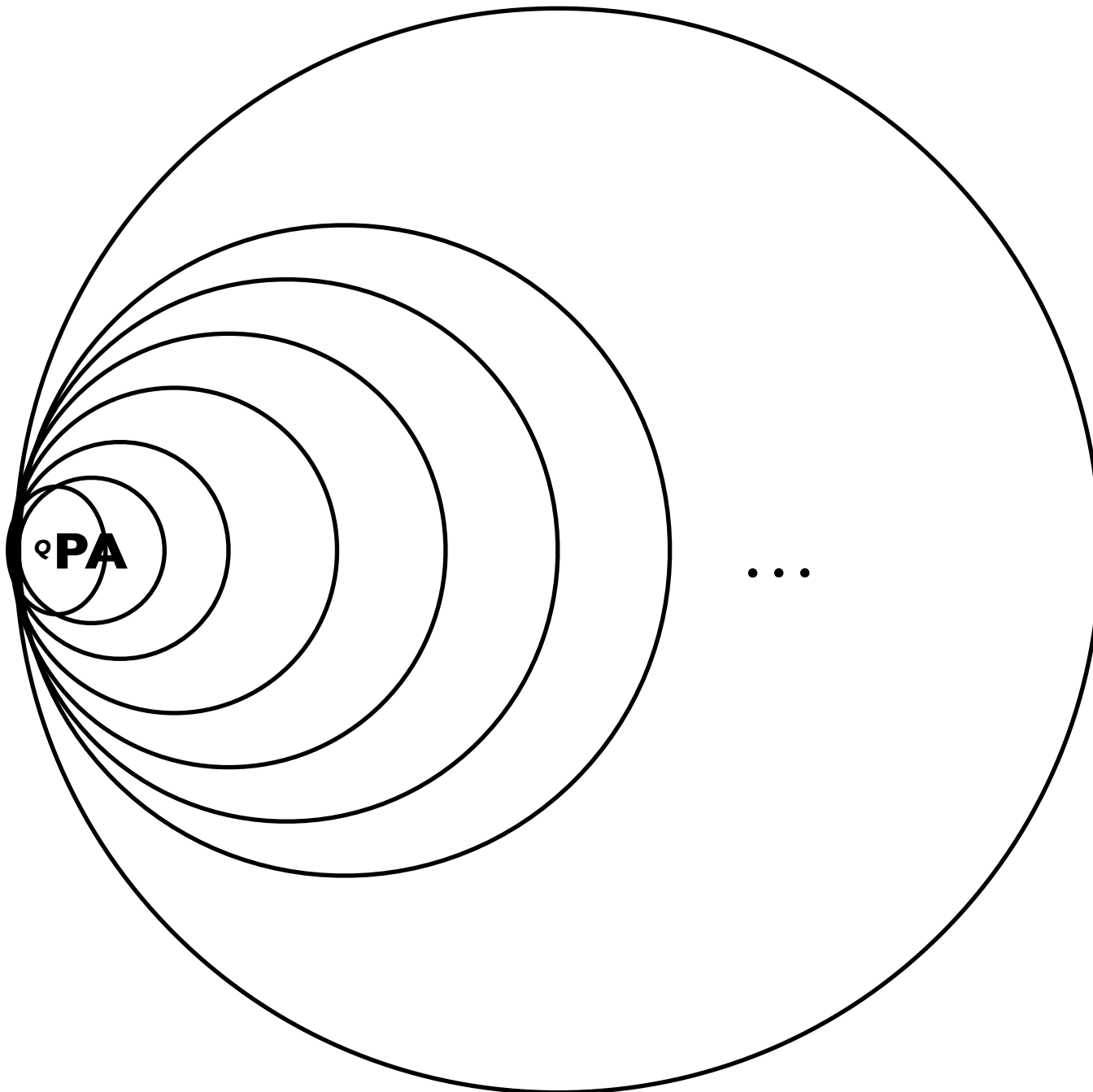
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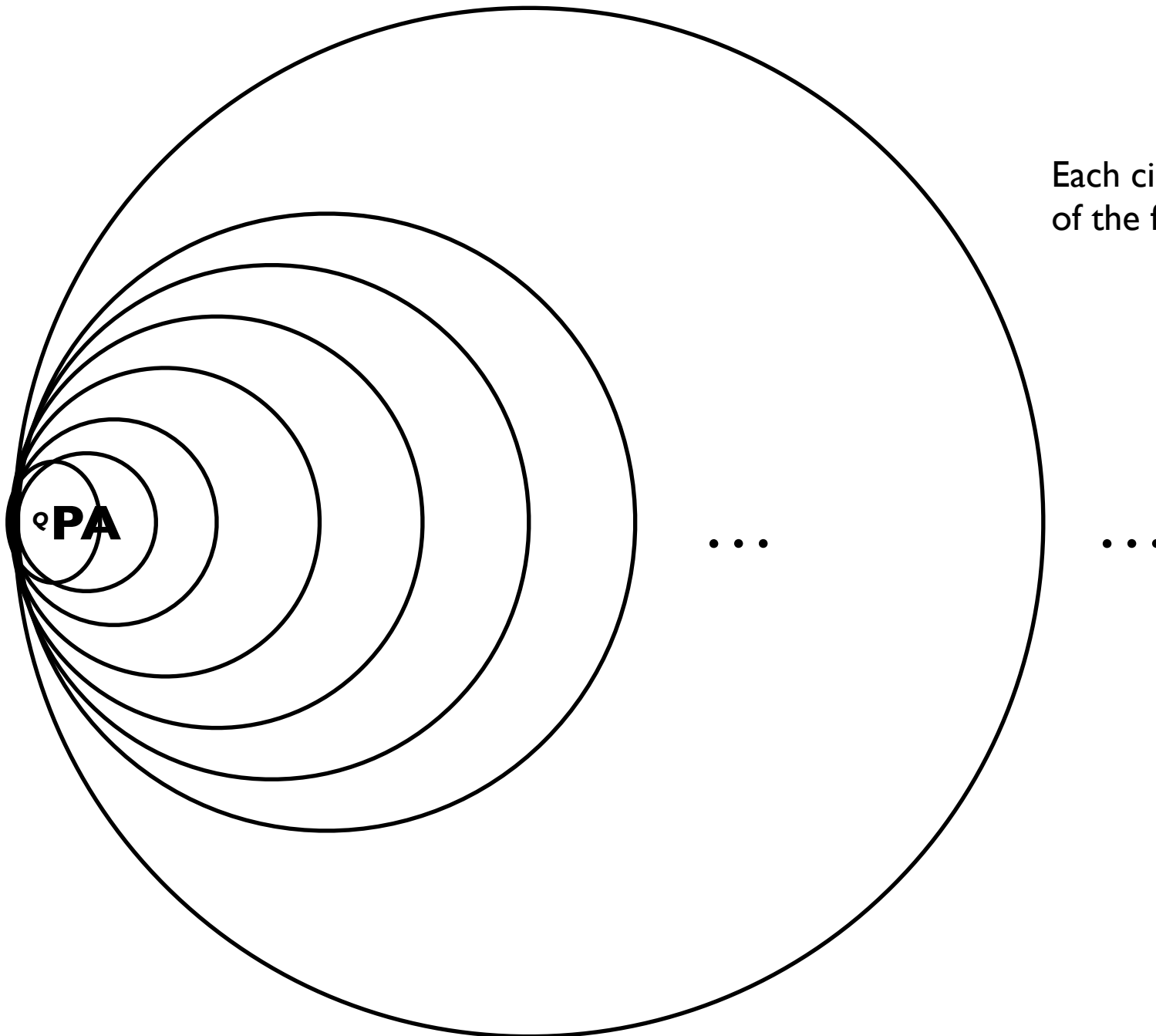
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Each circle is a larger part
of the formal sciences.

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Gödel Numbering

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Problem: How do we enable a formula to refer to other formulae and itself (and also other objects like proofs, terms etc.), in a perfectly consistent way?

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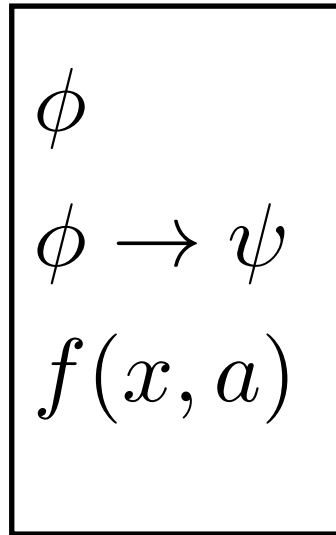
Object-level objects
in the language of \mathcal{L}_1

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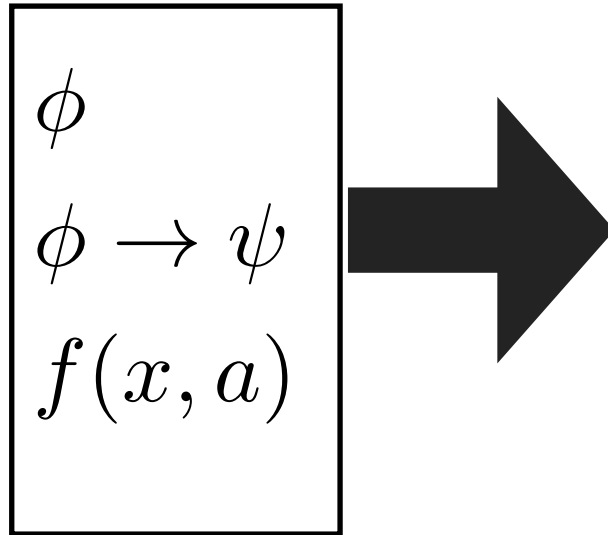
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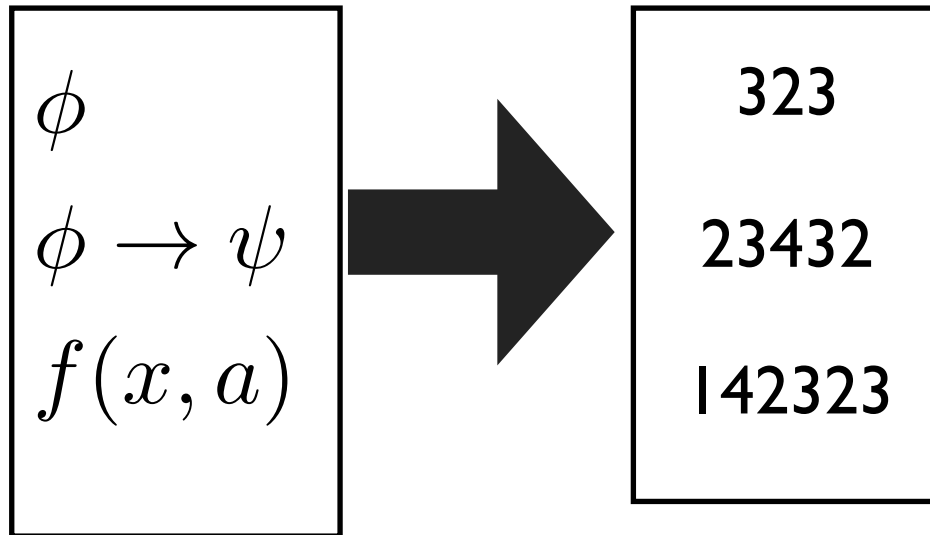
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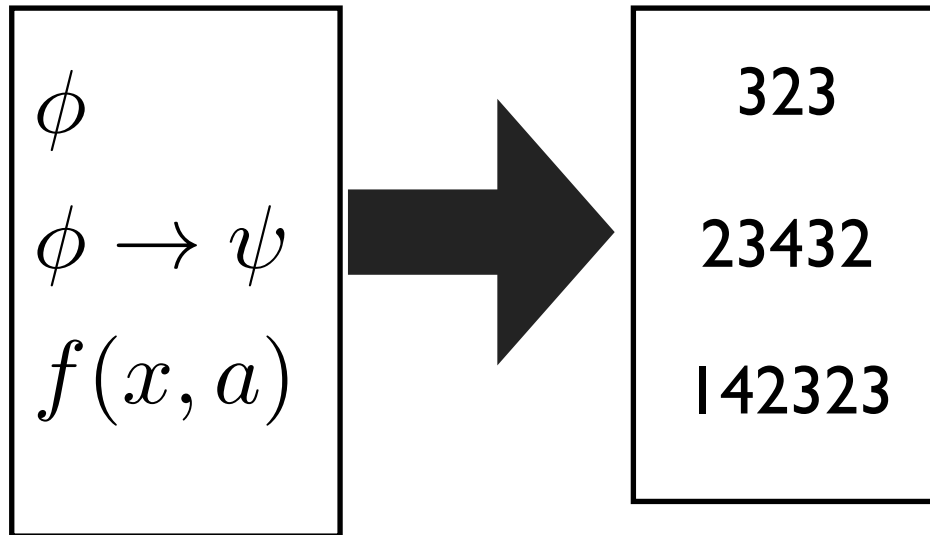
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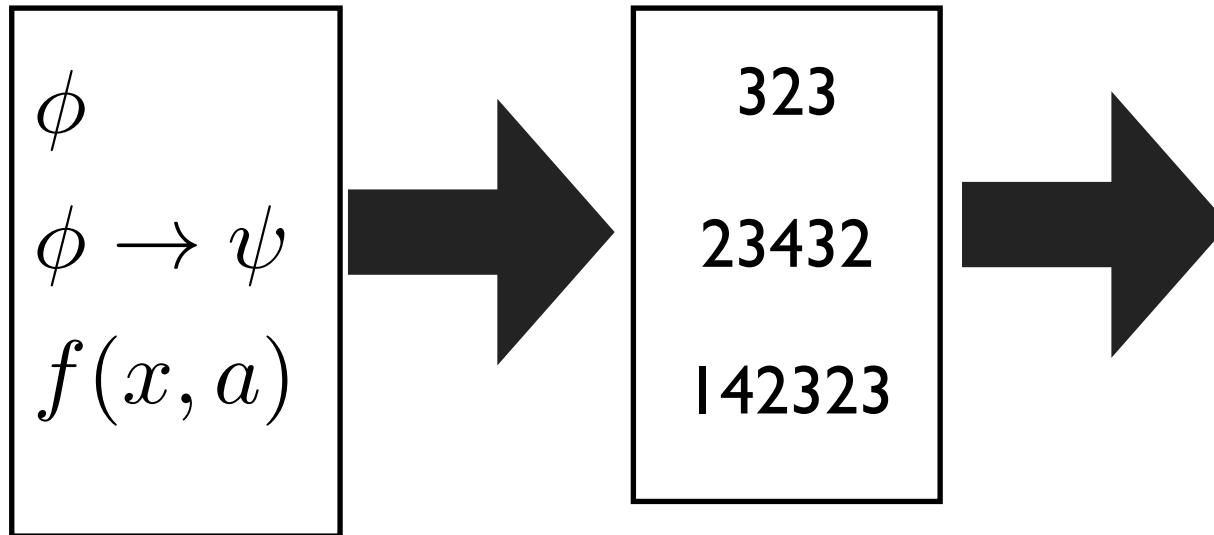
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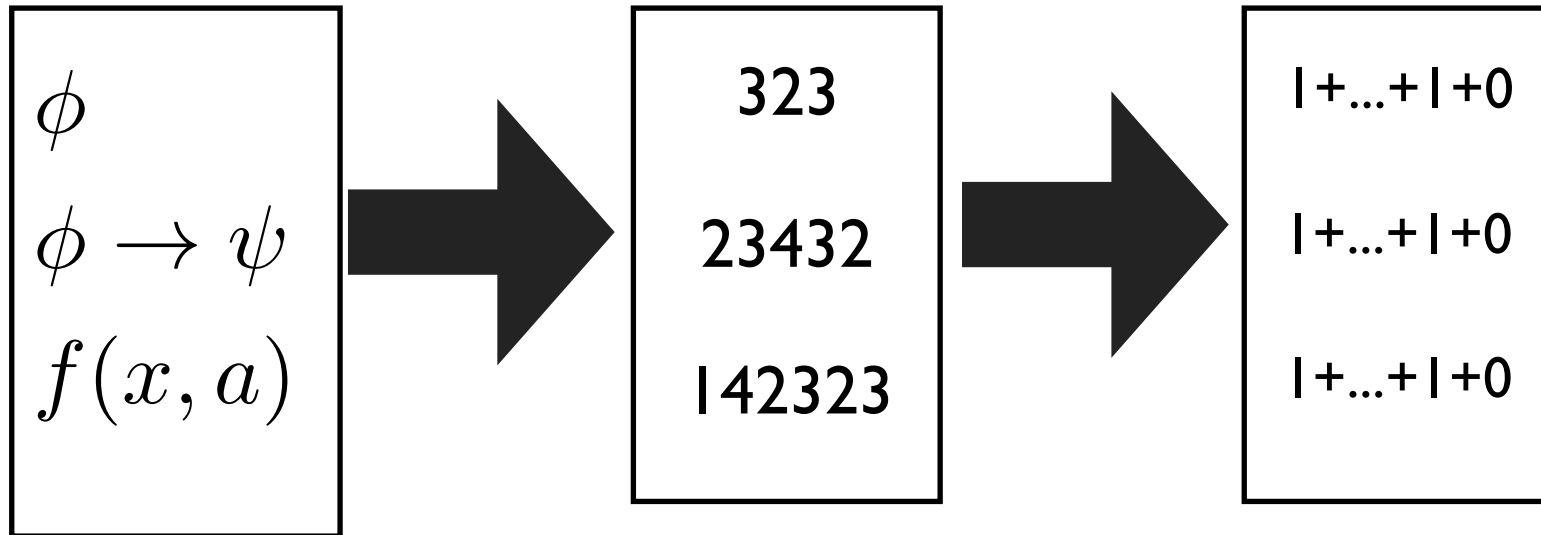
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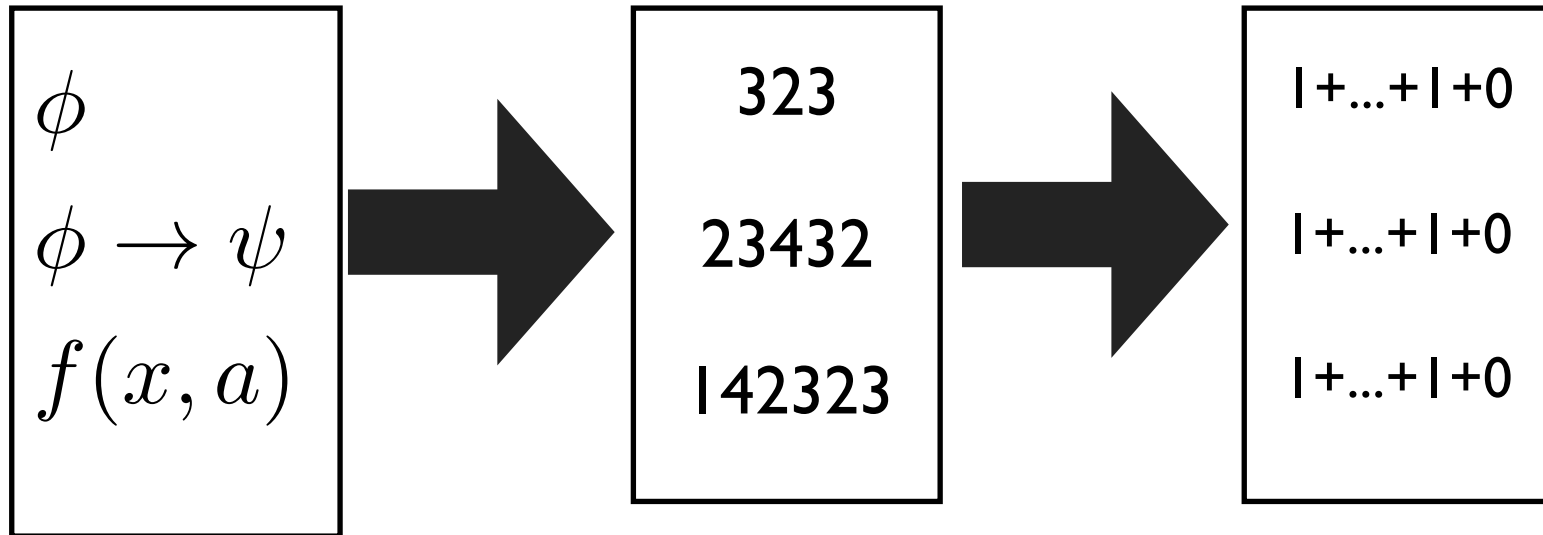
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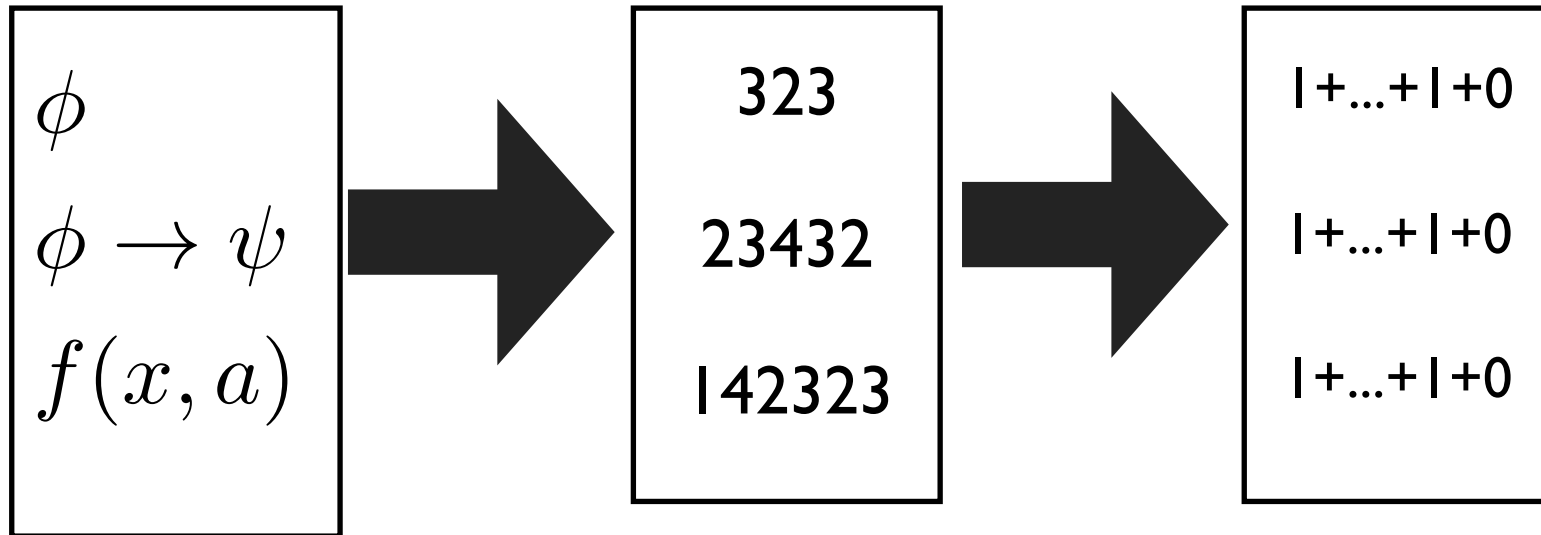
Gödel numeral

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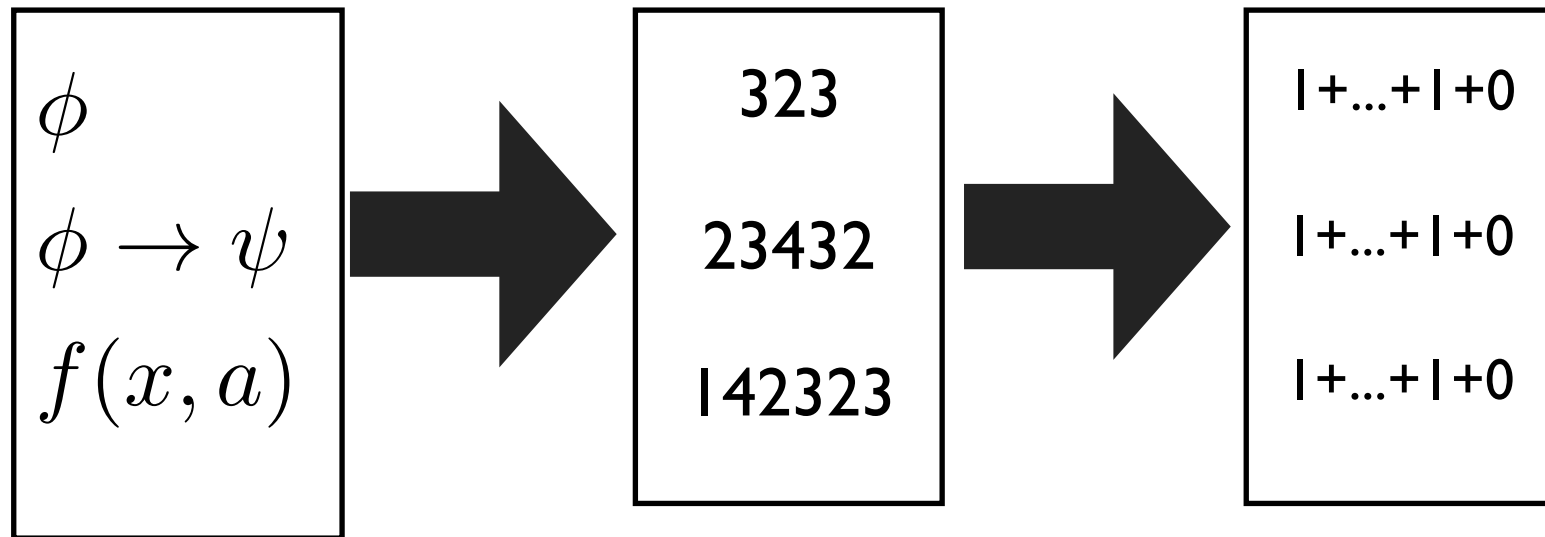
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n^ϕ

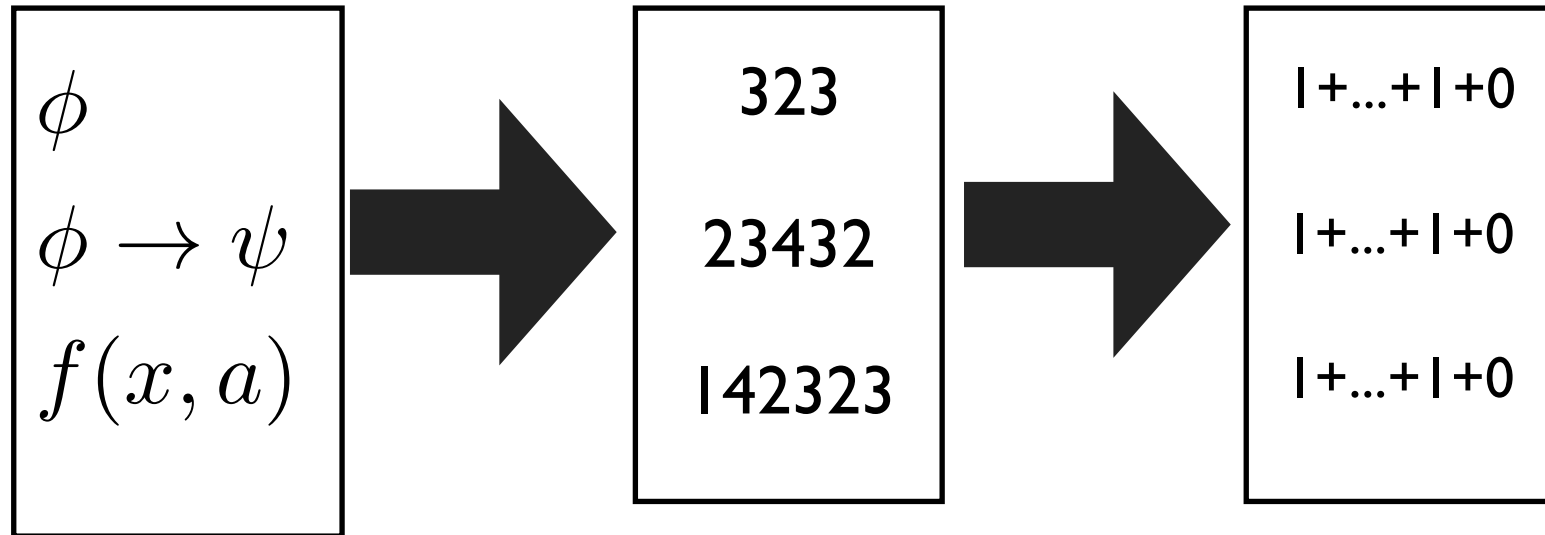
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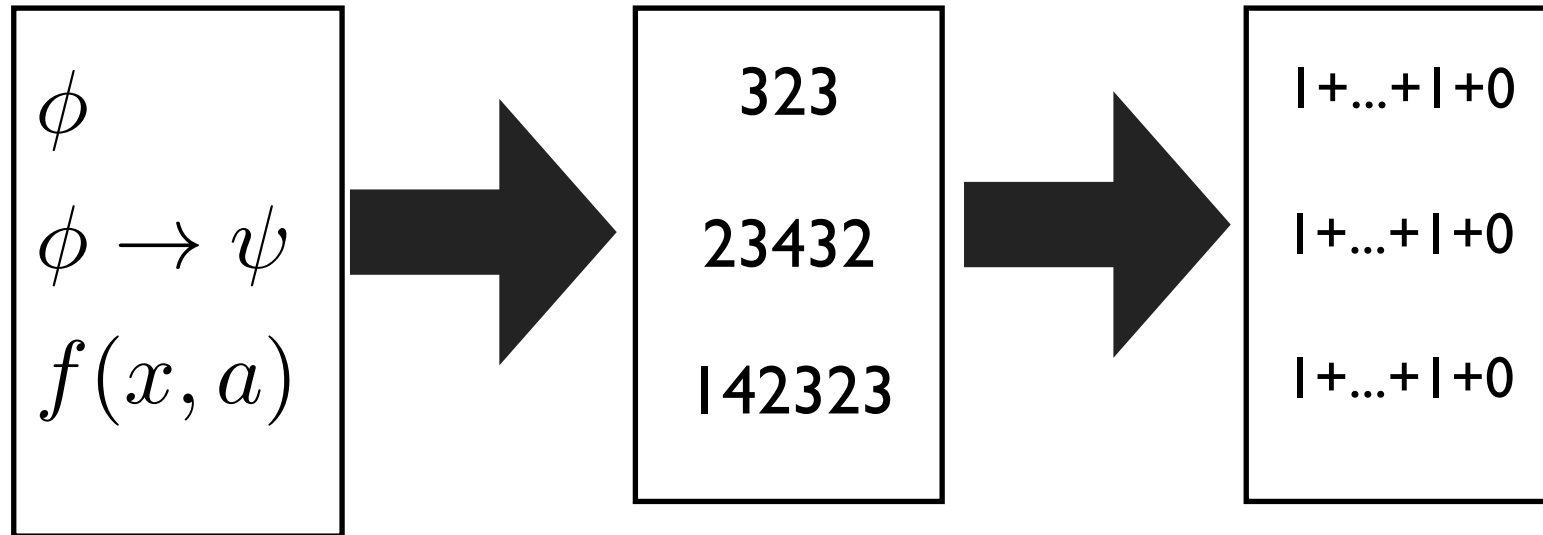
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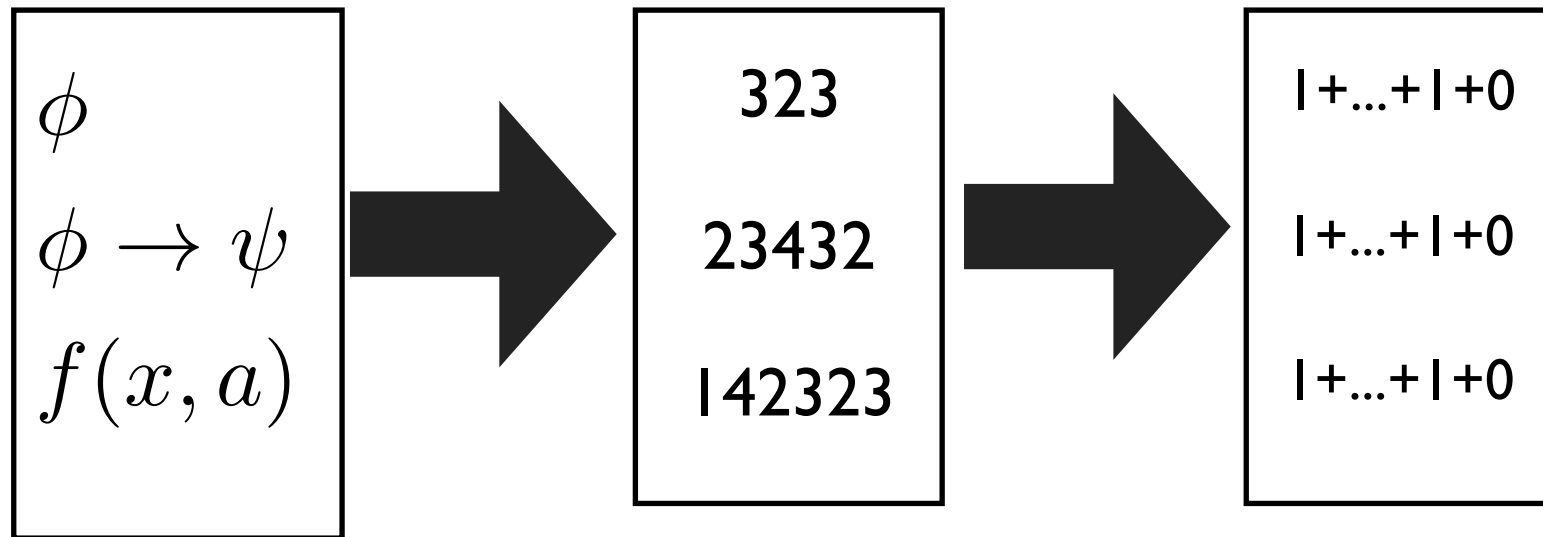
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S will sometimes conflate.

n^ϕ

\hat{n}^ϕ

(or just “ ϕ ”)

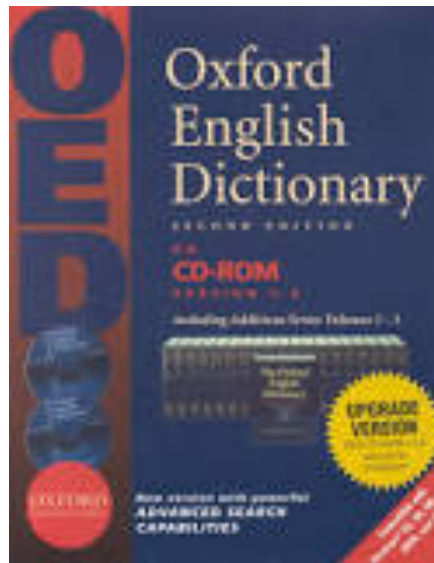
Gödel Numbering, the Easy Way

Gödel Numbering, the Easy Way

Just realize that every entry in a dictionary is named by a number n , and by the same basic lexicographic ordering, every computer program, formula, etc. is named by a number m in a lexicographic ordering going from 1, to 2, to ...

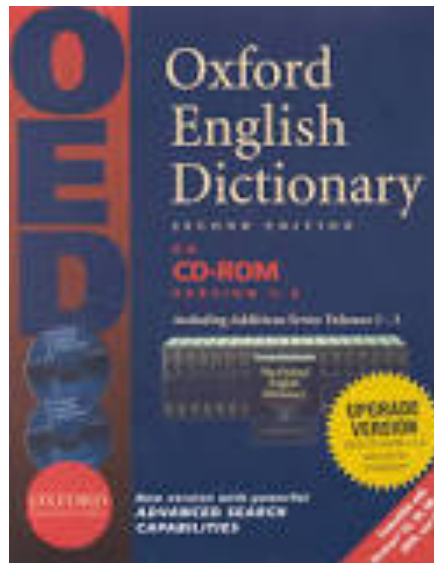
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So, gimcrack is named by some positive integer k . Hence, I can just refer to this word as “ k ” Or in the notation I prefer: k^{gimcrack} .

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Or, every syntactically valid computer program in Clojure that you will ever write can be uniquely denoted by some number m in the lexicographic ordering of all syntactically valid such programs. So your program π can just be coded as a numeral m^π in a formal language that captures arithmetic (i.e., an *arithmetic language*).

Gödel's First Incompleteness Theorem

Let Φ be a set of arithmetic sentences that is

- (i) consistent (i.e. no contradiction $\phi \wedge \neg\phi$ can be deduced from Φ);
- (ii) s.t. an algorithm is available to decide whether or not a given string u is a member of Φ ; and
- (iii) sufficiently expressive to logicize all of the operations of a standard computing machine (e.g. a Turing machine, register machine, KU machine, etc.).

Then there is an “undecidable” arithmetic sentence \mathcal{G} from Gödel that can't be proved from Φ , nor can the negation of this sentence (i.e. $\neg\mathcal{G}$) be proved from Φ !

Alas, that's painfully verbose.

Gödel's First Incompleteness Theorem

pocket theorem

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Suppose $\Phi \supseteq \mathbf{PA}$ (= Φ contains \mathbf{PA}) is

- (i) Φ is *consistent*: no contradiction $\phi \wedge \neg\phi$ can be proved from Φ (Con Φ);
- (ii) Membership of strings in Φ is *algorithmically decidable*; and
- (iii) Φ has *capturability*: it can capture functions and relations over \mathbb{N} (Cap Φ).

Then there is a sentence \mathcal{G} s.t. $\Phi \not\vdash \mathcal{G}$ and $\Phi \not\vdash \neg\mathcal{G}$. (\mathcal{G} is an “undecidable” sentence.)

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To prove G I, we shall
allow ourselves ...

The Fixed Point Theorem (FPT)

“The Self-Ascription Theorem” (GSAT)

Assume that Φ is a set of arithmetic sentences such that $\text{Cap } \Phi$. Then for every arithmetic formula $\psi(x)$ with one free variable x , there is an arithmetic sentence ϕ s.t.

$$\Phi \vdash \phi \leftrightarrow \psi(\hat{n}^\phi).$$

We can intuitively understand ϕ to be saying:

“I have the property ascribed to me by the formula ψ .”

“I *heard* there was no free lunch!”

[W]e “would hope that such a deep theorem would have an insightful proof. No such luck. I am going to write down a sentence ... and verify that it works. What I won't do is give you a satisfactory explanation for why I write down the particular formula that I do. I write down the formula because Gödel wrote down the formula, and Gödel wrote down the formula because, when he played the logic game he was able to see seven or eight moves ahead, whereas you and I are only able to see one or two moves ahead. I don't know anyone who thinks he has a fully satisfying understanding of why the Self-referential Lemma [= FPT] works. It has a rabbit-out-of-a-hat quality for everyone.”

—V. McGee, 2002; as quoted in (Salehi 2020)

The Fixed Point Theorem (FPT)

“The Self-Ascription Template Theorem”

Assume that Φ is a set of arithmetic sentences such that $\text{Cap } \Phi$. Then for every arithmetic formula $\psi(x)$ with one free variable x , there is an arithmetic sentence ϕ s.t.

$$\Phi \vdash \phi \leftrightarrow \psi(\hat{n}^\phi).$$

We can intuitively understand ϕ to be saying:

“I have the property ascribed to me by the formula ψ .”

Ok; so let's do it ...

Proof: Let Φ be a set of arithmetic sentences that's a superset of **PA**, and suppose the antecedent of $\boxed{\text{GI}}$ holds, i.e. (i)–(iii) hold. We must show that there exists an arithmetic sentence s.t. neither it nor its negation can be proved from Φ . In homage to Gödel, we label this sentence ' \mathcal{G} .'

For *reductio*, assume the opposite: that for every arithmetic sentence ϕ , either $\Phi \vdash \phi$ or $\Phi \vdash \neg\phi$.

Next, let's use the formula $\pi(v)$ to capture provability in Φ , where the variable v is a placeholder ready to receive the Gödel numeral \hat{n}^ϕ of a formula ϕ that's proved. We thus have:

$$(GI.1) \quad \Phi \vdash \pi(\hat{n}^\phi) \text{ iff } \Phi \vdash \phi.$$

Now we bring on stage Gödel's self-ascribing sentence \mathcal{G} , via his Self-Ascription Template Theorem (SATL); specifically, we have:

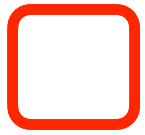
$$(GI.2) \quad \Phi \vdash \mathcal{G} \leftrightarrow \neg\pi(\hat{n}^\mathcal{G}).$$

"I'm not provable from Φ !"

We have two cases to consider, just like what we did in the original Liar Paradox. The first is that \mathcal{G} is provable from Φ ; suppose this holds. Then from (1) right-to-left, and *modus ponens* = conditional elim., with left-to-right (by biconditional elimination) on (3) — contradiction!

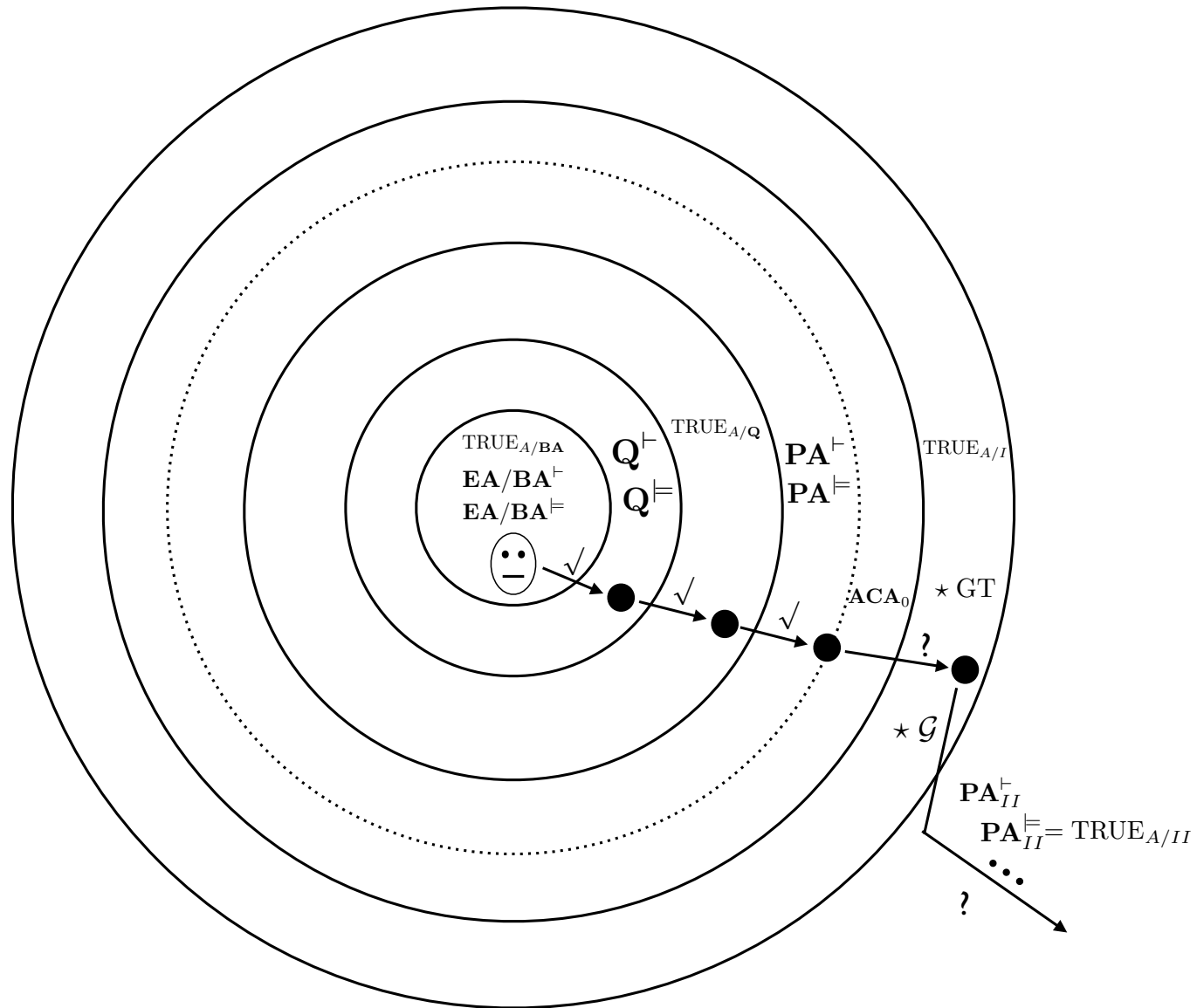
How about the second case, viz. that $\Phi \vdash \neg\mathcal{G}$? With this and (3) by *modus tollens* and simplification of a double negation we deduce $\Phi \vdash \pi(\hat{n}^\mathcal{G})$, which with (1) by biconditional elimination yields $\Phi \vdash \mathcal{G}$ — contradiction!, which is ruled out by the assumption that Φ is consistent. **QED**

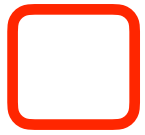
“Silly abstract nonsense! There aren’t any concrete examples of \mathcal{G} !”



Astrologic:

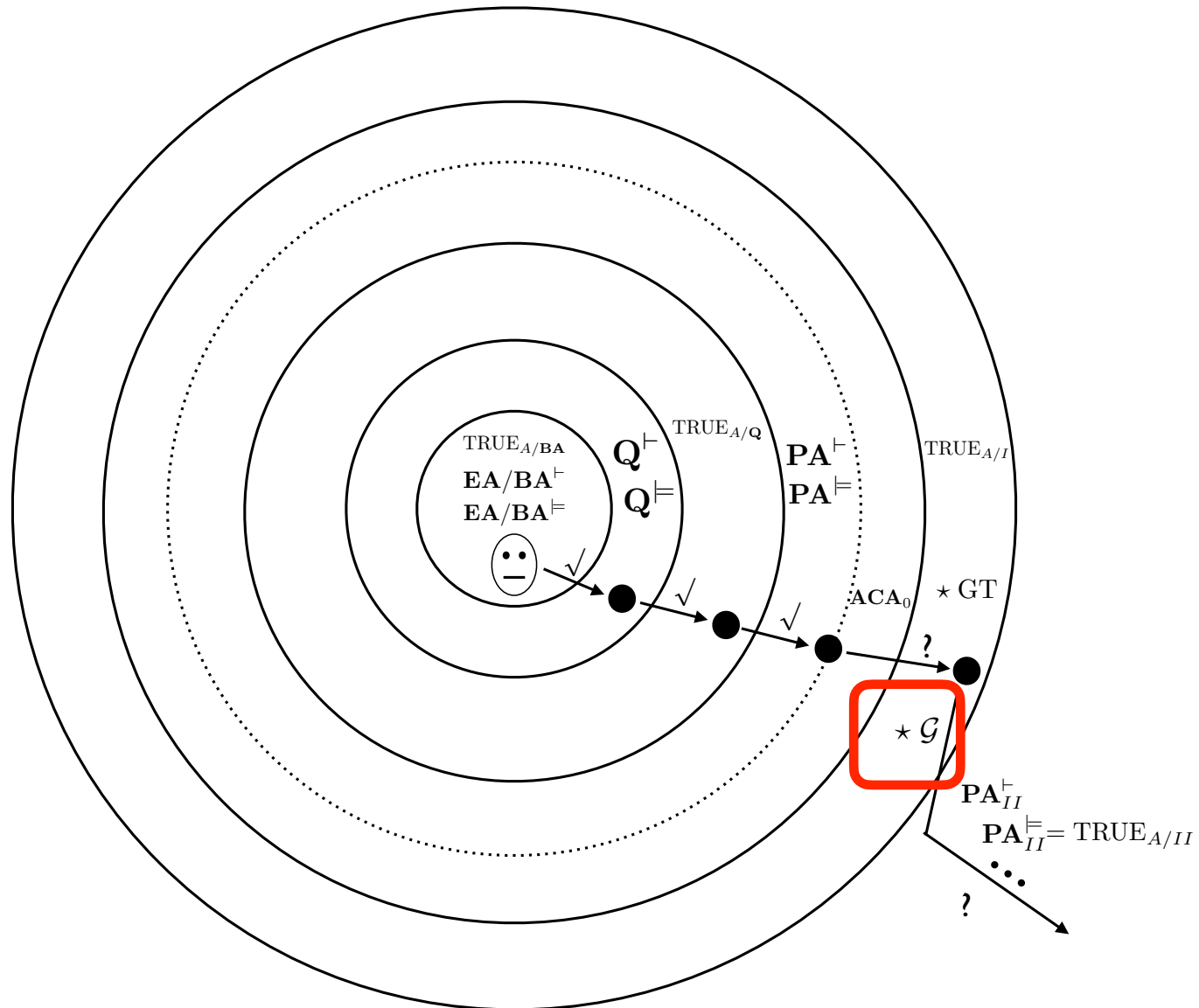
Rational Aliens Will be on the Same “Race Track”!





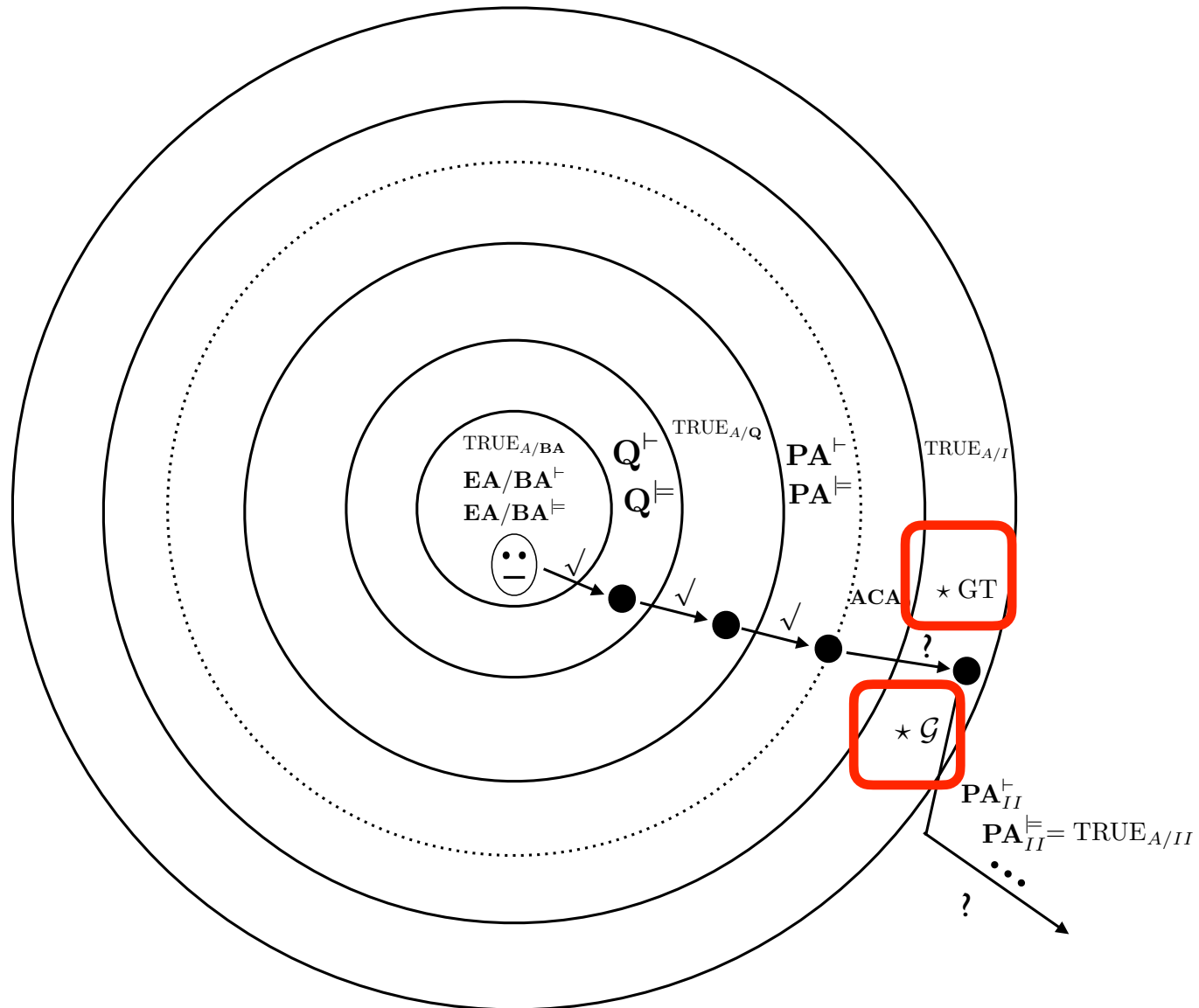
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Ah, but e.g.: Goodstein's Theorem!

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The Goodstein Sequence goes to zero!

Pure base n representation of a number r

- Represent r as only sum of powers of n in which the exponents are also powers of n , etc.

$$266 = 2^{2^{(2^{2^0} + 2^0)}} + 2^{(2^{2^0} + 2^0)} + 2^{2^0}$$

Grow Function

$Grow_k(n)$:

1. Take the pure base k representation of n
2. Replace all k by $k + 1$. Compute the number obtained.
3. Subtract one from the number

Example of **Grow**

$Grow_2(19)$

$$19 = 2^{2^{2^{2^0}}} + 2^{2^0} + 2^0$$

$$3^{3^{3^{3^0}}} + 3^{3^0} + 3^0$$

$$3^{3^{3^{3^0}}} + 3^{3^0} + 3^0 - 1$$

7625597484990

Goodstein Sequence

- For any natural number m

m

$Grow_2(m)$

$Grow_3(Grow_2(m))$

$Grow_4(Grow_3(Grow_2(m))),$

...

Sample Values

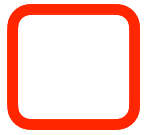
Sample Values

m										
2	2	2	1	0						

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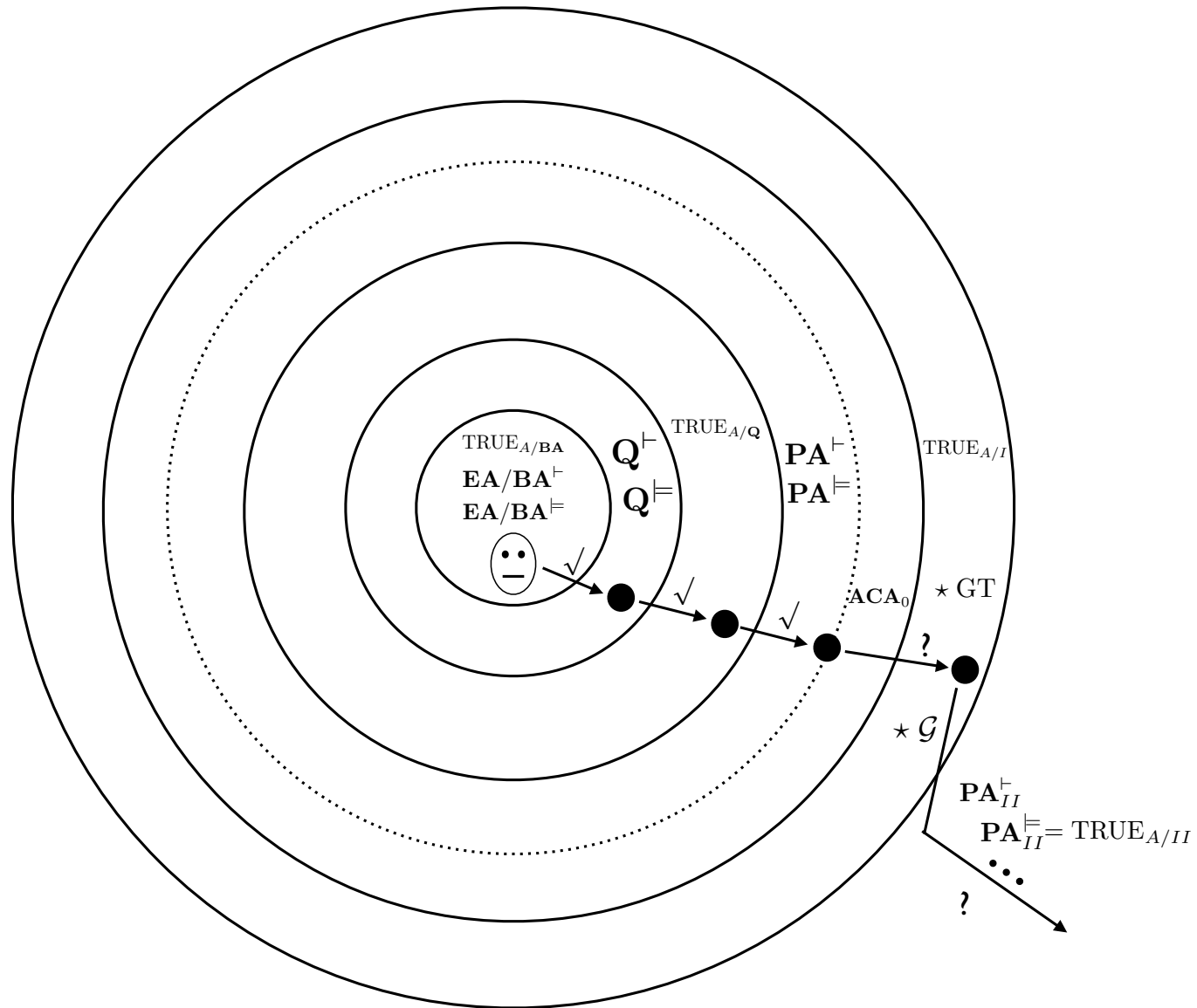
m										
2	2	2	1	0						
3	3	3	3	2	1	0				
4	4	26	41	60	83	109	139	...	11327 (96th term)	...
5	15	$\sim 10^{13}$	$\sim 10^{155}$	$\sim 10^{2185}$	$\sim 10^{36306}$	10^{695975}	$10^{15151337}$...		

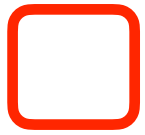
This sequence actually goes to zero!



Astrologic:

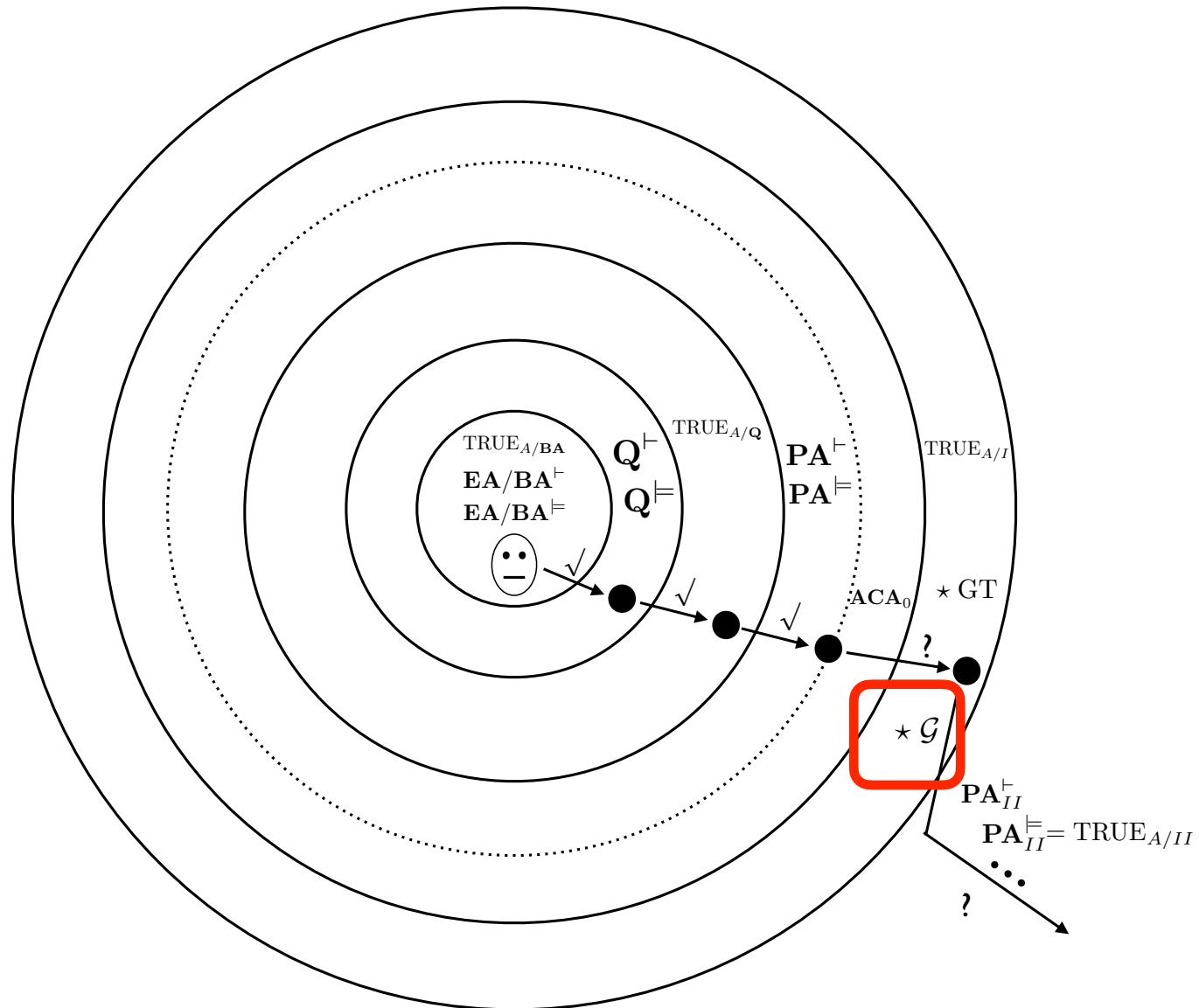
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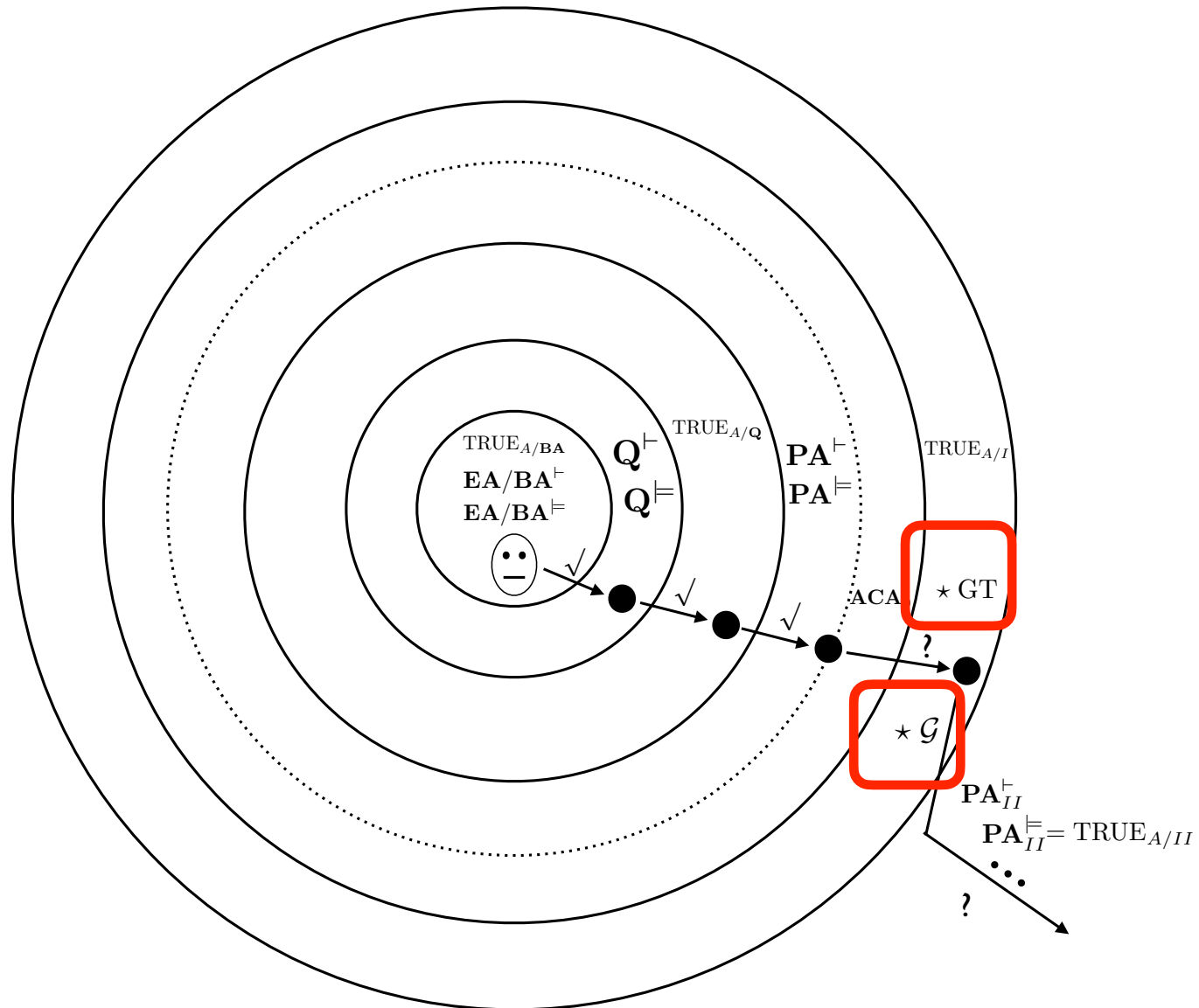
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*Med nok penger, kan
logikk løse alle problemer.*