Propositional Calculus II:

Two more Rules of Inference/Inference Schemata
(`conditional elim` = *modus ponens;*
proof by cases = *disjunction elimination*),
Applying Them to Additional Motivating Problems

Selmer Bringsjord

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Lally School of Management & Technology
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Troy, New York 12180 USA

Intro to Logic
2/5/2026

Logic-and-AI in the news

…

Apropos of Selmer’s (Last Class Mtg) Conveying Imperative:
“Learn all you can about AI!” (Demis Hassabis)

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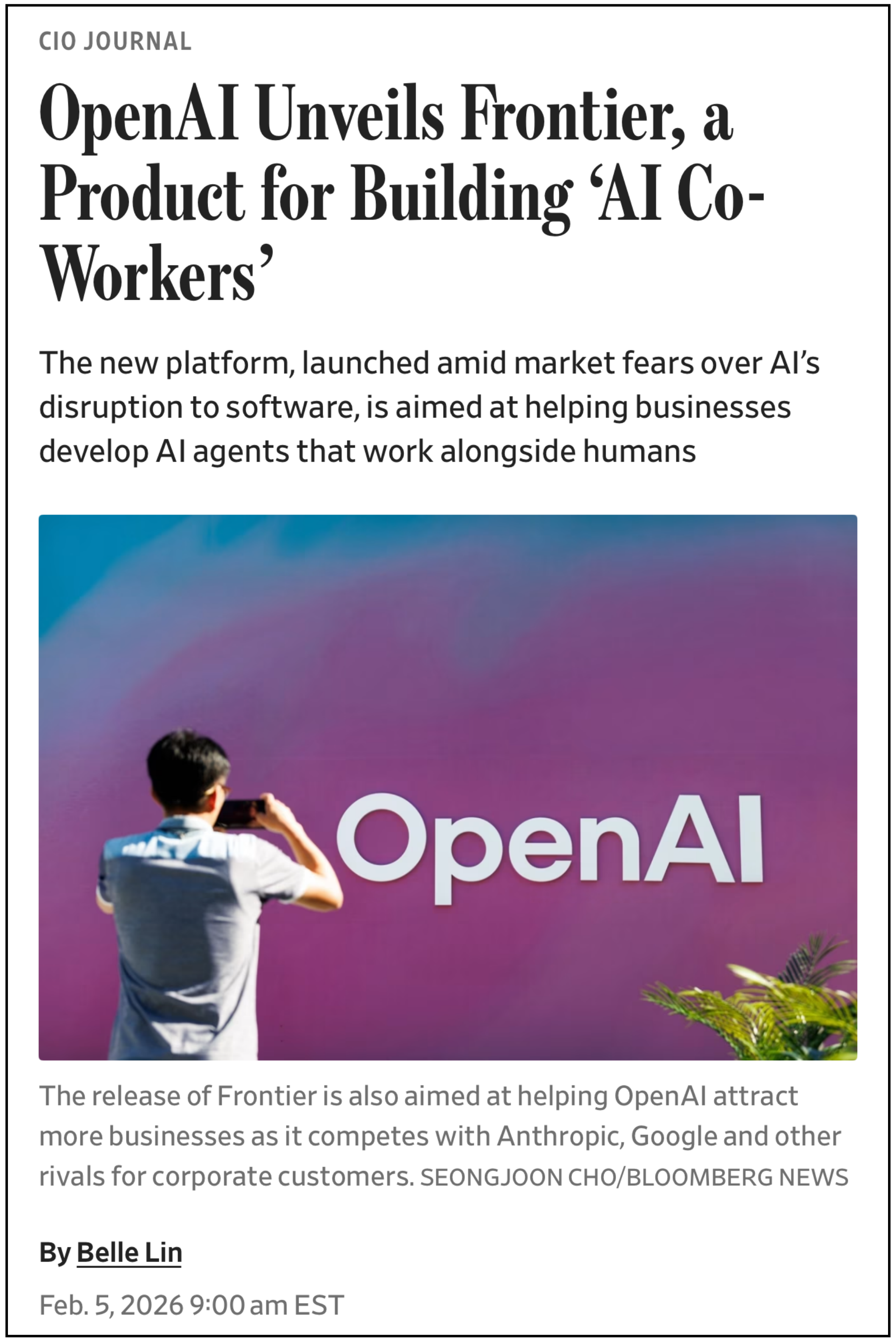

CIO JOURNAL

OpenAI Unveils Frontier, a Product for Building ‘AI Co-Workers’

The new platform, launched amid market fears over AI’s disruption to software, is aimed at helping businesses develop AI agents that work alongside humans

The release of Frontier is also aimed at helping OpenAI attract more businesses as it competes with Anthropic, Google and other rivals for corporate customers. SEONGJOON CHO/BLOOMBERG NEWS

By Belle Lin

Feb. 5, 2026 9:00 am EST

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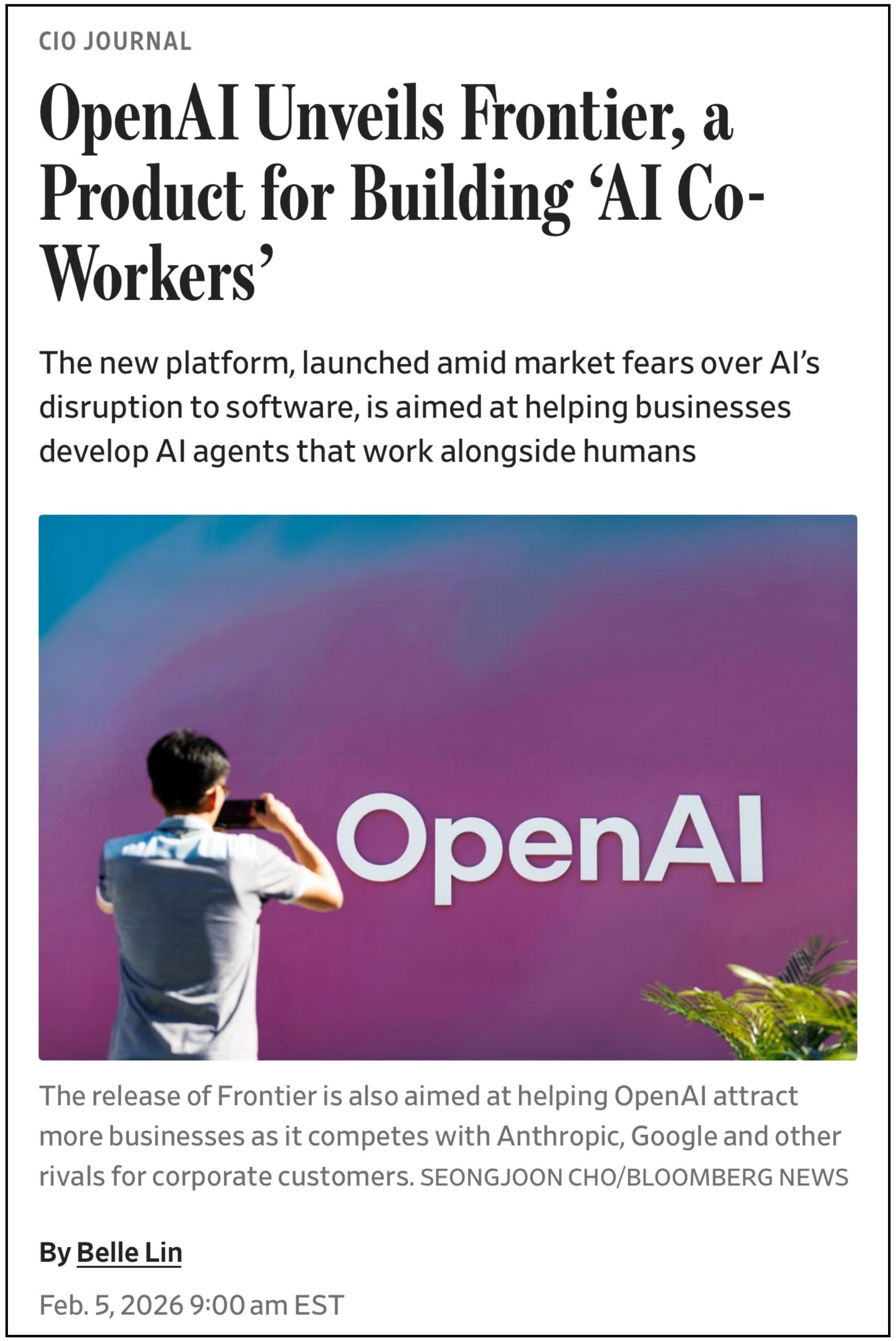

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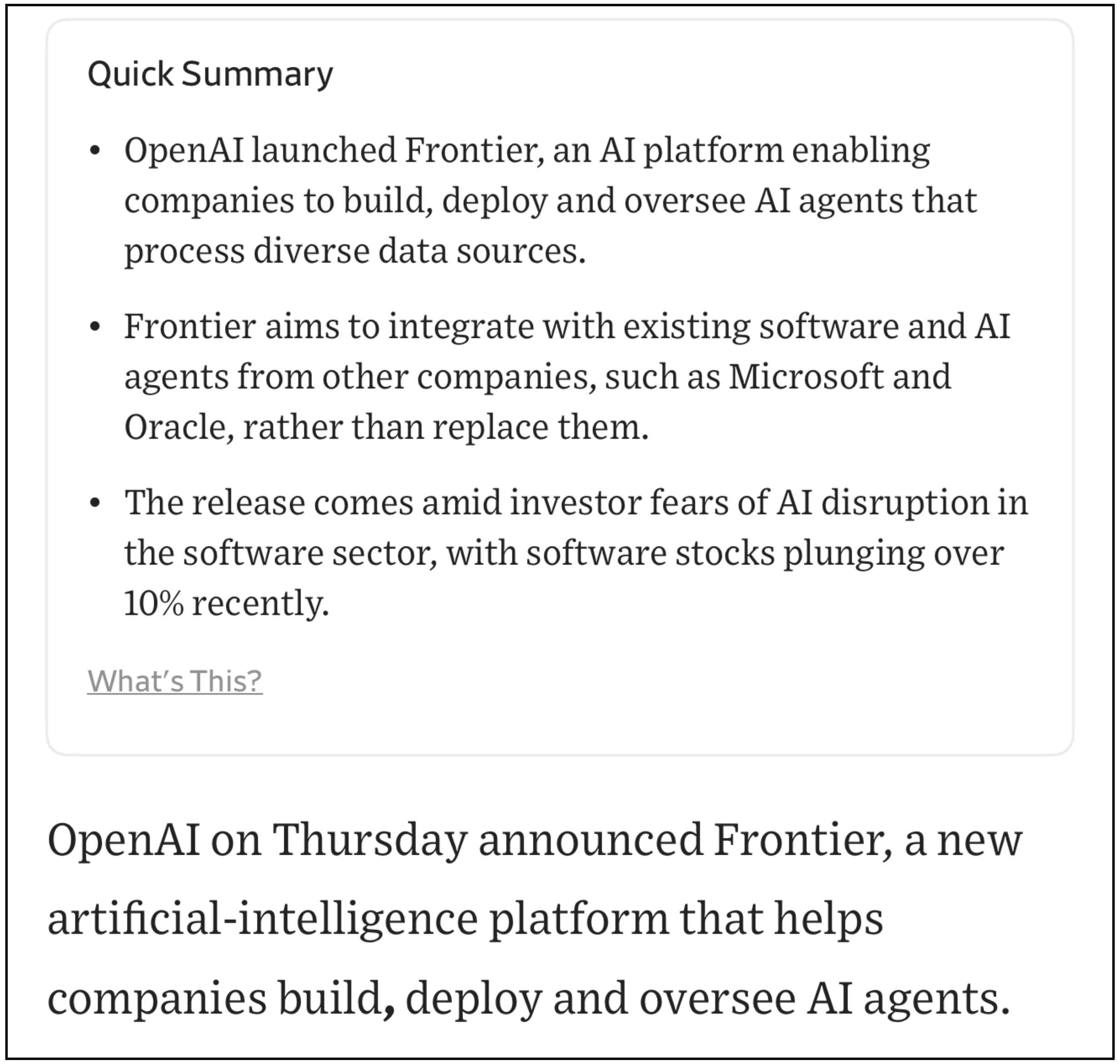

Quick Summary

- OpenAI launched Frontier, an AI platform enabling companies to build, deploy and oversee AI agents that process diverse data sources.
- Frontier aims to integrate with existing software and AI agents from other companies, such as Microsoft and Oracle, rather than replace them.
- The release comes amid investor fears of AI disruption in the software sector, with software stocks plunging over 10% recently.

What’s This?

OpenAI on Thursday announced Frontier, a new artificial-intelligence platform that helps companies build, deploy and oversee AI agents.

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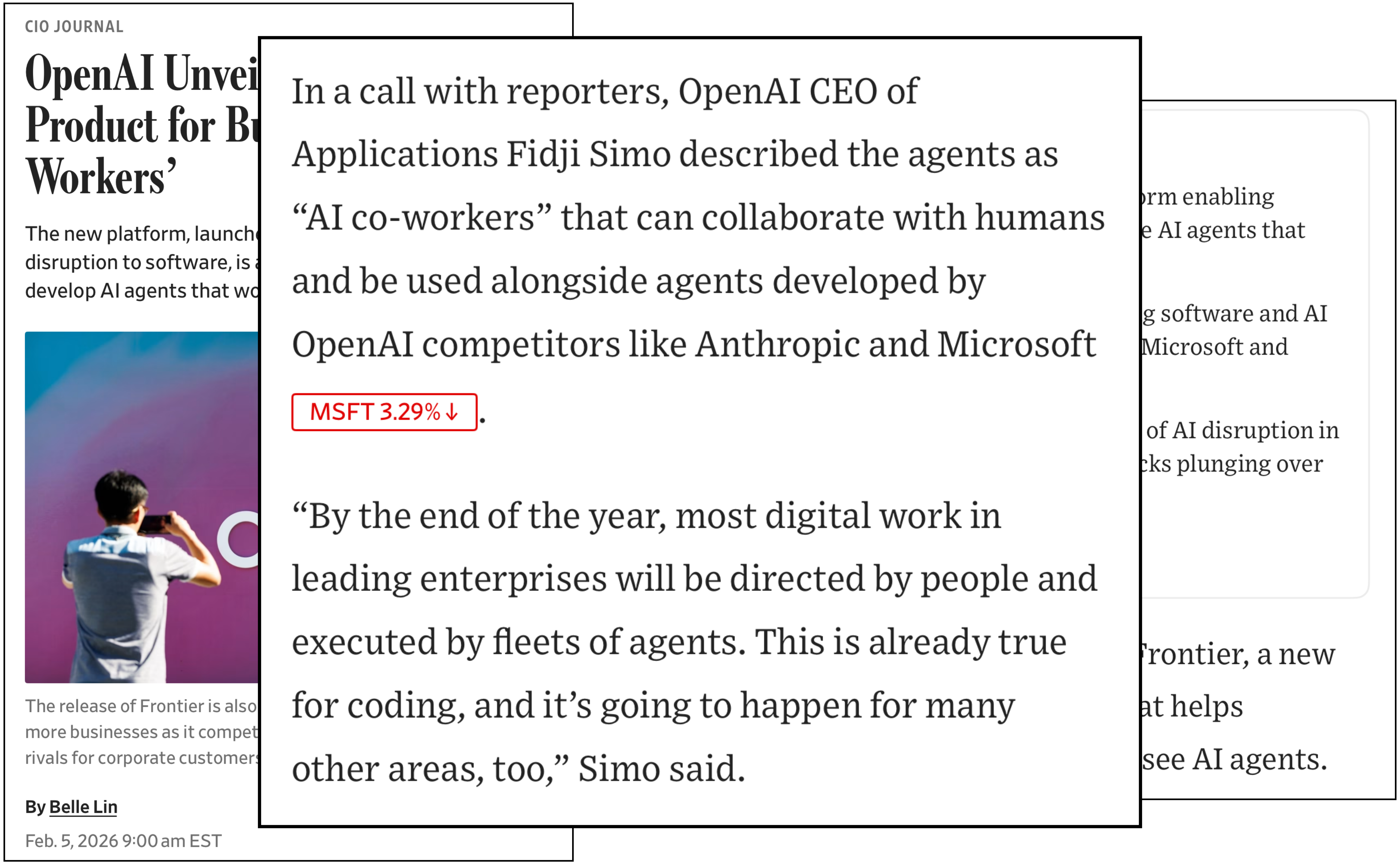

CIO JOURNAL

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In a call with reporters, OpenAI CEO of Applications Fidji Simo described the agents as "AI co-workers" that can collaborate with humans and be used alongside agents developed by OpenAI competitors like Anthropic and Microsoft MSFT 3.29%↓.

"By the end of the year, most digital work in leading enterprises will be directed by people and executed by fleets of agents. This is already true for coding, and it's going to happen for many other areas, too," Simo said.

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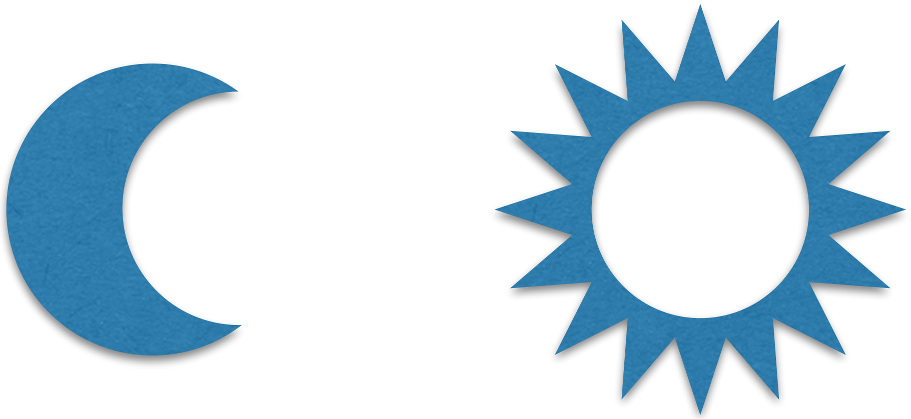
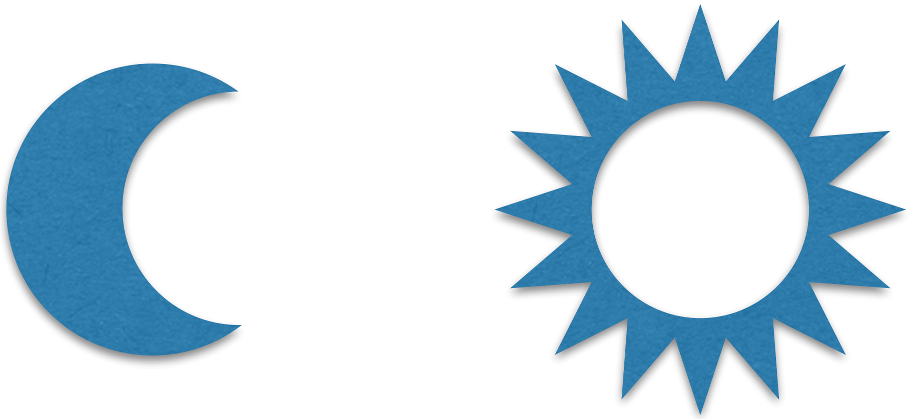
Logistics, again …

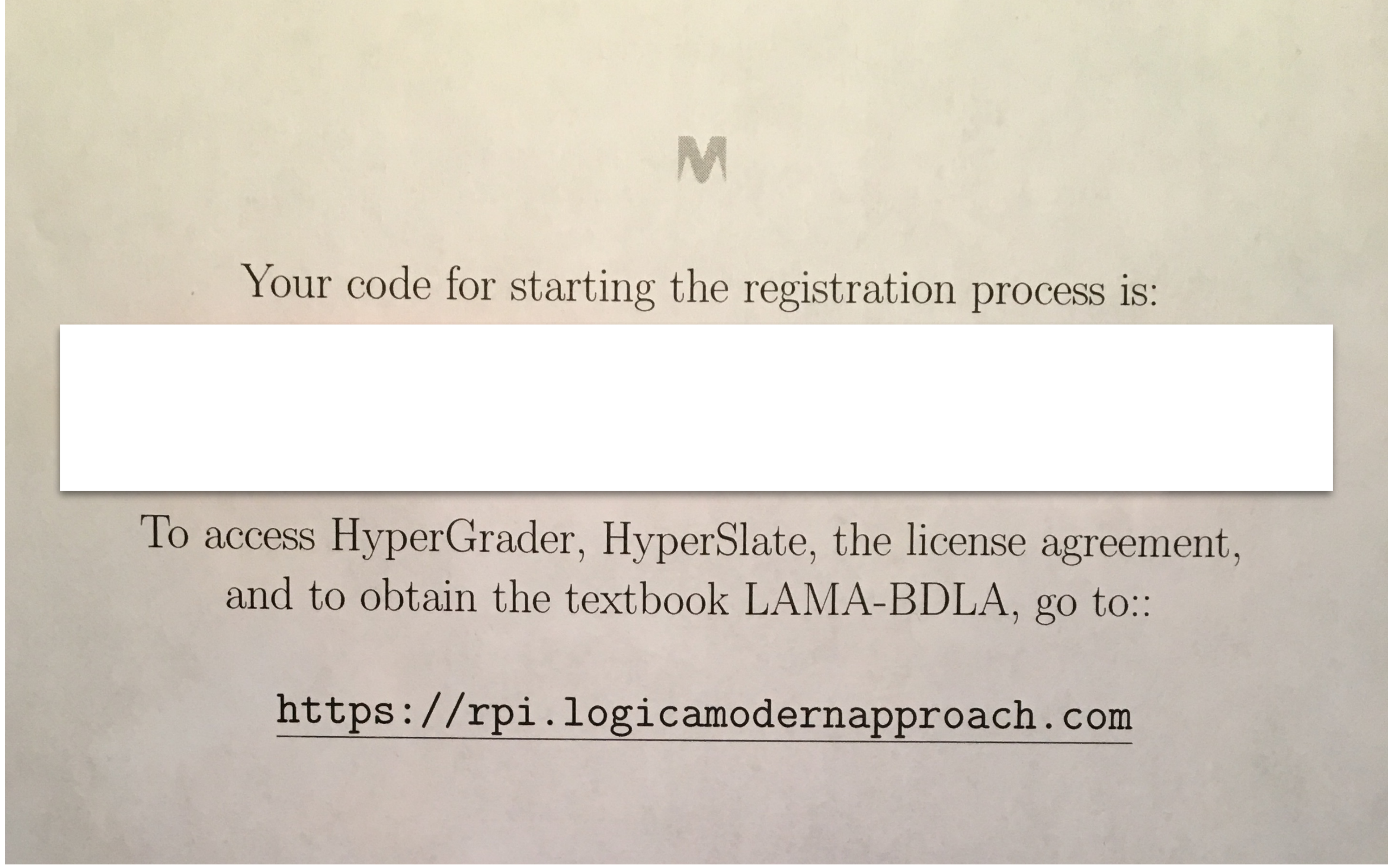
New Release of HyperGrader®!!





Markdown ⊃ LaTeX!

The Starting Code to Purchase in Bookstore



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Your code for starting the registration process is:

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To access HyperGrader, HyperSlate, the license agreement,
and to obtain the textbook LAMA-BDLA, go to::

https://rpi.logicamodernapproach.com

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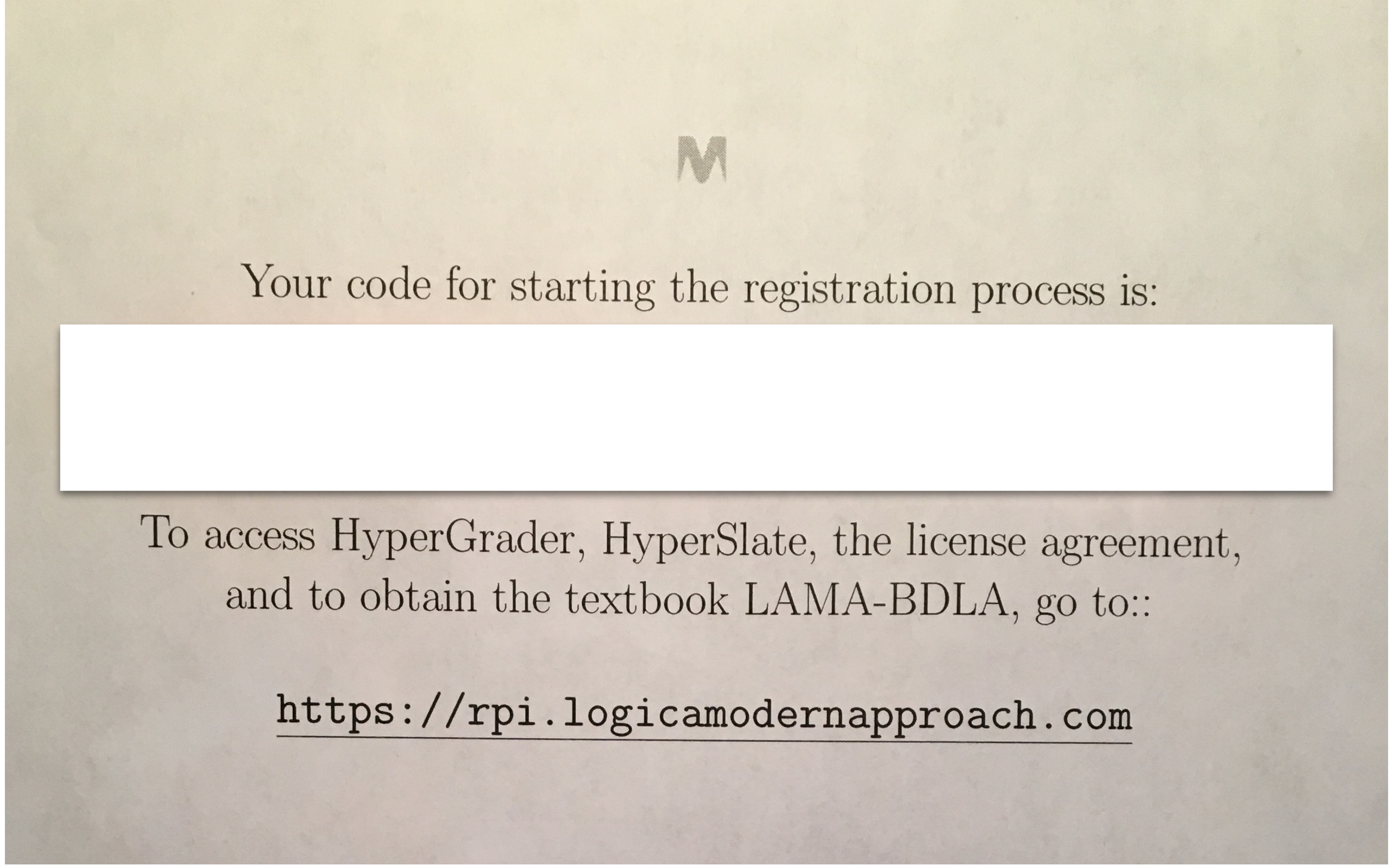

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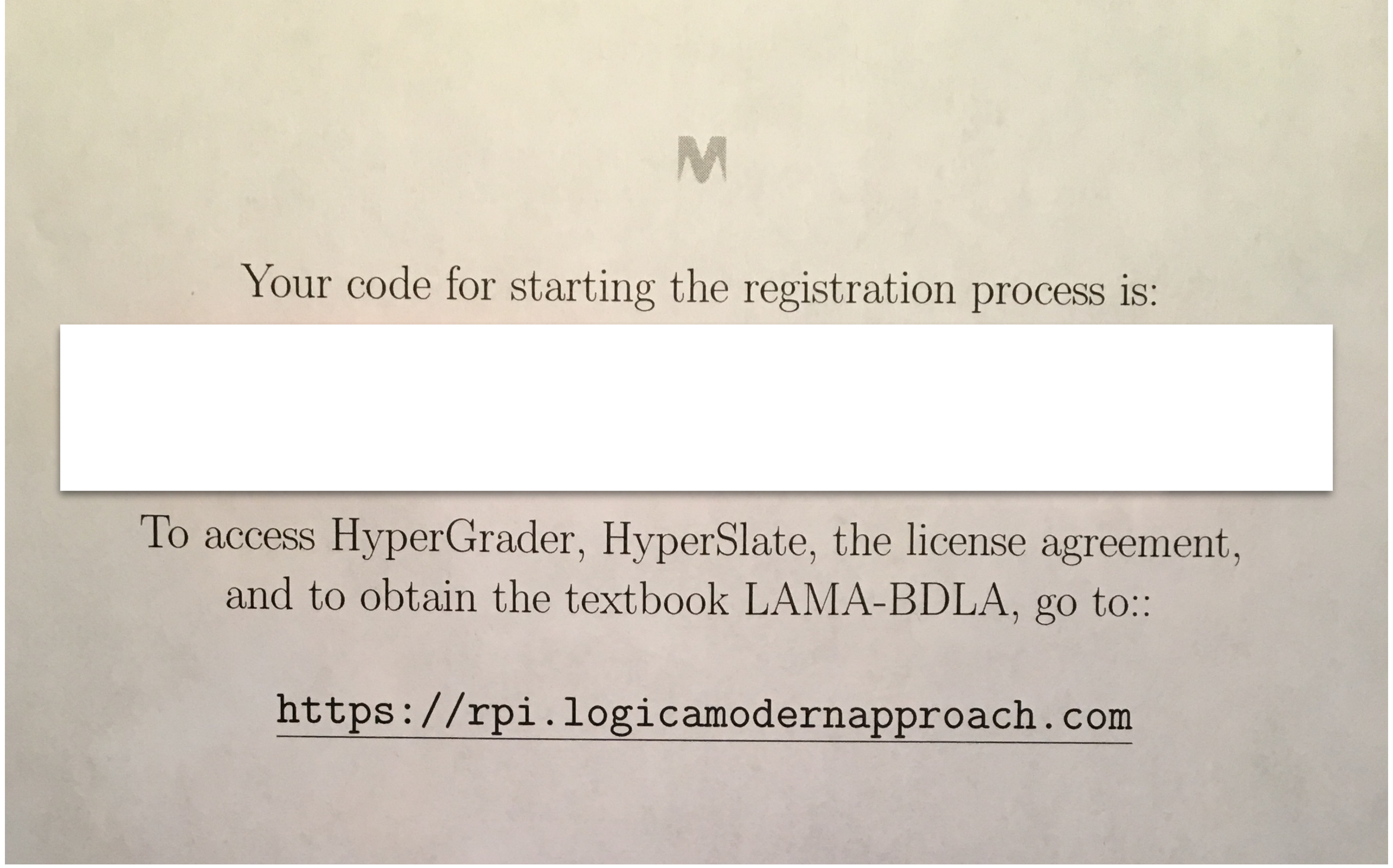

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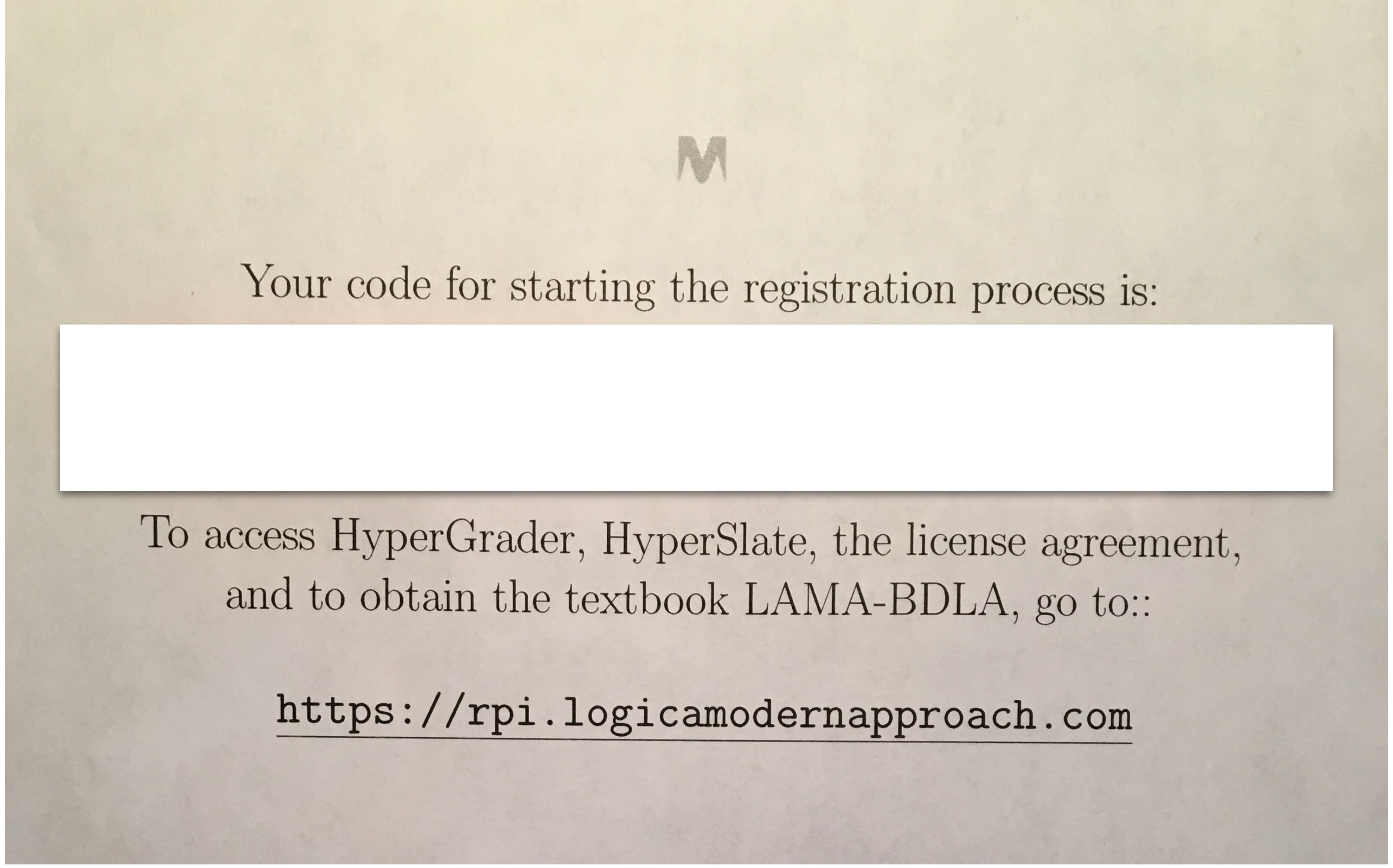

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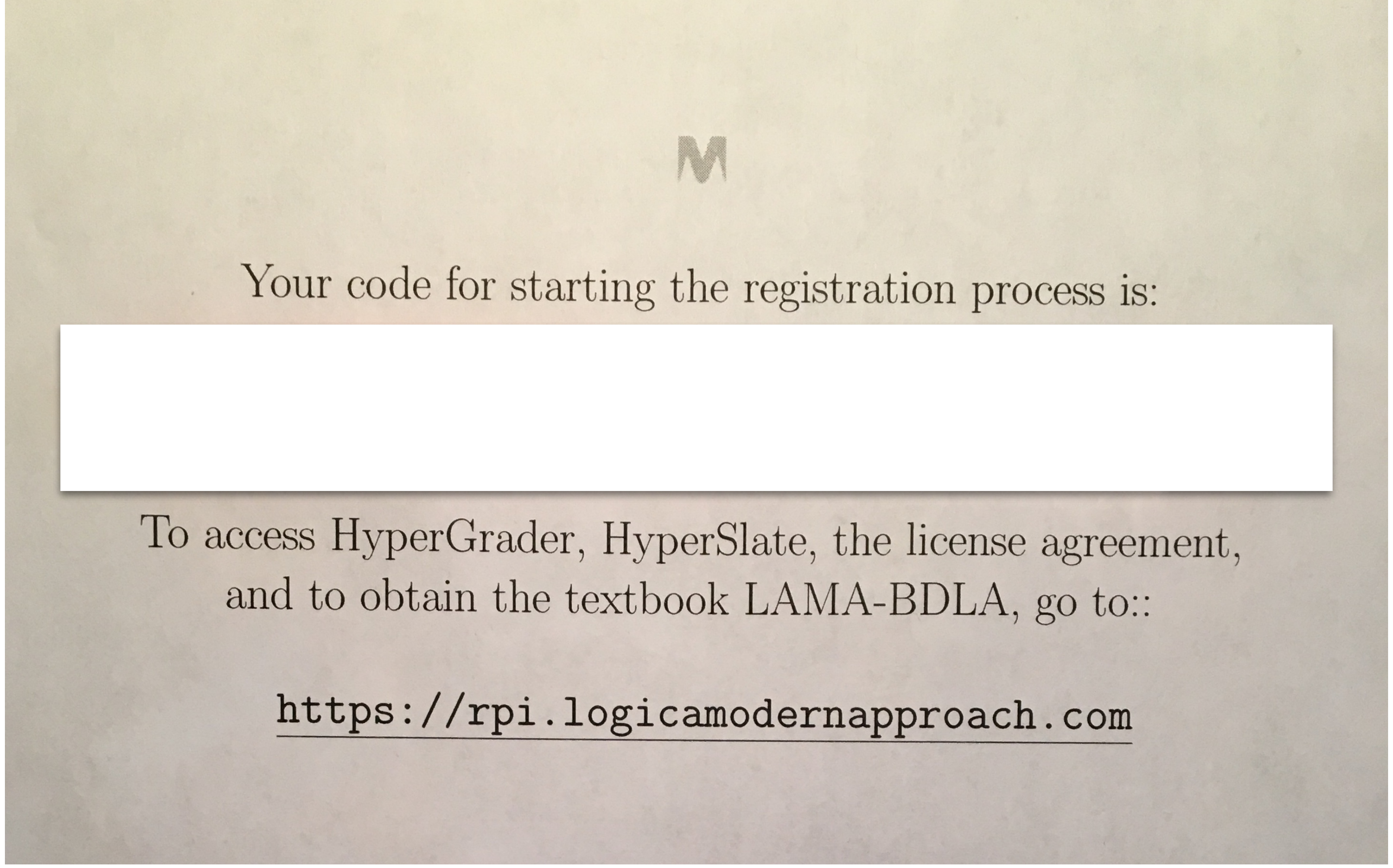
The email address you enter is case-sensitive!

Your OS and browser must be fully up-to-date; Chrome is the best choice, browser-wise (though I use Safari).

The Starting Code Purchased in Bookstore Should By Now've Been Used to Register & Subsequently Sign In

First two Prop-Calc Practice problems:

`switching_conjuncts_fine`, `switching_disjuncts_fine`



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E-Housekeeping Pts (again)

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- Must input your RIN. (This is your "University ID.")

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E-Housekeeping Pts (again)

- Must input your RIN. (This is your "University ID.")
- Make sure OS fully up-to-date.
- Make sure browser fully up-to-date.
- Chrome best (but I use Safari).
- Always work in the same browser window with multiple tabs; must do this with email and HyperGrader® & HyperSlate®.

Propositional Calculus II:

Two more Rules of Inference/Inference Schemata

(`conditional elim` = *modus ponens;*

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Intro to Logic
2/5/2026

Last time we introduced and and lauded the power of **oracles**, and questions …
and now … picking up where we left off …

"NYS 3" Revisited

Given the statements

$\neg\neg c$

$c \rightarrow a$

$\neg a \vee b$

$b \rightarrow d$

$\neg(d \vee e)$

which one of the following statements must also be true?

$\neg c$

e

h

$\neg a$

all of the above

“NYS 3” Revisited

Given the statements

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$b \rightarrow d$

$\neg(d \vee e)$

which one of the following statements must also be true?

$\neg c$

e

h

$\neg a$

[✓ all of the above]

“NYS 3” Revisited

Given the statements

$\neg\neg c$

$c \rightarrow a$

$\neg a \vee b$

$b \rightarrow d$

$\neg(d \vee e)$

After last class, should have explored if you are registered … Show in HyperSlate® as I did that each of the first four options can be proved using the PC entailment (= provability) oracle.

which one of the following statements must also be true?

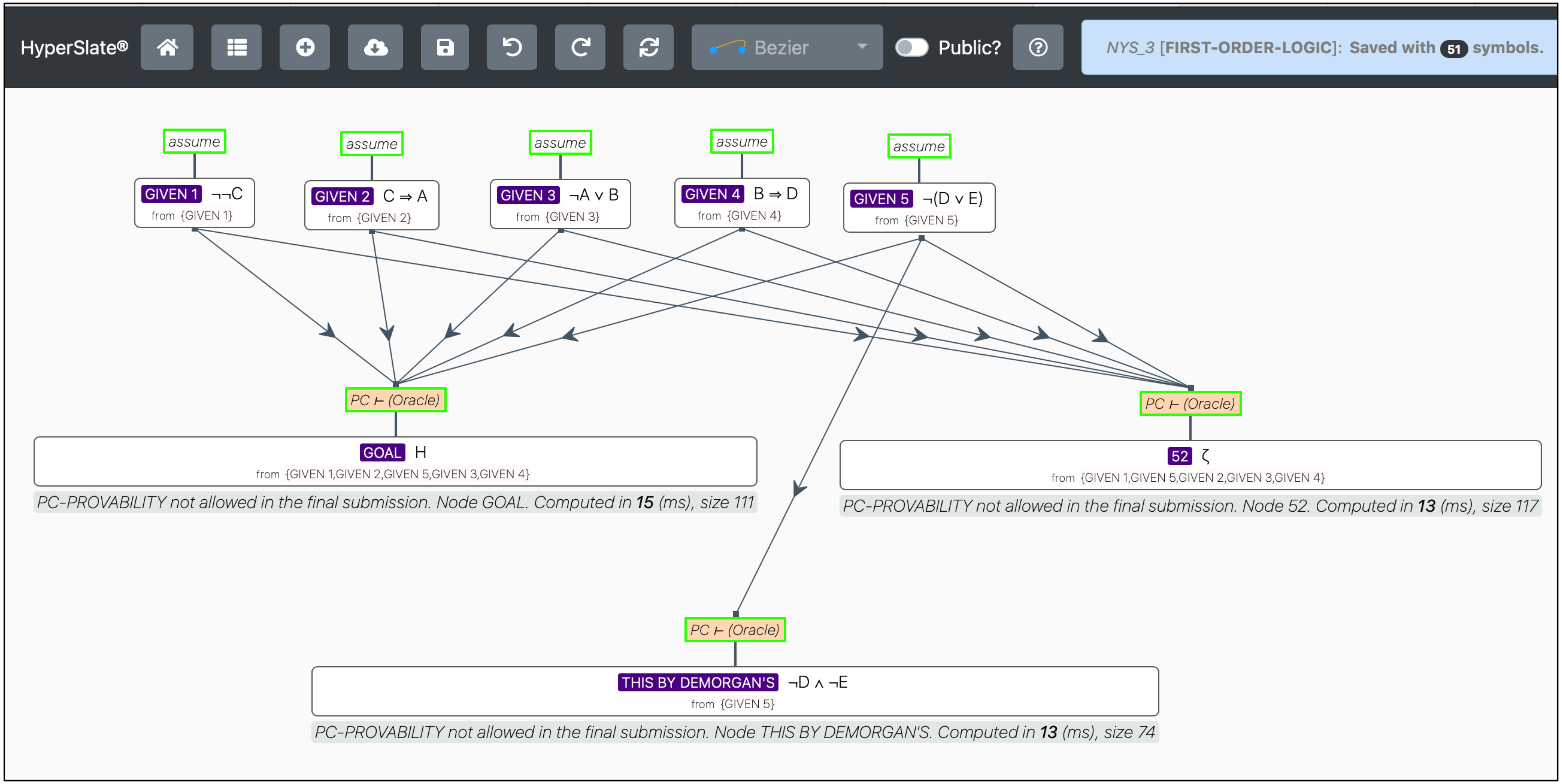
$\neg c$

e

h

$\neg a$

all of the above



NYS_3 **[FIRST-ORDER-LOGIC]: Saved with 51 symbols.**

PC-PROVABILITY not allowed in the final submission. Node GOAL. Computed in ***15*** *(ms), size 111*

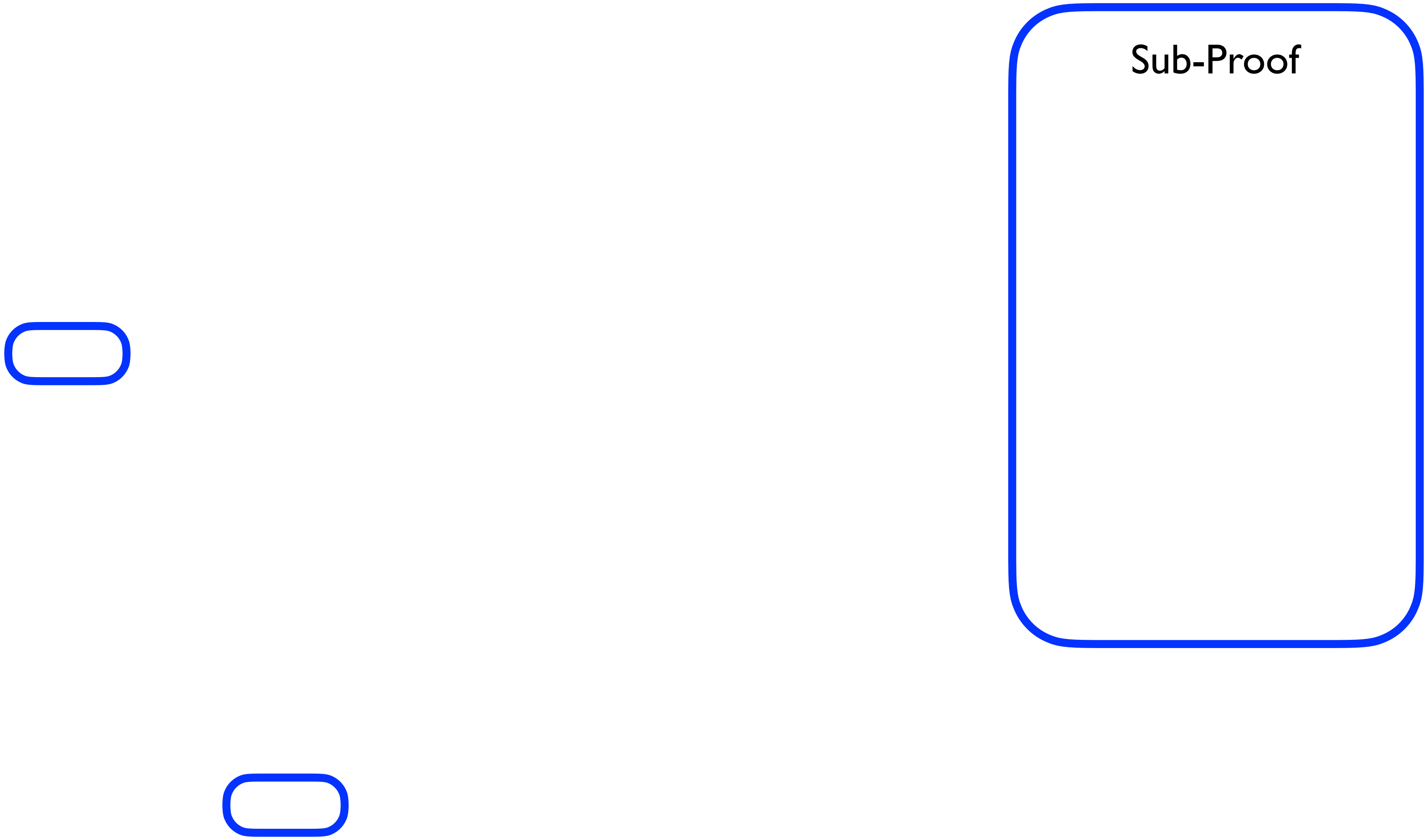
PC-PROVABILITY not allowed in the final submission. Node 52. Computed in ***13*** *(ms), size 117*

PC-PROVABILITY not allowed in the final submission. Node THIS BY DEMORGAN'S. Computed in ***13*** *(ms), size 74*

Proof Plan …

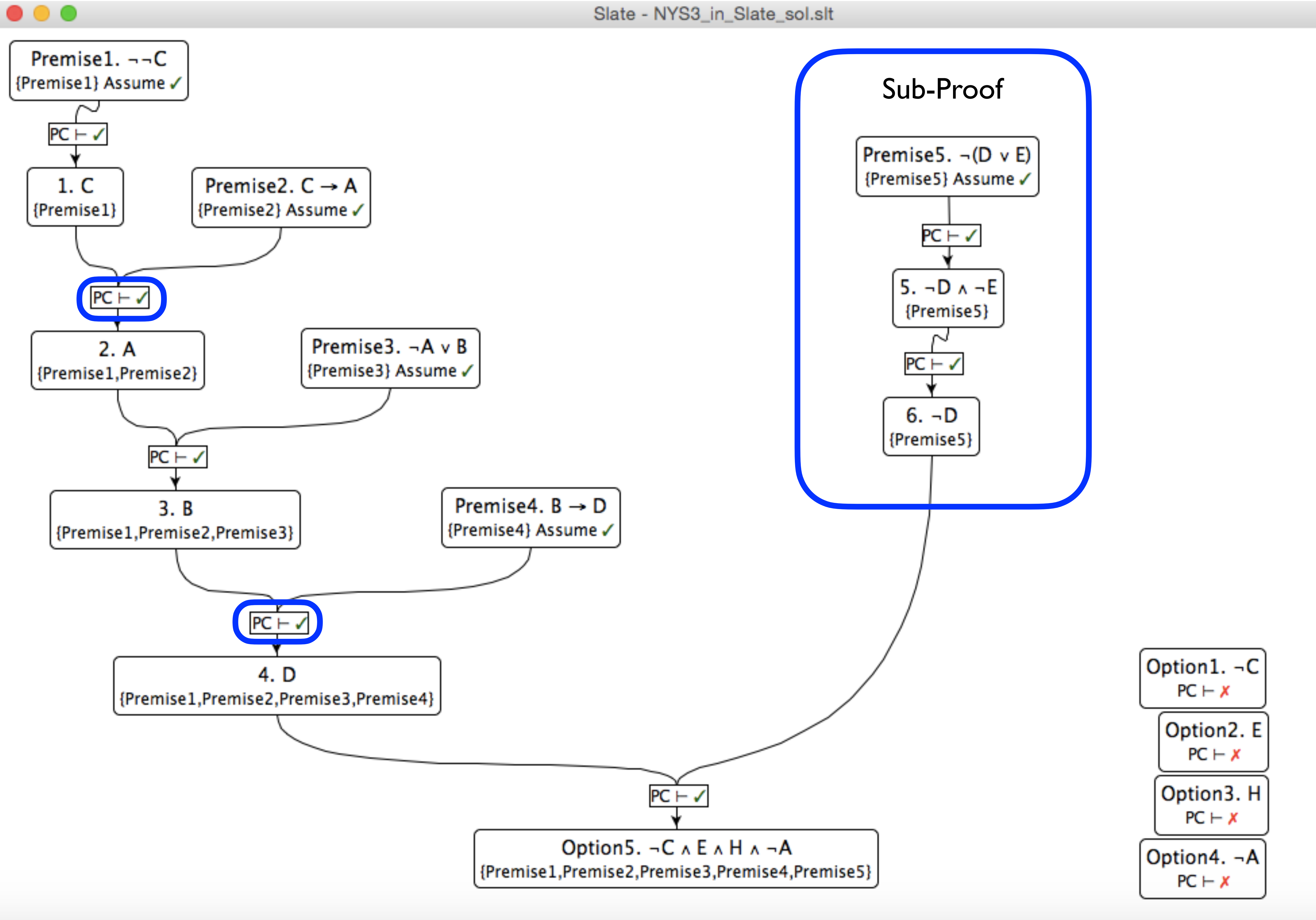
Proof Plan …

Proof Plan …

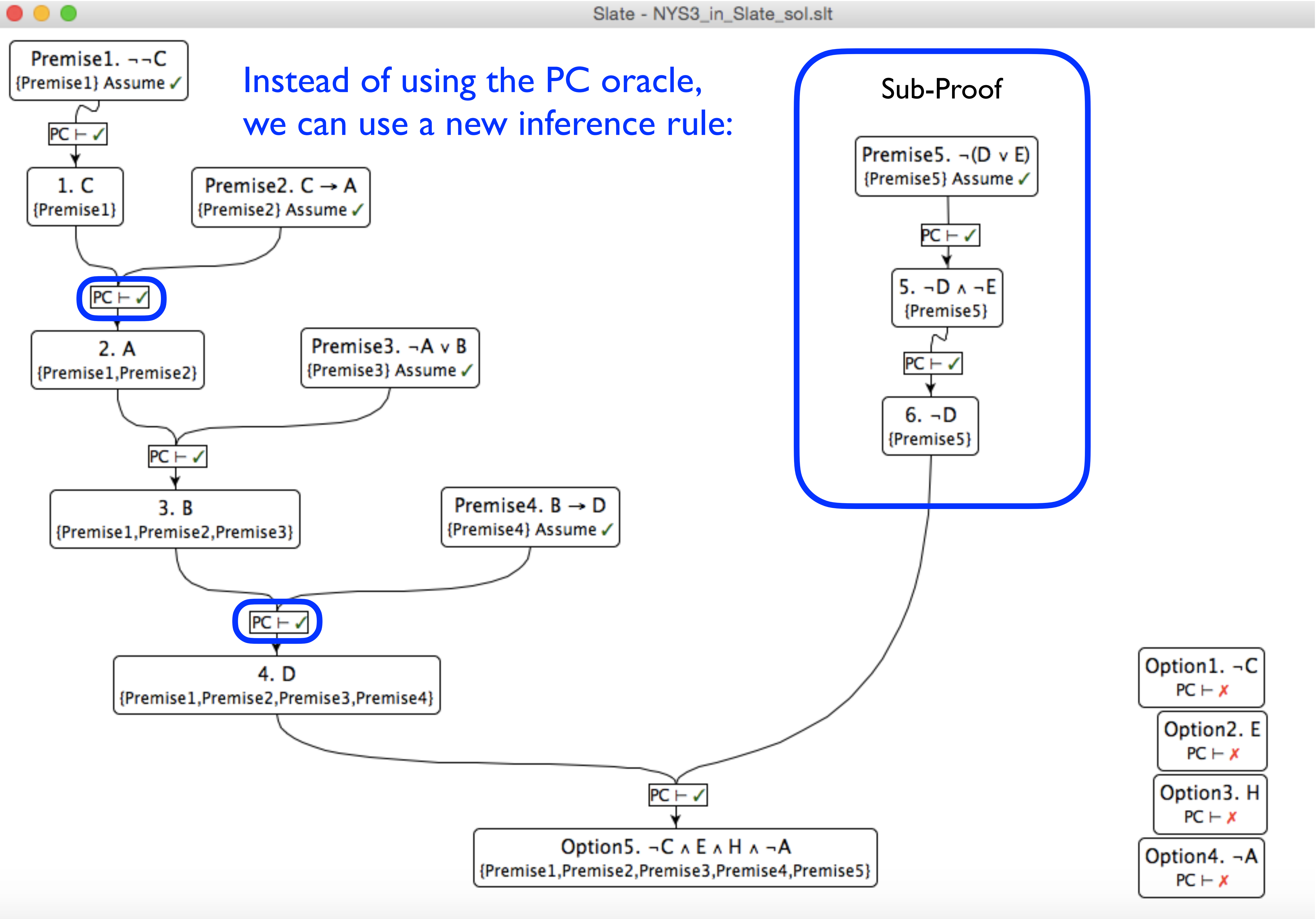


Sub-Proof

Proof Plan …


Slate - NYS3_in_Slate_sol.slt
Premise1. ¬¬C
{Premise1} Assume ✓
PC ⊢ ✓
1. C
{Premise1}
Premise2. C → A
{Premise2} Assume ✓
PC ⊢ ✓
2. A
{Premise1,Premise2}
Premise3. ¬A ∨ B
{Premise3} Assume ✓
PC ⊢ ✓
3. B
{Premise1,Premise2,Premise3}
Premise4. B → D
{Premise4} Assume ✓
PC ⊢ ✓
4. D
{Premise1,Premise2,Premise3,Premise4}
Sub-Proof
Premise5. ¬(D ∨ E)
{Premise5} Assume ✓
PC ⊢ ✓
5. ¬D ∧ ¬E
{Premise5}
PC ⊢ ✓
6. ¬D
{Premise5}
PC ⊢ ✓
Option5. ¬C ∧ E ∧ H ∧ ¬A
{Premise1,Premise2,Premise3,Premise4,Premise5}
Option1. ¬C
PC ⊢ ✗
Option2. E
PC ⊢ ✗
Option3. H
PC ⊢ ✗
Option4. ¬A
PC ⊢ ✗

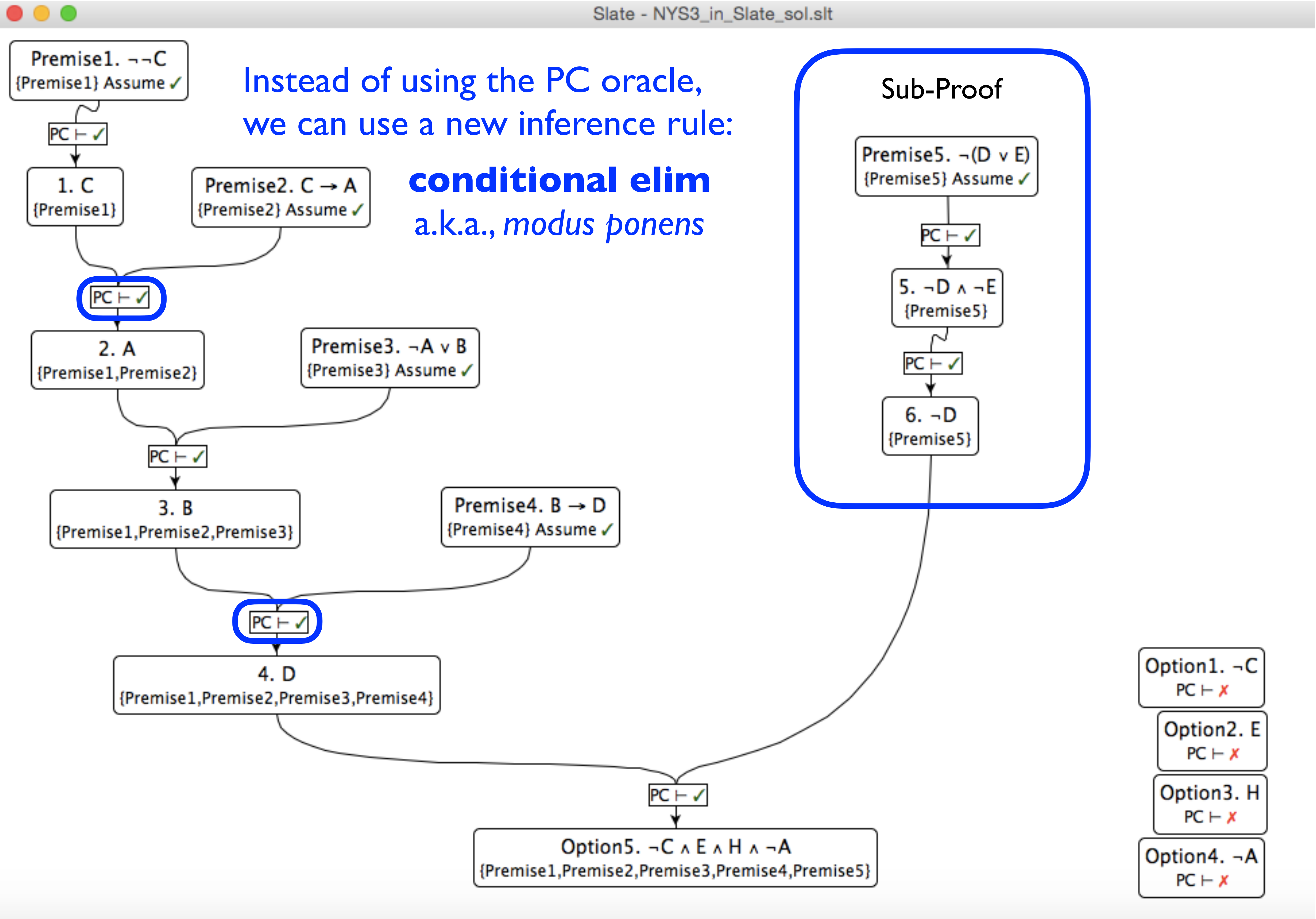
Proof Plan …



Instead of using the PC oracle, we can use a new inference rule:

Sub-Proof

Proof Plan …

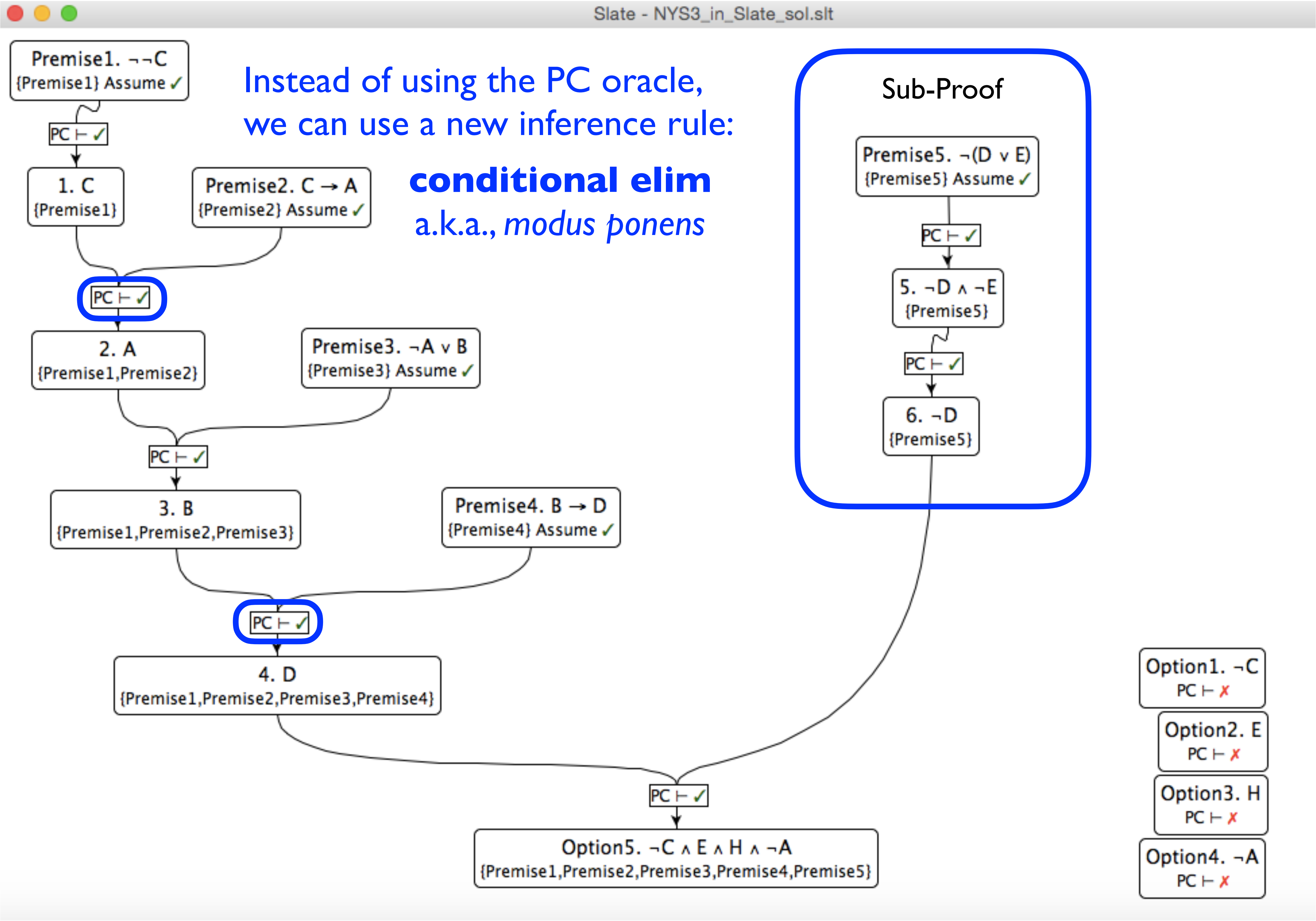


Instead of using the PC oracle, we can use a new inference rule:

conditional elim

a.k.a., *modus ponens*

Proof Plan …



Slate - NYS3_in_Slate_sol.slt

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PC ⊢ ✓

1. C
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Premise2. C → A
{Premise2} Assume ✓

PC ⊢ ✓

2. A
{Premise1,Premise2}

Premise3. ¬A ∨ B
{Premise3} Assume ✓

PC ⊢ ✓

3. B
{Premise1,Premise2,Premise3}

Premise4. B → D
{Premise4} Assume ✓

PC ⊢ ✓

4. D
{Premise1,Premise2,Premise3,Premise4}

Sub-Proof

Premise5. ¬(D ∨ E)
{Premise5} Assume ✓

PC ⊢ ✓

5. ¬D ∧ ¬E
{Premise5}

PC ⊢ ✓

6. ¬D
{Premise5}

PC ⊢ ✓

Option5. ¬C ∧ E ∧ H ∧ ¬A
{Premise1,Premise2,Premise3,Premise4,Premise5}

Option1. ¬C
PC ⊢ ✗

Option2. E
PC ⊢ ✗

Option3. H
PC ⊢ ✗

Option4. ¬A
PC ⊢ ✗

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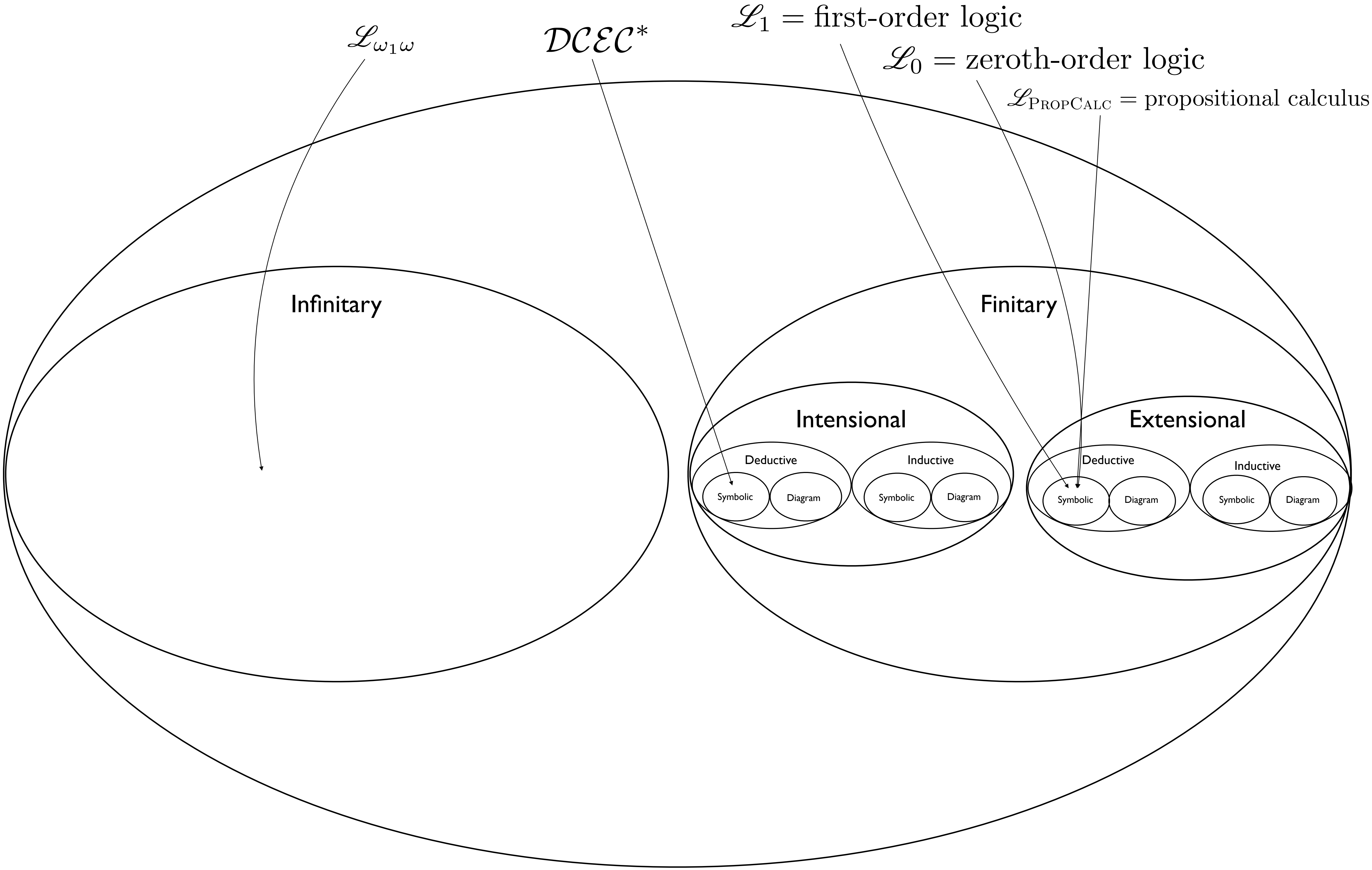
The Universe of Logics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Symbolic
Diagram

What *is* Logic?

What *is* Logic?

- The key to becoming rational.
- "The science and engineering of reasoning." — so the not-unreasonable slogan goes.
- The only invincible subject there is.
- The basis for the formal sciences (from mathematics to game theory to decision theory to probability calculi to axiomatic physics ….) — and hence the basis for disciplines based on the formal sciences, e.g., …
  - Engineering! Computer Science!
  - Mathematics itself: see "reverse mathematics"!
- The way of escape from "shallow" content and context to pure, immaterial, and immortal form and structure (which is why the exotic, imaginary, and seemingly non-sensical is so pedagogically useful).
- The most challenging subject there is.
- One of the chief differentiators between dogs and monkeys versus you (let alone bears and you); and mindless machines (like Deep Blue & Watson) versus you.
- A key to riches.
- The key to divining the meaning of life (and other such big questions).
- The better way to program computers; and fundamentally the *only* way to *reliably* program computers.
- One of two fundamental approaches to studying minds, and replicating/simulating minds in machines…
- The thing many creatures of fiction have mastered — have you (as a New Yorker)?…
- …

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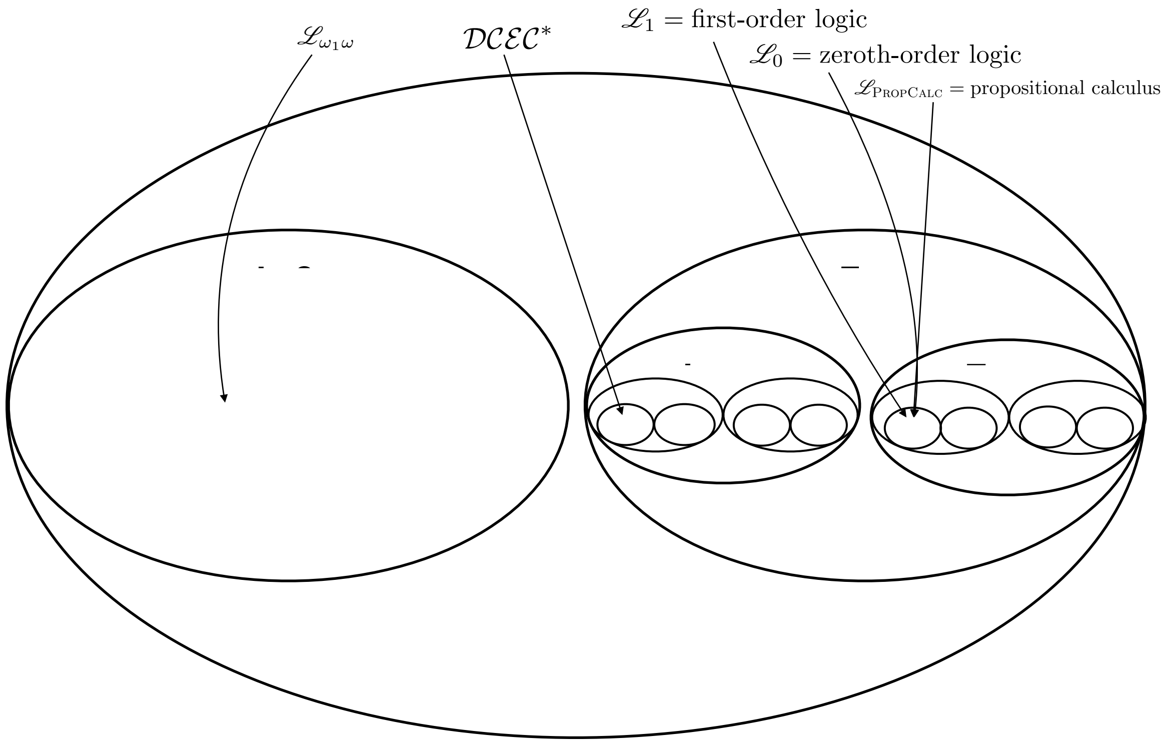
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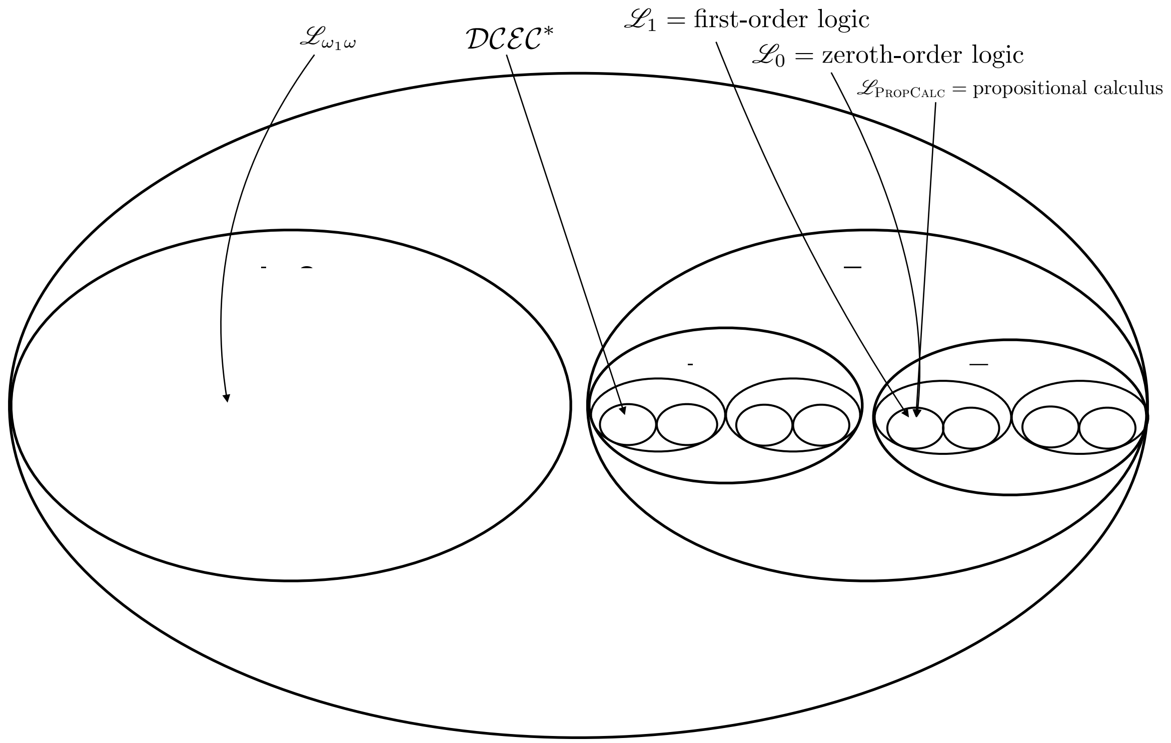
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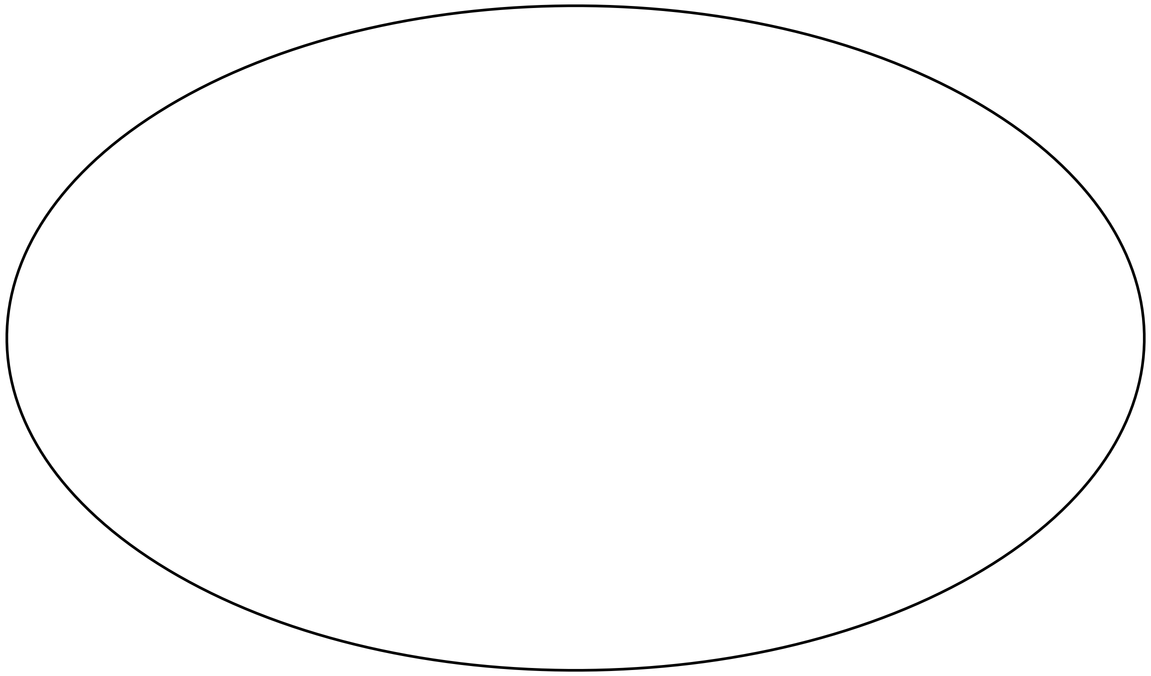
The Physical Universe



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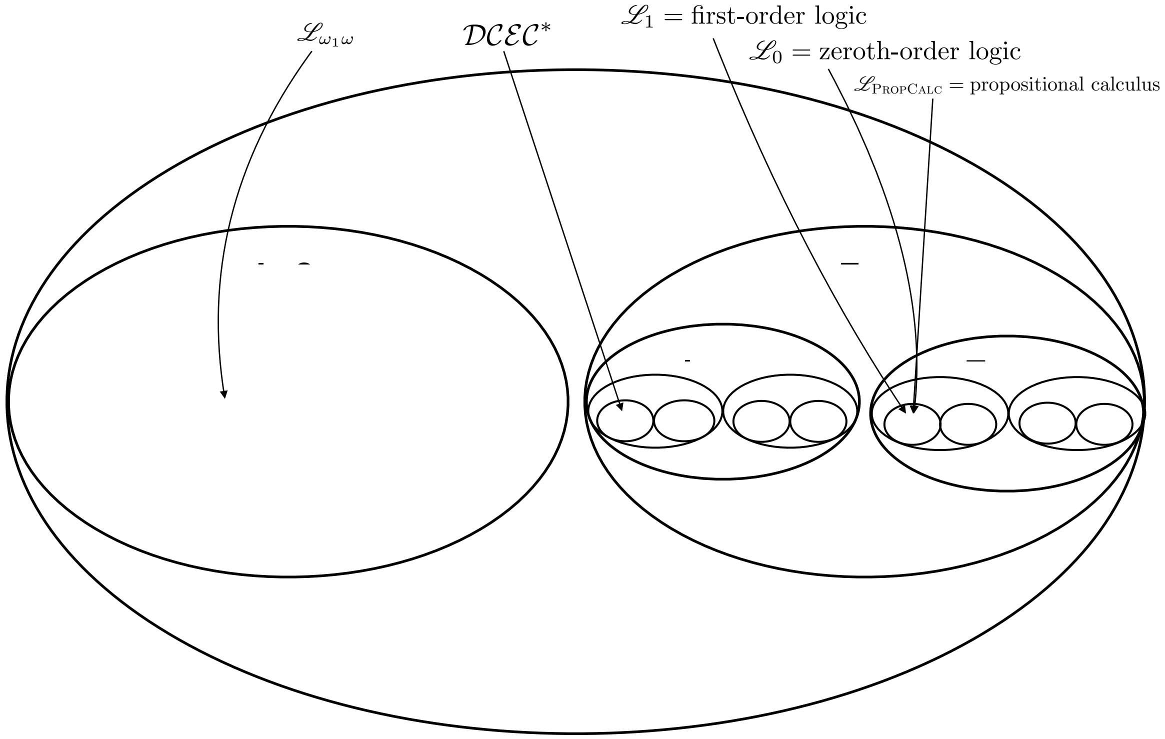
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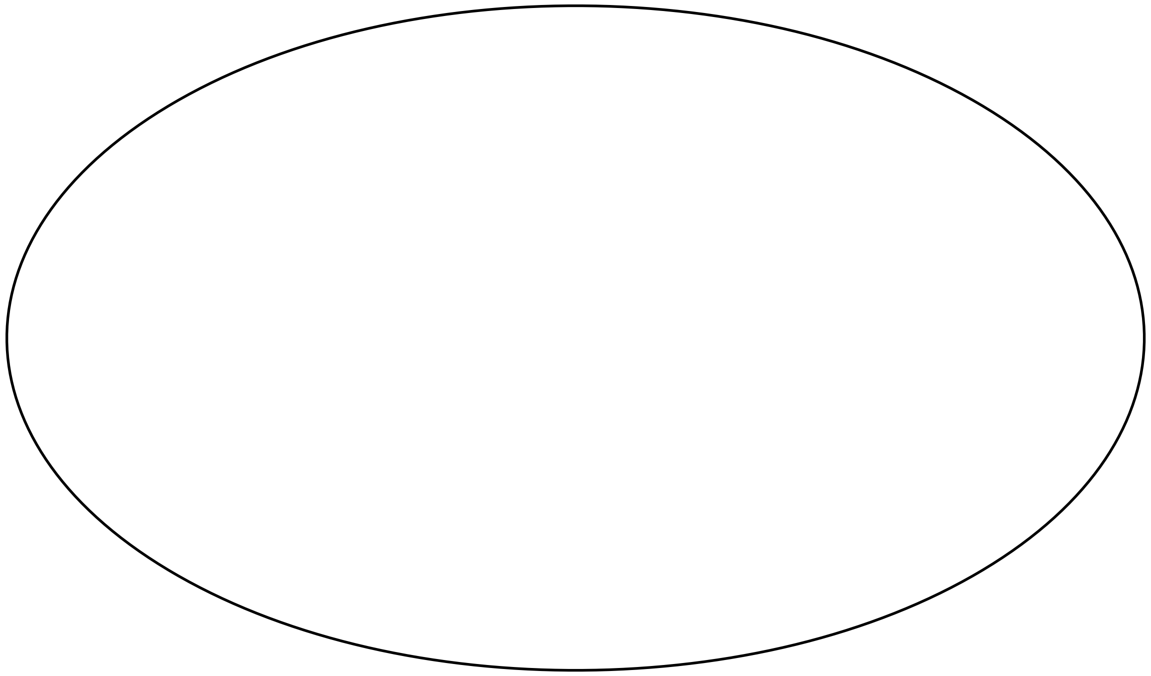
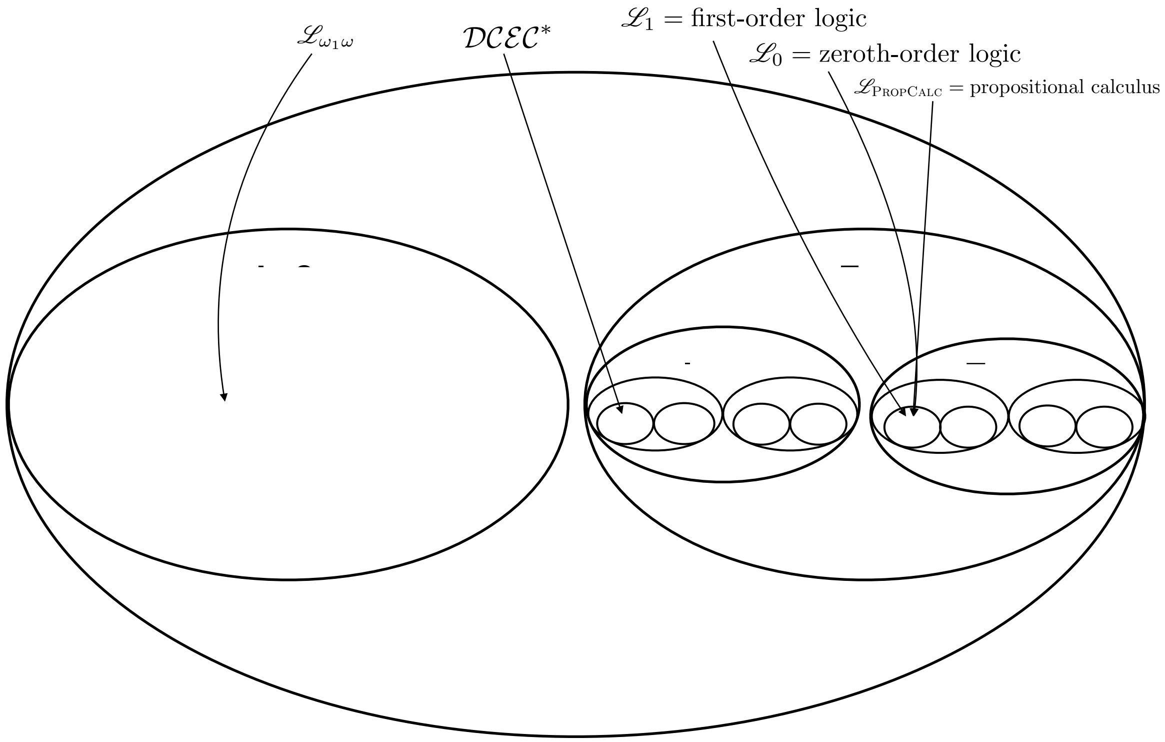


Non-Physical

The Universe of Logics

The Physical Universe



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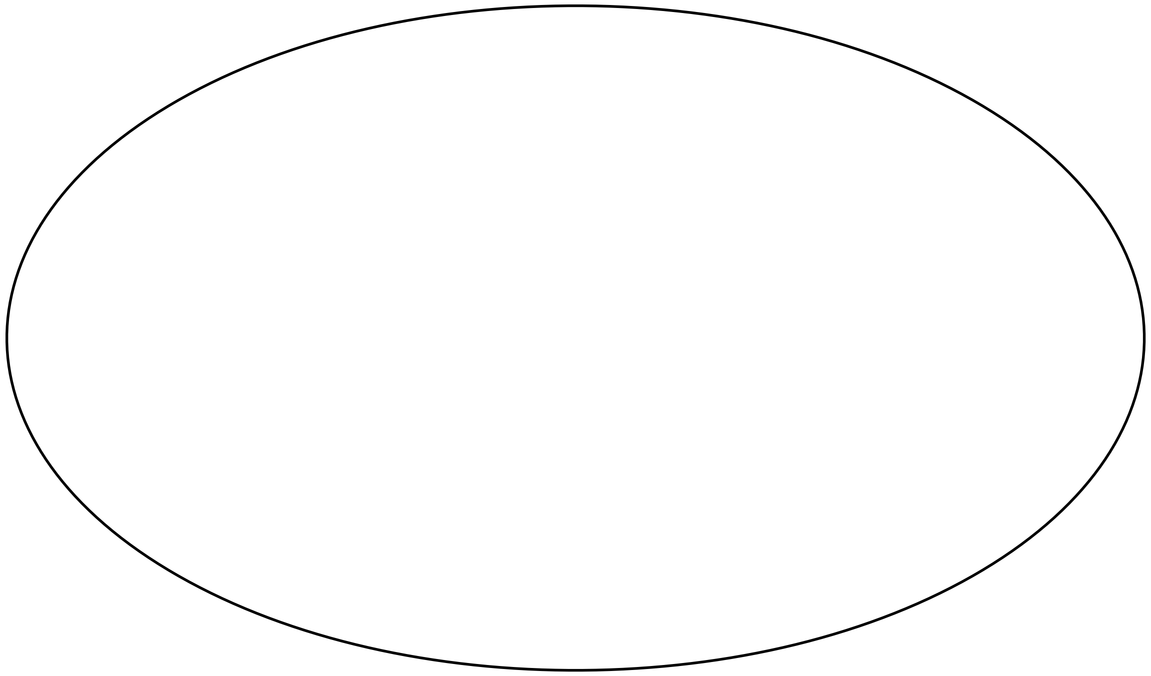

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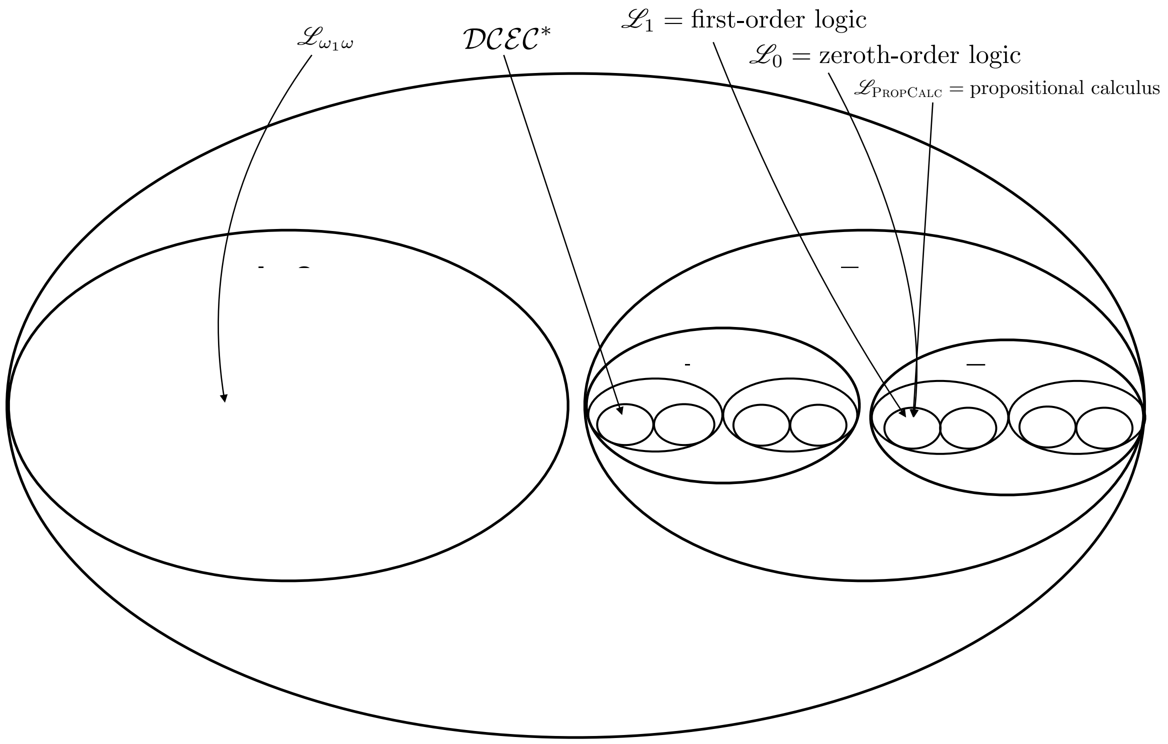
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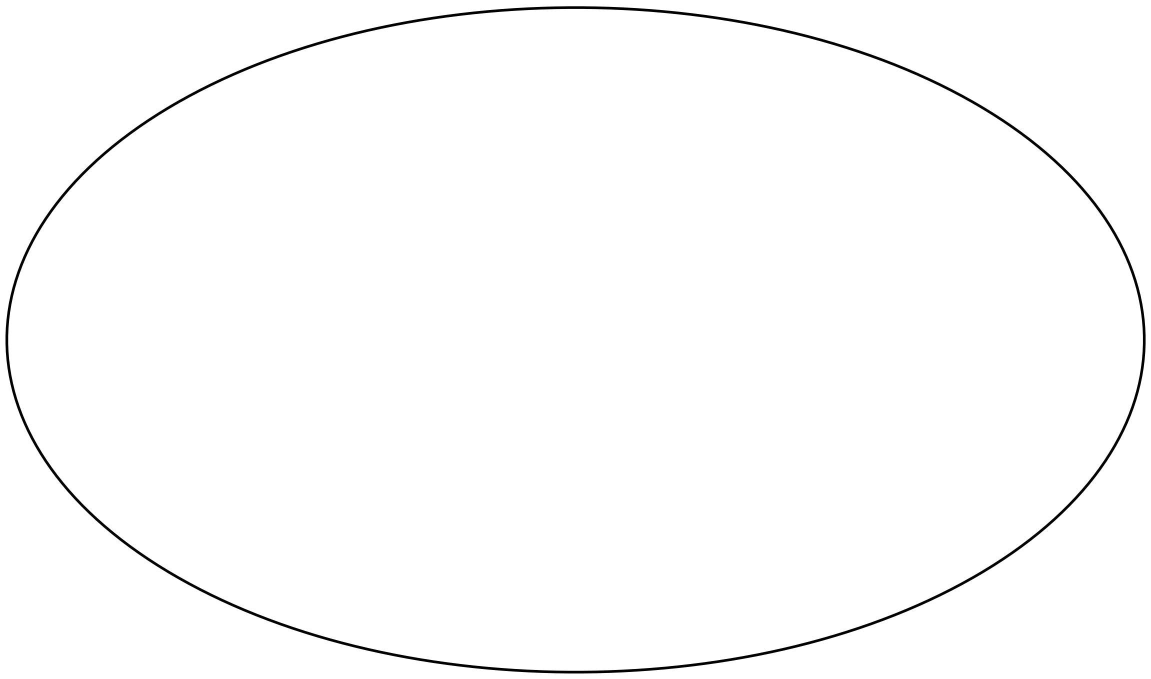

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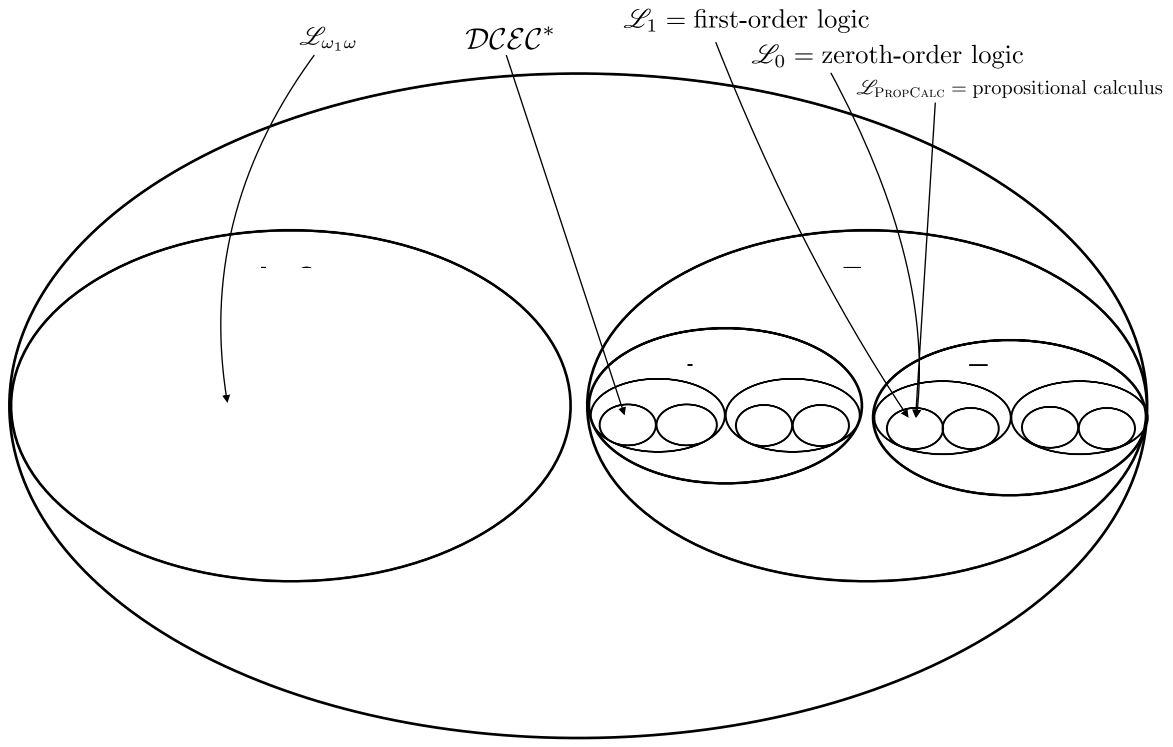

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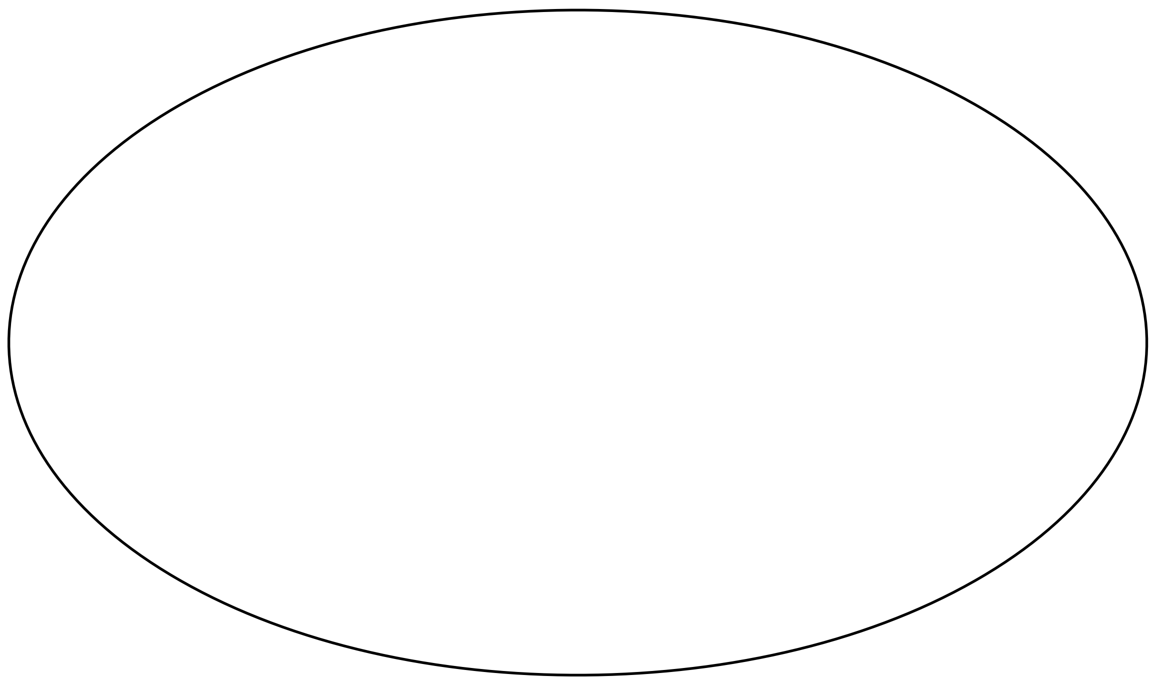

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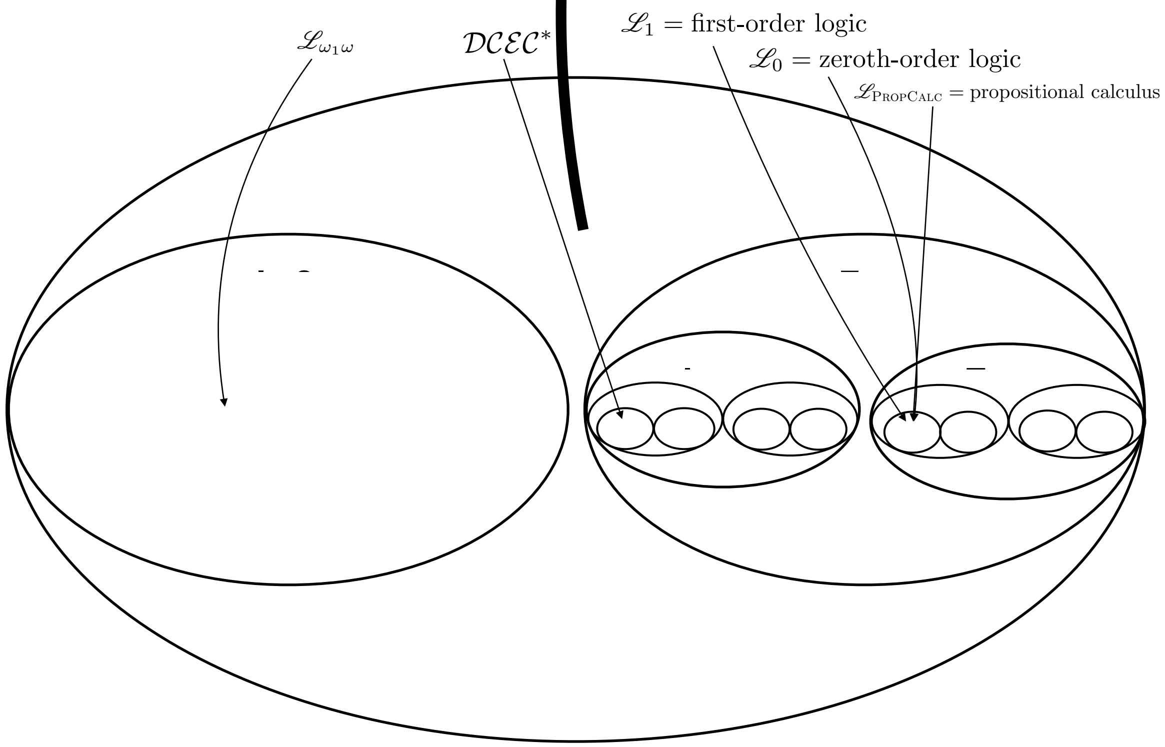
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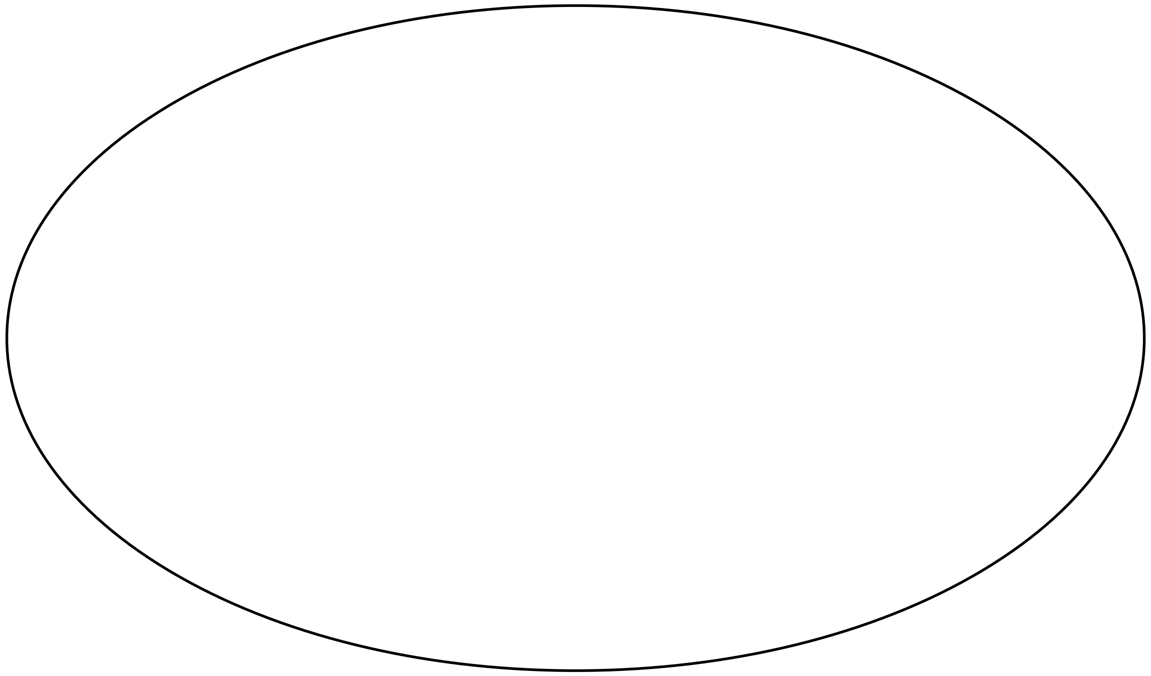

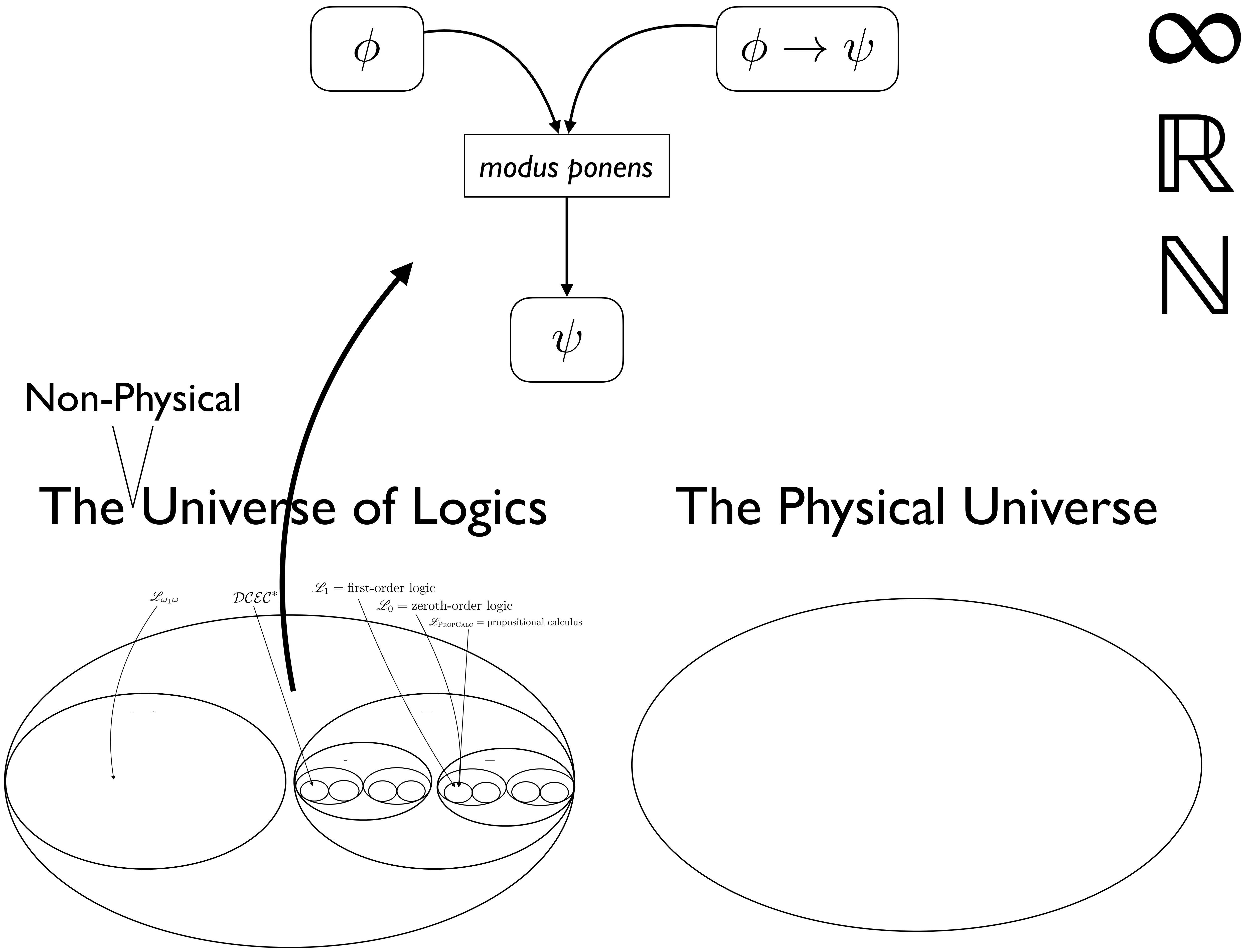

ϕ

$\phi \to \psi$

modus ponens

ψ

∞

$\mathbb{R}$

$\mathbb{N}$

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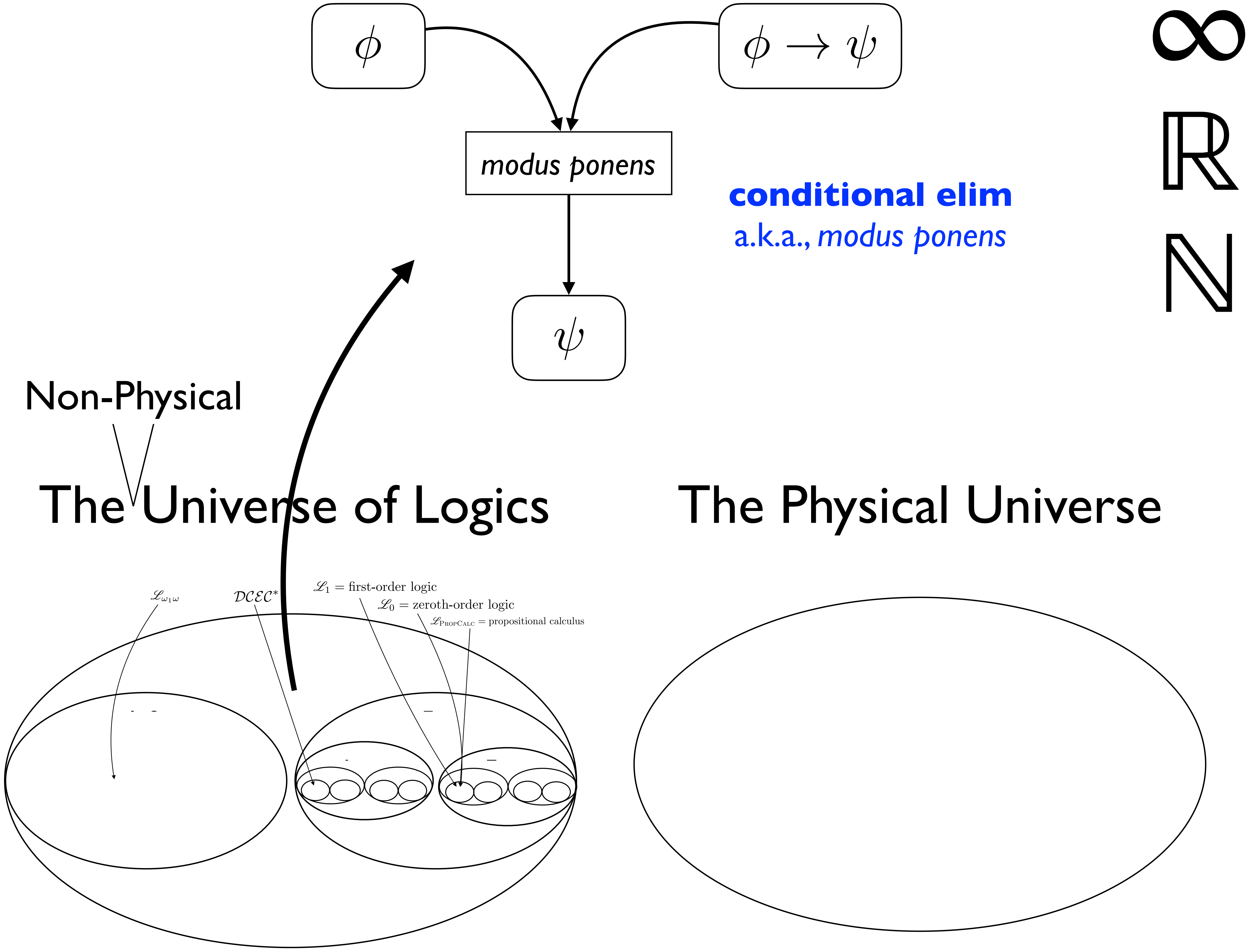

ϕ

$\phi \rightarrow \psi$

modus ponens

ψ

conditional elim

a.k.a., *modus ponens*

∞

ℝ

ℕ

Non-Physical

The Universe of Logics

The Physical Universe

$\mathscr{L}_{\omega_1\omega}$
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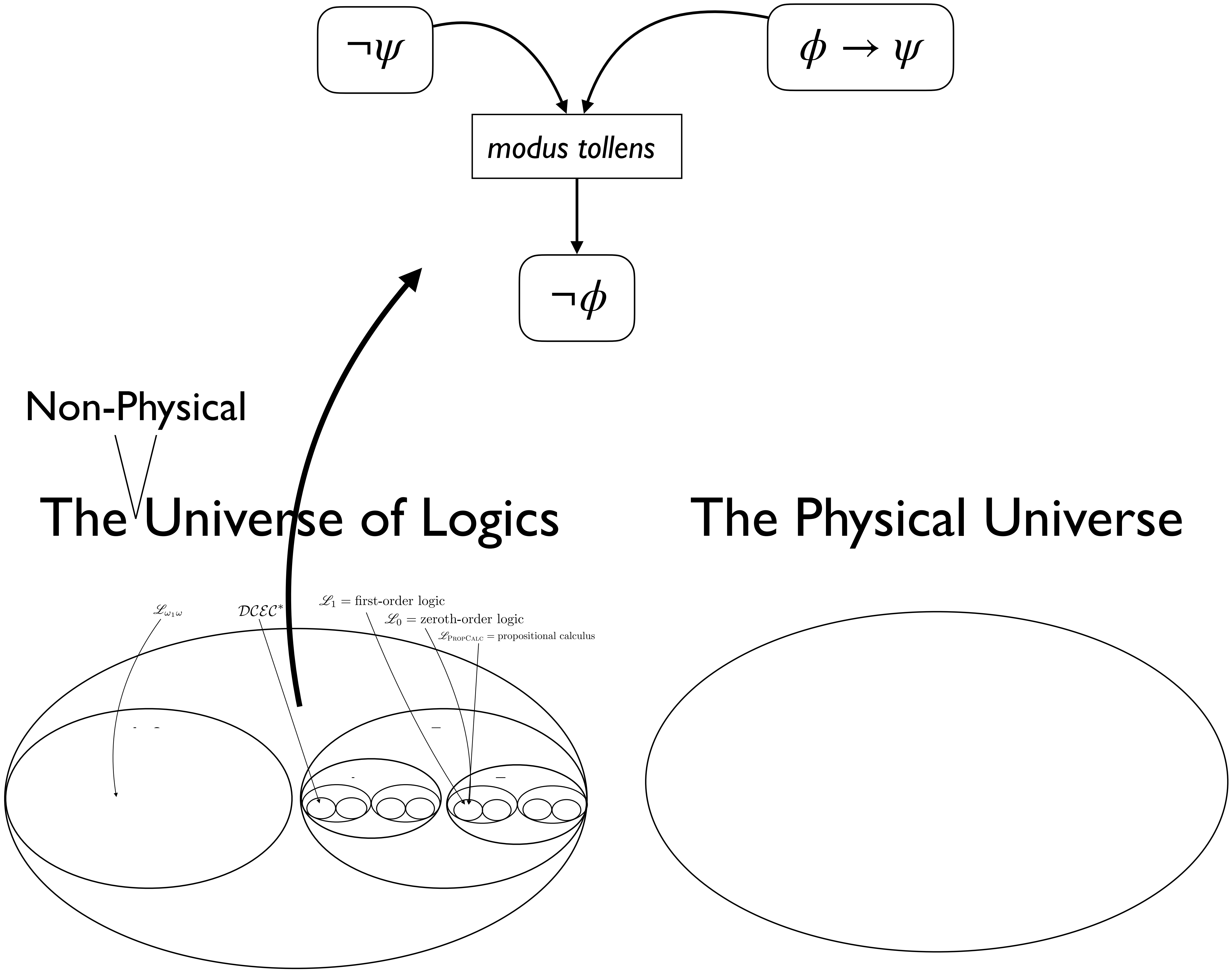

$\neg\psi$

$\phi \rightarrow \psi$

modus tollens

$\neg\phi$

Non-Physical

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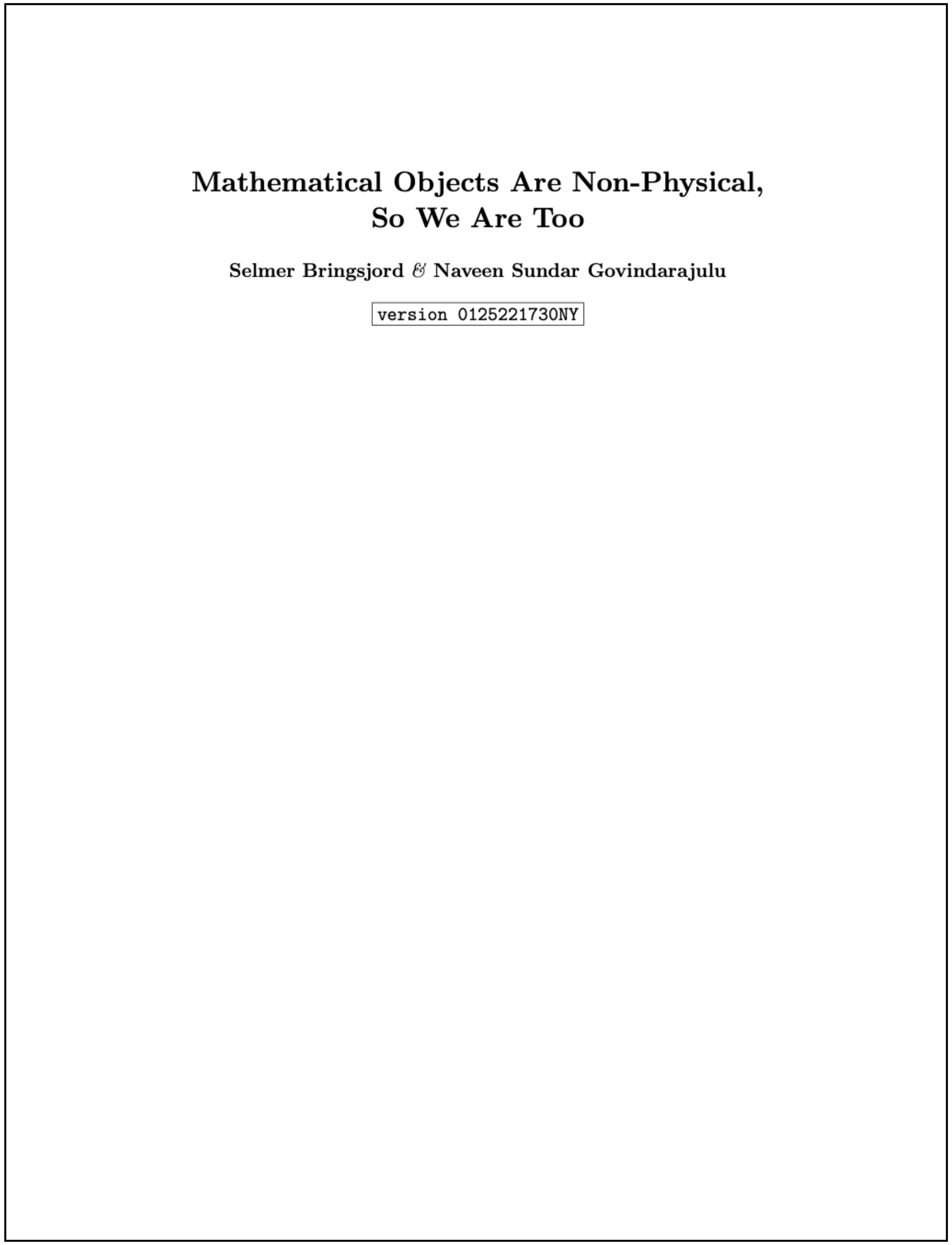

Mathematical Objects Are Non-Physical, So We Are Too

Selmer Bringsjord *&* **Naveen Sundar Govindarajulu**

`version 0125221730NY`

Mathematical Objects Are Non-Physical, So We Are Too

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itself is at bottom a simple recursive one. (There are now numerous variants, but we ignore this for efficiency.) The algorithm is to receive an array of ordered objects, for example

$$\left\langle \boxed{5} \;\; \boxed{9} \;\; \boxed{10} \;\; \boxed{7} \;\; \boxed{4} \;\; \boxed{3} \;\; \boxed{11} \;\; \boxed{8} \;\; \boxed{6} \right\rangle,$$

and to then produce as output the sorted version of this input, which in this case is:

$$\left\langle \boxed{3} \;\; \boxed{4} \;\; \boxed{5} \;\; \boxed{6} \;\; \boxed{7} \;\; \boxed{8} \;\; \boxed{9} \;\; \boxed{10} \;\; \boxed{11} \right\rangle.$$

So, what's the algorithm? In order to answer this question, we can't avoid resorting to what we can call *embodiments* or *tokens* of the general and abstract *type* $\mathcal{Q}$.[9] This terminology, and the associated concrete practice, is easy to grasp. For an example, we give one high-level embodiment/token $\hat{\mathcal{Q}}_1$ of $\mathcal{Q}$ that views the algorithm as a three-stage one.[10] Before supplying the example in question, we draw the reader's attention to what we just did with a bit of suggestive notation: We used "hat" $\hat{O}$ to indicate that what is being referred to an an embodiment of the thing O (in this case, of course, an algorithm). Hence, the hat in '$\hat{\mathcal{Q}}_1$' says that we have here an embodiment of the algorithm $\mathcal{Q}$ itself. Very well, and now to the embodiment in question itself:

I Pick the rightmost element in the array as the *pivot*.

II Partition the array so that all elements in the array less than the pivot are before it, and all elements greater than the pivot are placed after it.

III Recursively apply both I and II to the sub-array now before the pivot, as well as to the sub-array now after the pivot.

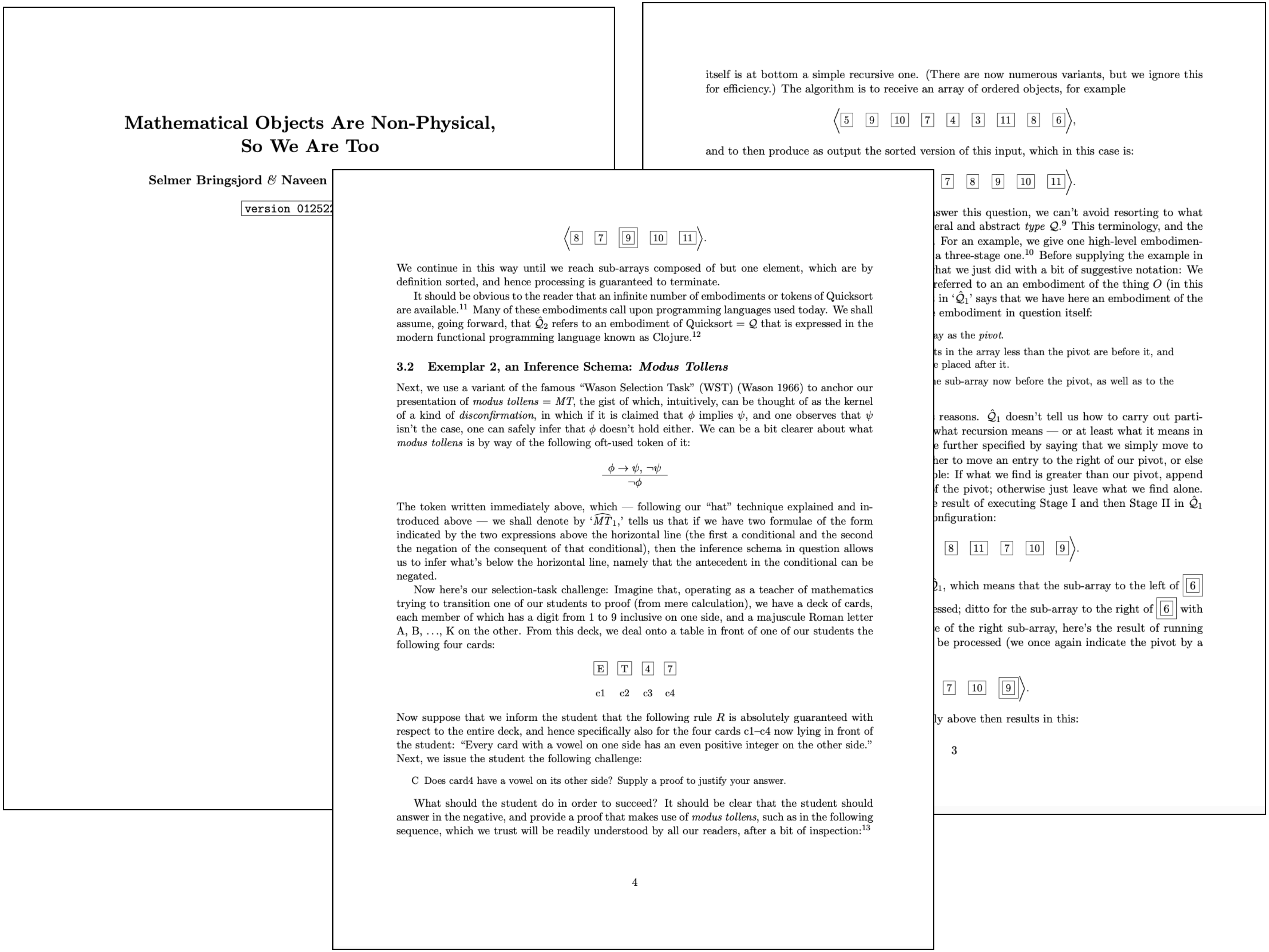
This is said to be 'high-level' for obvious reasons. $\hat{\mathcal{Q}}_1$ doesn't tell us how to carry out partitioning, and it relies on an understanding of what recursion means — or at least what it means in this context. But no worries: Stage II can be further specified by saying that we simply move to the left one entry at a time, and decide whether to move an entry to the right of our pivot, or else leave it where it is. And how to decide? Simple: If what we find is greater than our pivot, append it to whatever sub-sequence is to the right of the pivot; otherwise just leave what we find alone. Using a double-box to indicate our pivot, the result of executing Stage I and then Stage II in $\hat{\mathcal{Q}}_1$ on the initial input array will result in this configuration:

$$\left\langle \boxed{5} \;\; \boxed{4} \;\; \boxed{3} \;\; \boxed{\boxed{6}} \;\; \boxed{8} \;\; \boxed{11} \;\; \boxed{7} \;\; \boxed{10} \;\; \boxed{9} \right\rangle.$$

Now the algorithm calls for Stage III in $\hat{\mathcal{Q}}_1$, which means that the sub-array to the left of $\boxed{\boxed{6}}$ with $\boxed{3}$ as the pivot of this sub-array is processed; ditto for the sub-array to the right of $\boxed{\boxed{6}}$ with $\boxed{9}$ as the pivot of this sub-array. In the case of the right sub-array, here's the result of running Stage I, which is to be passed to Stage II to be processed (we once again indicate the pivot by a double-box):

$$\left\langle \boxed{8} \;\; \boxed{11} \;\; \boxed{7} \;\; \boxed{10} \;\; \boxed{\boxed{9}} \right\rangle.$$

Stage II applied to the input to it immediately above then results in this:



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`version 01252`

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and to then produce as output the sorted version of this input, which in this case is:

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We continue in this way until we reach sub-arrays composed of but one element, which are by definition sorted, and hence processing is guaranteed to terminate.

It should be obvious to the reader that an infinite number of embodiments or tokens of Quicksort are available.[11] Many of these embodiments call upon programming languages used today. We shall assume, going forward, that $\hat{\mathcal{Q}}_2$ refers to an embodiment of Quicksort $= \mathcal{Q}$ that is expressed in the modern functional programming language known as Clojure.[12]

3.2 Exemplar 2, an Inference Schema: *Modus Tollens*

Next, we use a variant of the famous "Wason Selection Task" (WST) (Wason 1966) to anchor our presentation of *modus tollens* $= MT$, the gist of which, intuitively, can be thought of as the kernel of a kind of *disconfirmation*, in which if it is claimed that ϕ implies ψ, and one observes that ψ isn't the case, one can safely infer that ϕ doesn't hold either. We can be a bit clearer about what *modus tollens* is by way of the following oft-used token of it:

$$\frac{\phi \rightarrow \psi,\ \neg\psi}{\neg\phi}$$

The token written immediately above, which — following our "hat" technique explained and introduced above — we shall denote by '$\widehat{MT}_1$,' tells us that if we have two formulae of the form indicated by the two expressions above the horizontal line (the first a conditional and the second the negation of the consequent of that conditional), then the inference schema in question allows us to infer what's below the horizontal line, namely that the antecedent in the conditional can be negated.

Now here's our selection-task challenge: Imagine that, operating as a teacher of mathematics trying to transition one of our students to proof (from mere calculation), we have a deck of cards, each member of which has a digit from 1 to 9 inclusive on one side, and a majuscule Roman letter A, B, . . ., K on the other. From this deck, we deal onto a table in front of one of our students the following four cards:

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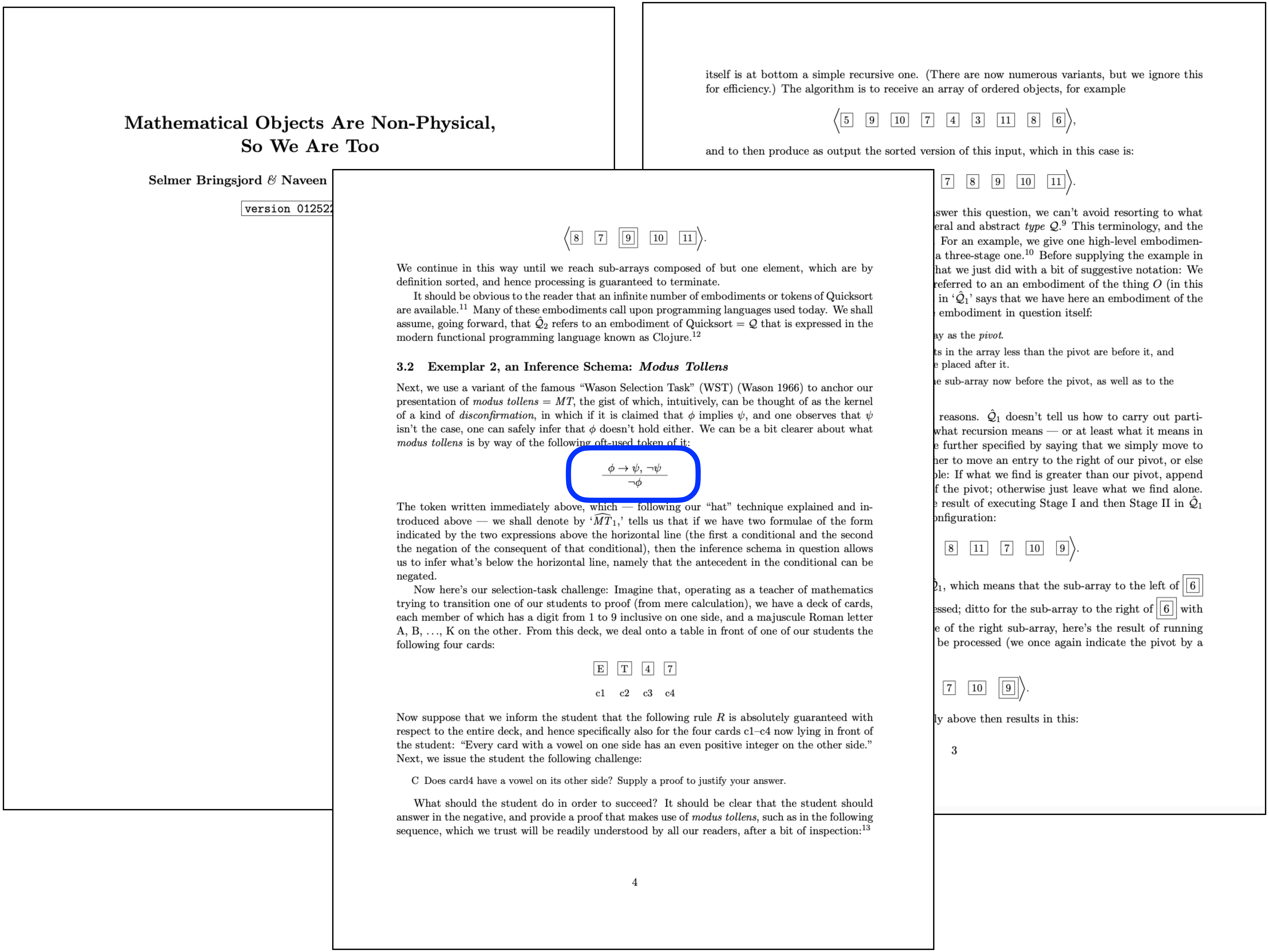

Mathematical Objects Are Non-Physical, So We Are Too

Selmer Bringsjord & Naveen

version 01252

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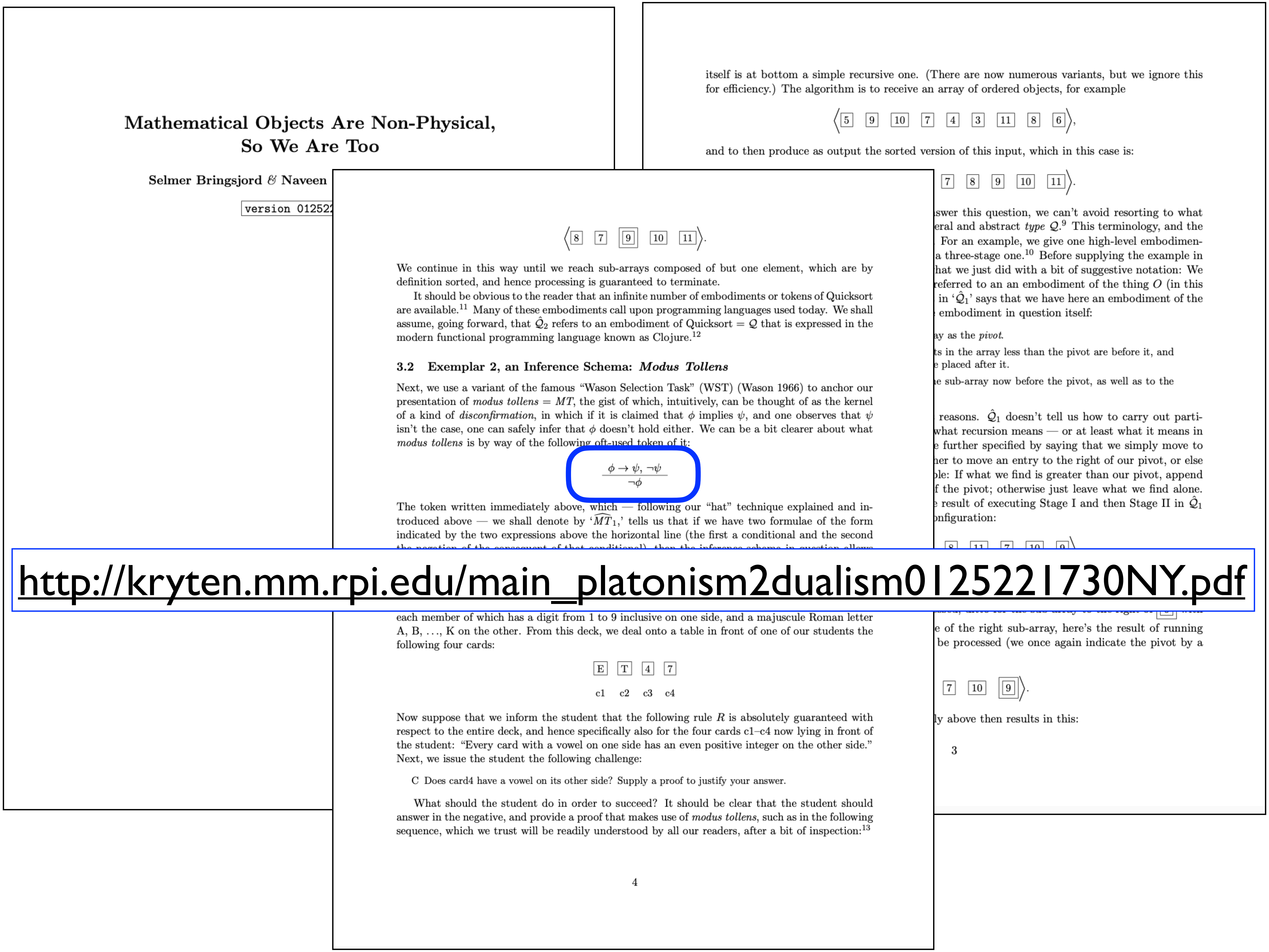

Mathematical Objects Are Non-Physical, So We Are Too

Selmer Bringsjord & Naveen

version 012522

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http://kryten.mm.rpi.edu/main_platonism2dualism0125221730NY.pdf

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4

Next problem (King-Ace) …

King-Ace 2

Suppose that the following premise is true:

If there is a king in the hand, then there is an ace in the hand; or if there isn't a king in the hand, then there is an ace; but not both of these if-then statements are true.

What can you infer from this premise?

King-Ace 2

Suppose that the following premise is true:

If there is a king in the hand, then there is an ace in the hand; or if there isn't a king in the hand, then there is an ace; but not both of these if-then statements are true.

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There is an ace in the hand.

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~~There is an ace in the hand.~~

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What can you infer from this premise?

NO! ~~There is an ace in the hand.~~

King-Ace 2

Suppose that the following premise is true:

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What can you infer from this premise?

NO! ~~There is an ace in the hand.~~ NO!

King-Ace 2

Suppose that the following premise is true:

If there is a king in the hand, then there is an ace in the hand; or if there isn't a king in the hand, then there is an ace; but not both of these if-then statements are true.

What can you infer from this premise?

NO! ~~There is an ace in the hand.~~ NO!

In fact, what you *can* infer is that there *isn't* an ace in the hand!

King-Ace Solved

Proposition: There is *not* an ace in the hand.

Proof: We know that at least one of the if-thens (i.e., at least one of the **conditionals**) is false. So we have two cases to consider, viz., that *K* => *A* is false, and that ¬*K* => *A* is false. Take first the first case; accordingly, suppose that *K* => *A* is false. Then it follows that *K* is true (since when a conditional is false, its antecedent holds but its consequent doesn't), and *A* is false. Now consider the second case, which consists in ¬*K* => *A* being false. Here, in a direct parallel, we know ¬*K* and, once again, ¬*A*. In both of our two cases, which are exhaustive, there is no ace in the hand. The proposition is established. **QED**

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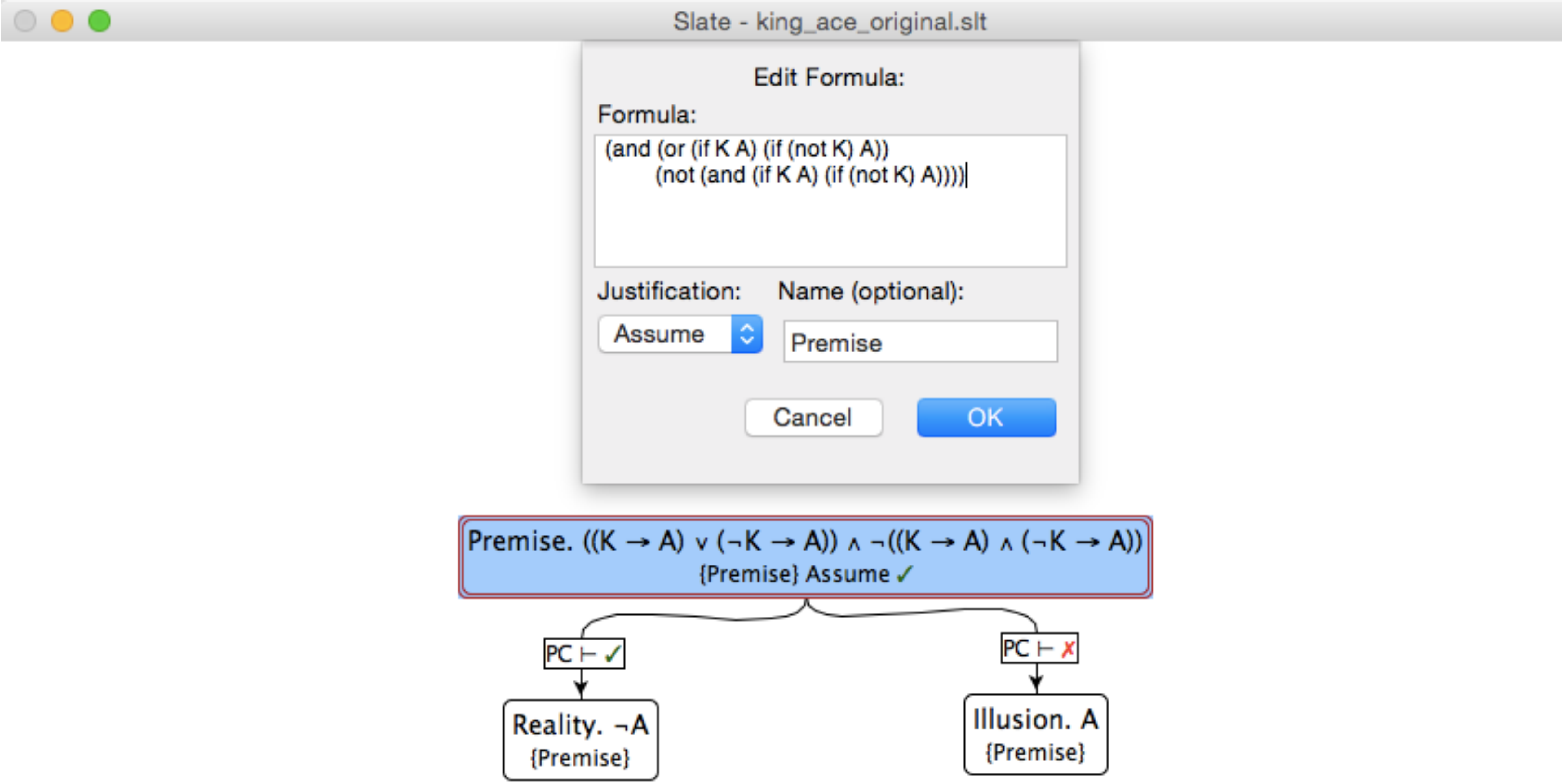


Slate - king_ace_original.slt

Edit Formula:

Formula:

```
(and (or (if K A) (if (not K) A))
      (not (and (if K A) (if (not K) A))))
```

Justification: Assume

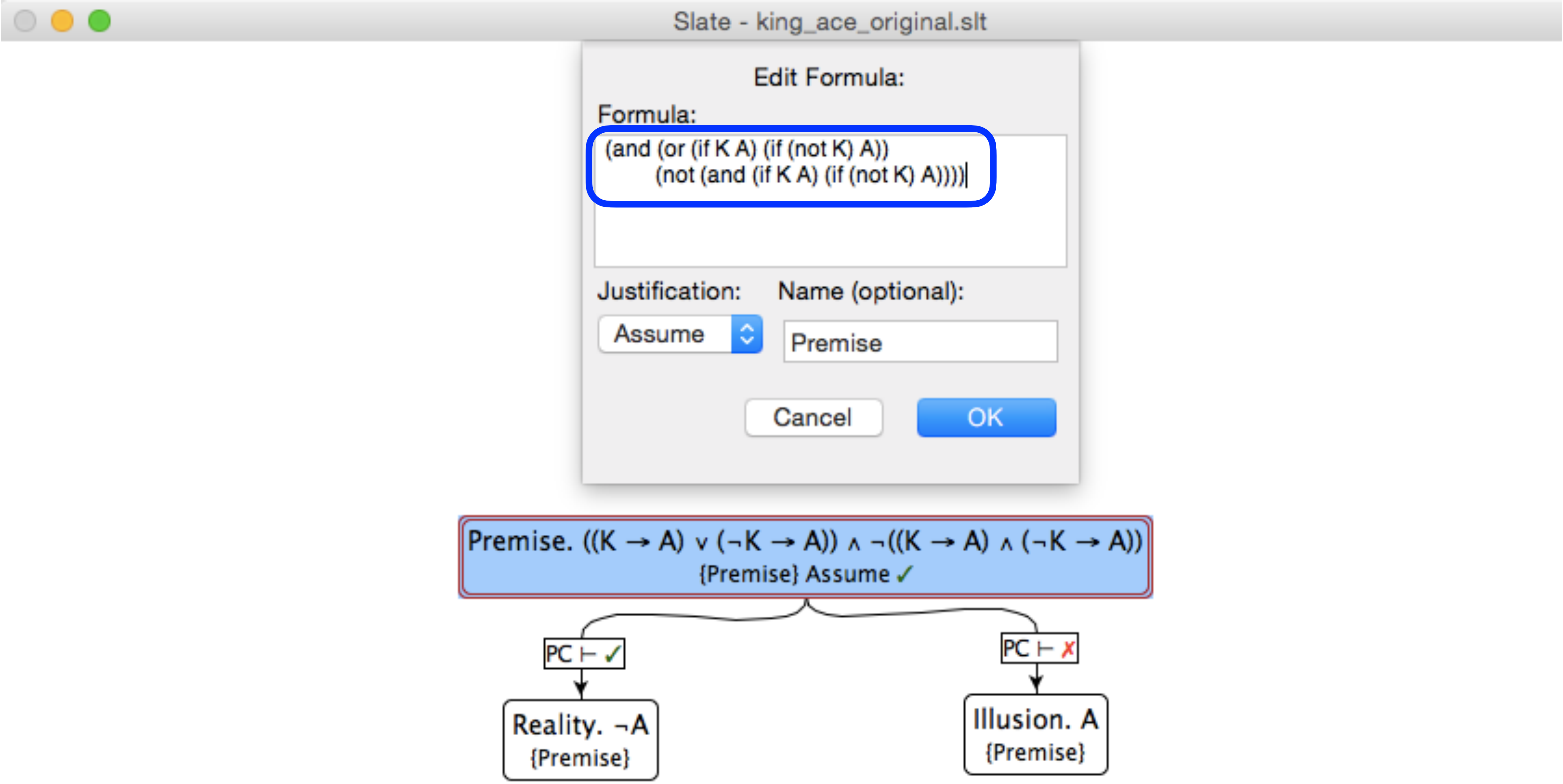
Name (optional): Premise

Premise. $((K \rightarrow A) \vee (\neg K \rightarrow A)) \wedge \neg((K \rightarrow A) \wedge (\neg K \rightarrow A))$
{Premise} Assume ✓

Reality. $\neg A$
{Premise}

Illusion. A
{Premise}

Study the S-expression



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We need another rule of inference to crack this problem … …

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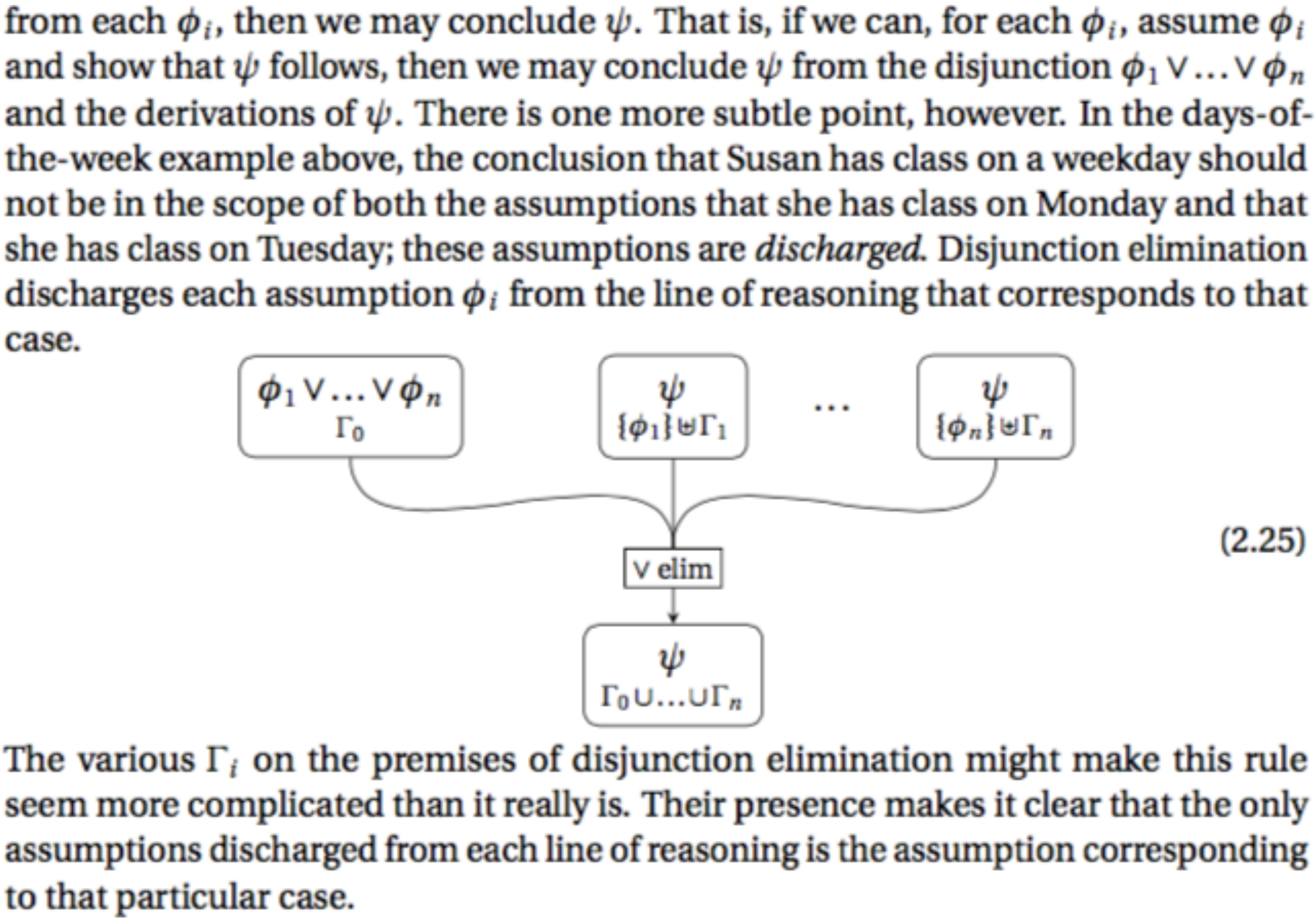
`disjunction elimination`

From ~ p. 54 in LAMA-BDLA



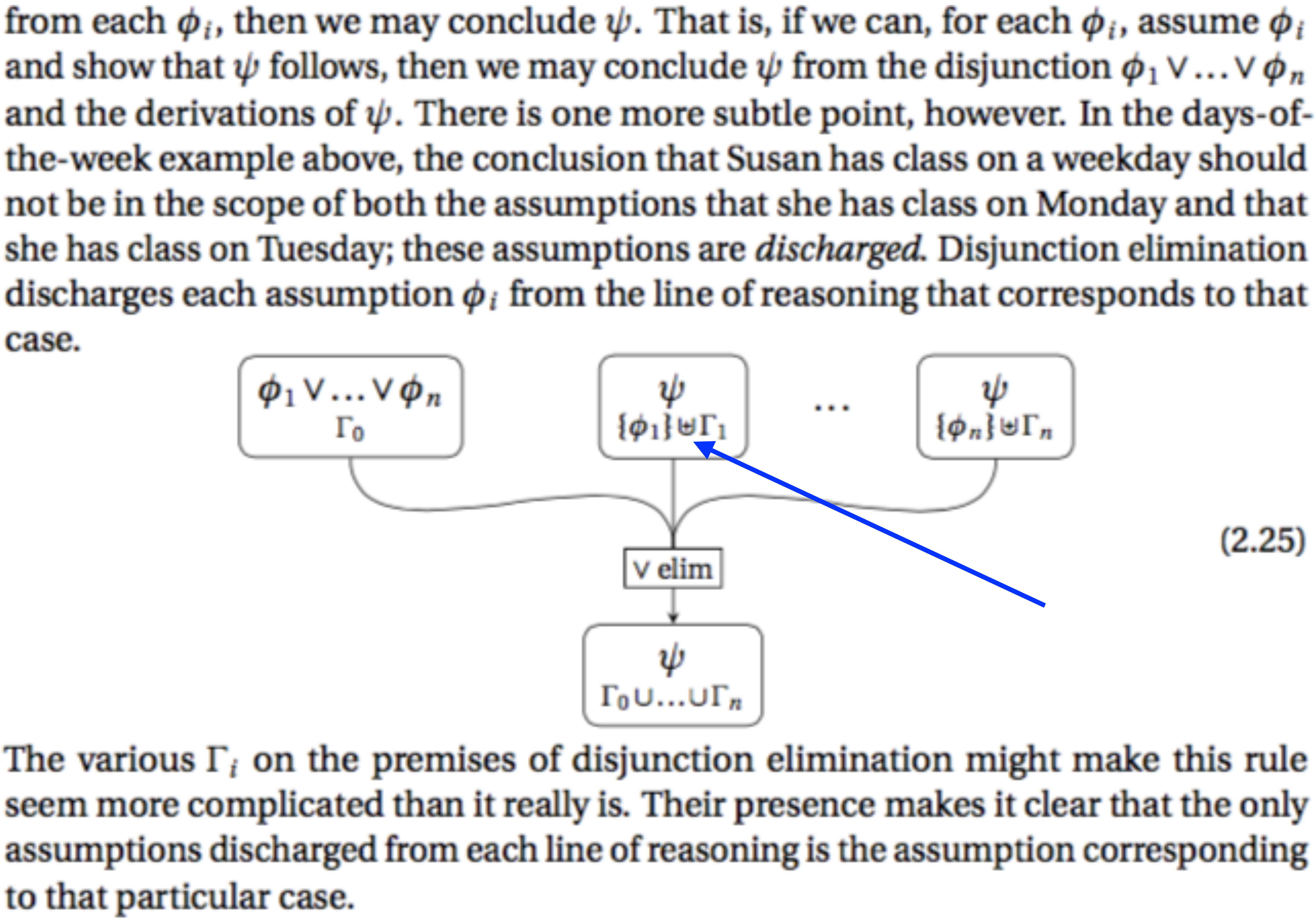
from each ϕ_i, then we may conclude ψ. That is, if we can, for each ϕ_i, assume ϕ_i and show that ψ follows, then we may conclude ψ from the disjunction $\phi_1 \vee \ldots \vee \phi_n$ and the derivations of ψ. There is one more subtle point, however. In the days-of-the-week example above, the conclusion that Susan has class on a weekday should not be in the scope of both the assumptions that she has class on Monday and that she has class on Tuesday; these assumptions are *discharged*. Disjunction elimination discharges each assumption ϕ_i from the line of reasoning that corresponds to that case.

$\phi_1 \vee \ldots \vee \phi_n$
Γ_0
ψ
$\{\phi_1\} \uplus \Gamma_1$
$\ldots$
ψ
$\{\phi_n\} \uplus \Gamma_n$
$\vee$ elim
ψ
$\Gamma_0 \cup \ldots \cup \Gamma_n$

(2.25)

The various Γ_i on the premises of disjunction elimination might make this rule seem more complicated than it really is. Their presence makes it clear that the only assumptions discharged from each line of reasoning is the assumption corresponding to that particular case.

From ~ p. 54 in LAMA-BDLA



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$\phi_1 \vee \ldots \vee \phi_n$
Γ_0
ψ
$\{\phi_1\} \uplus \Gamma_1$
…
ψ
$\{\phi_n\} \uplus \Gamma_n$
∨ elim
ψ
$\Gamma_0 \cup \ldots \cup \Gamma_n$

(2.25)

The various Γ_i on the premises of disjunction elimination might make this rule seem more complicated than it really is. Their presence makes it clear that the only assumptions discharged from each line of reasoning is the assumption corresponding to that particular case.

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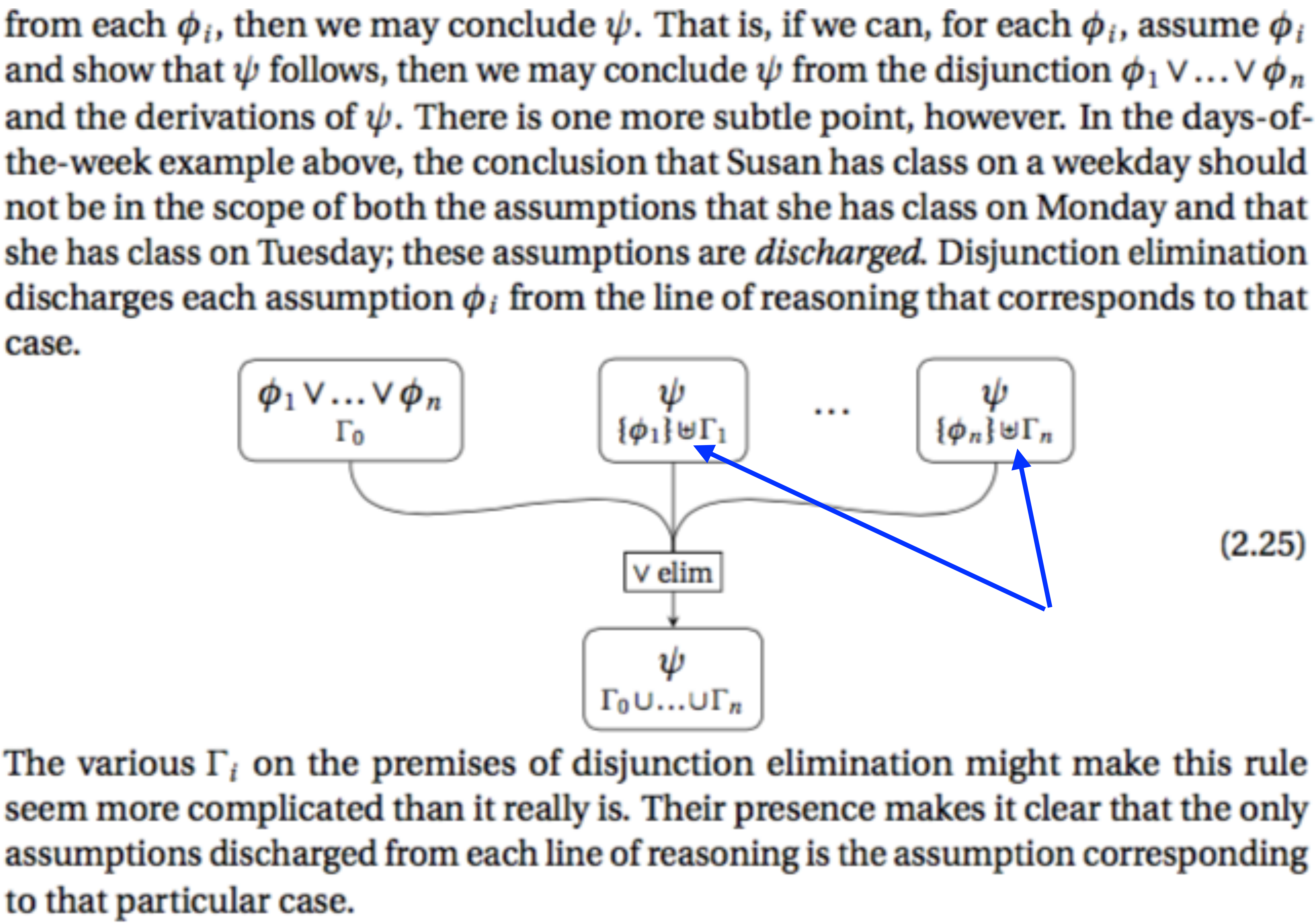

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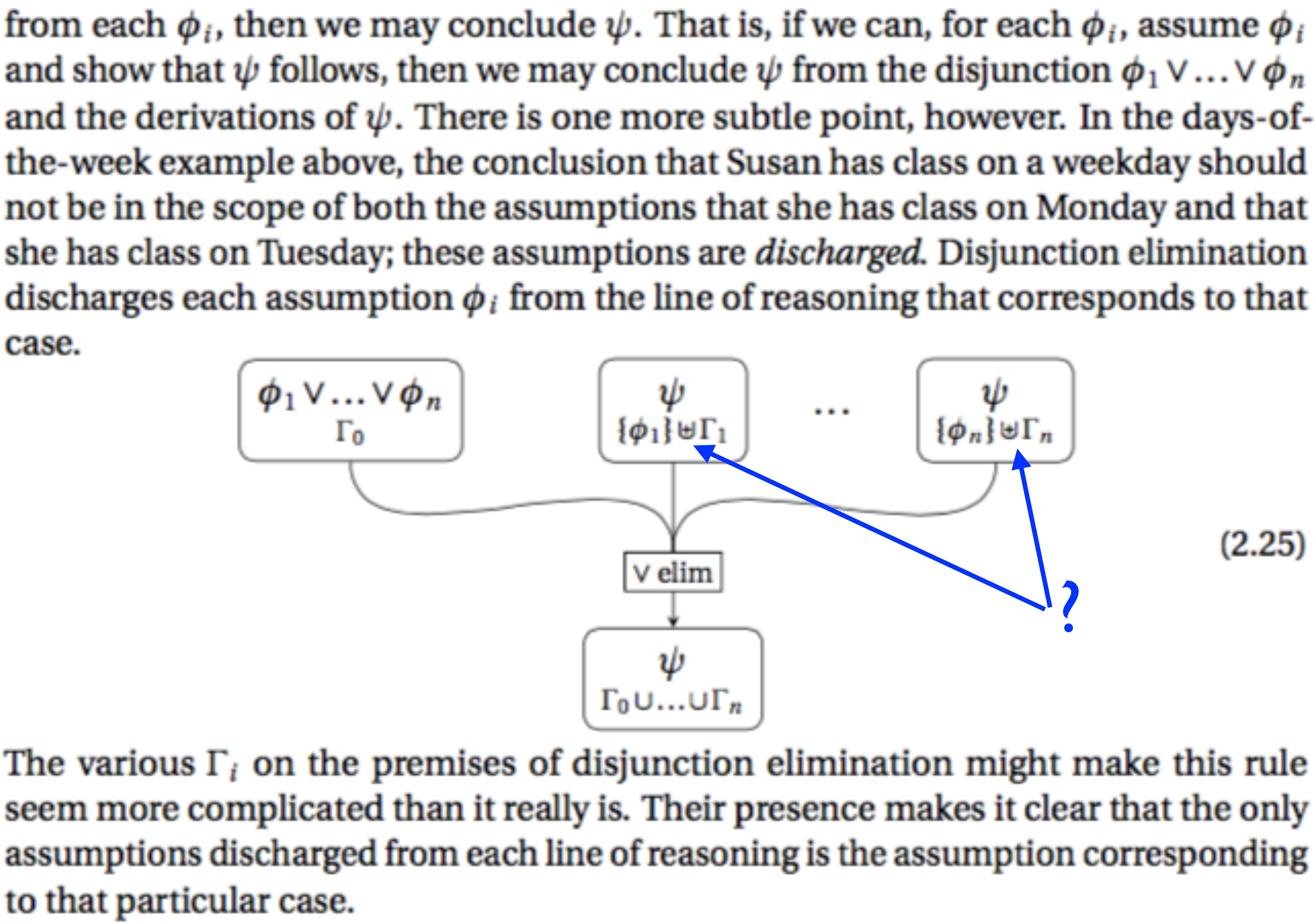

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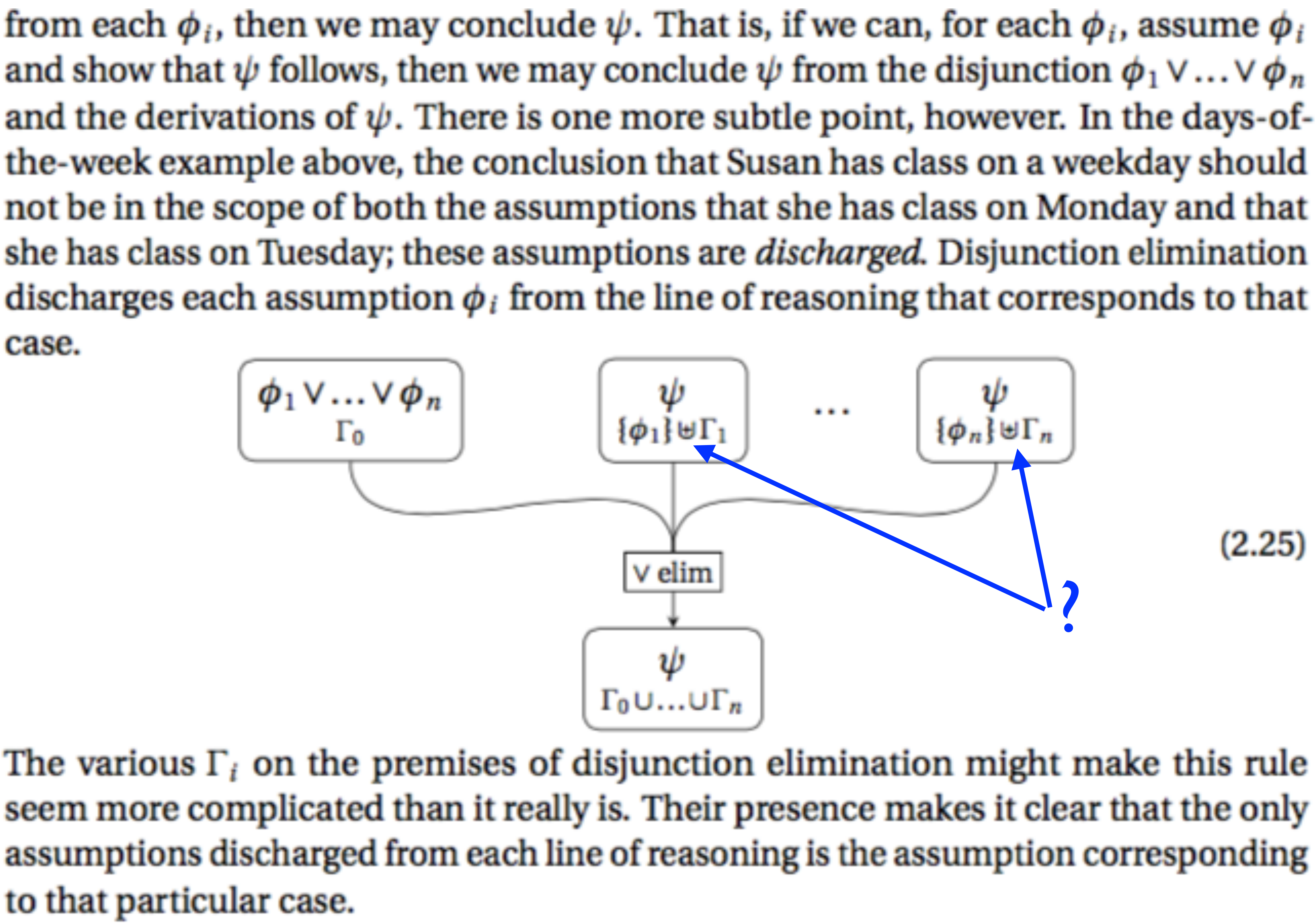

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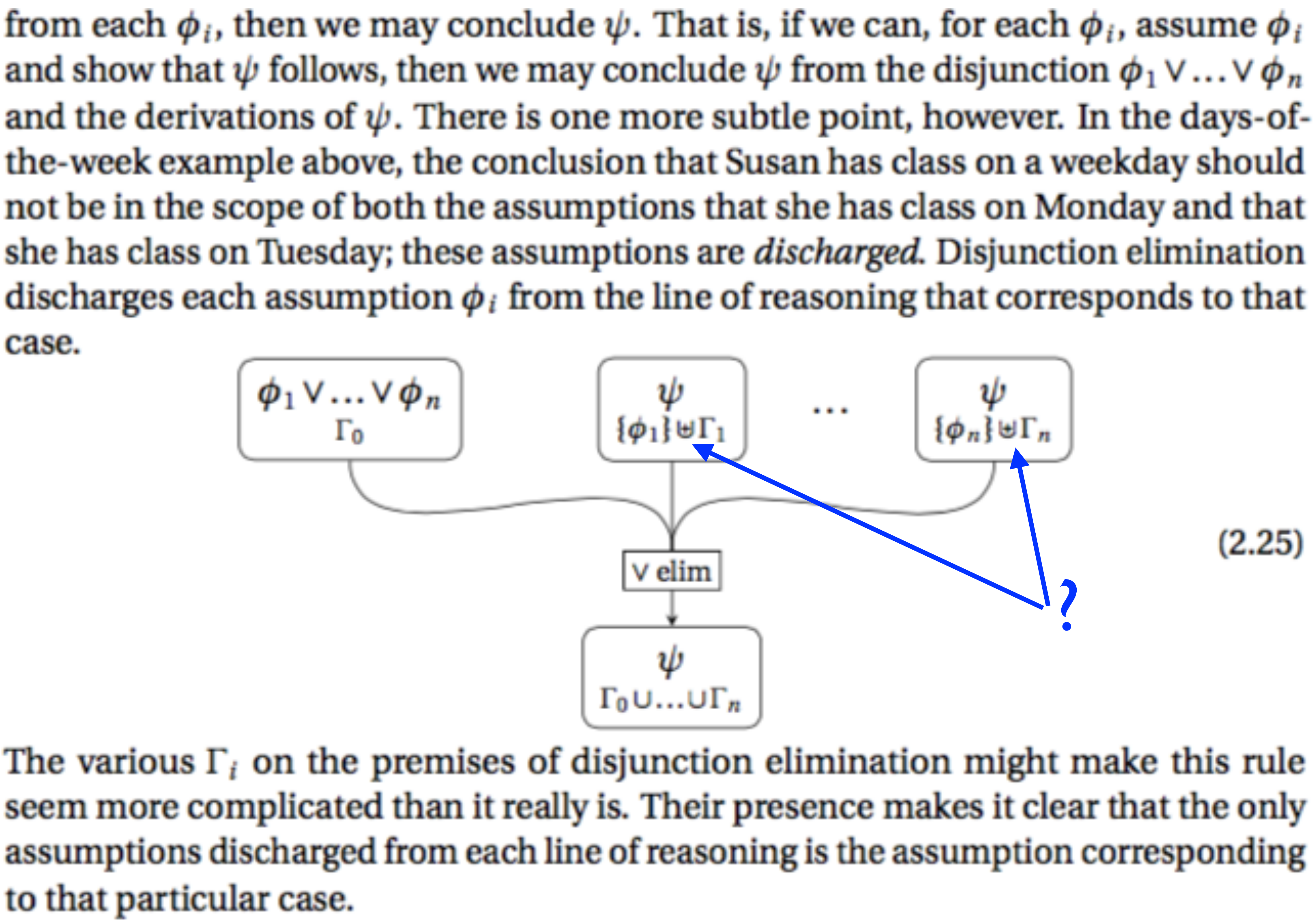

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King-Ace 2

Suppose that the following premise is true:

If there is a king in the hand, then there is an ace in the hand; or if there isn't a king in the hand, then there is an ace; but not both of these if-then statements are true.

What can you infer from this premise?

NO! ~~There is an ace in the hand.~~ NO!

In fact, what you *can* infer is that there *isn't* an ace in the hand!

Future Required problem (on HyperGrader®): You will need to finish the proof in HyperSlate® — with no remaining use of an oracle.

Det er en ære å lære formell logikk!

(Det er en ære å lære formell logikk!)

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Further hands-on interaction in Part II of Class?