Review of Indirect Proof & Tertium Non Datur; Rebuilding the Foundations of Math via (the “Theory”) ZFC; ~~ZFC to Axiomatized Arithmetic (the “Theories” BA and PA)~~

Selmer Bringsjord

Rensselaer AI & Reasoning (RAIR) Lab
Department of Cognitive Science
Department of Computer Science
Lally School of Management & Technology
Rensselaer Polytechnic Institute (RPI)
Troy, New York 12180 USA

IFLAI1
3/16/2026

$1000 Problem From Last Time

From Ben Joffe:

≠

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Dear Prof. Selmer,

Find below my attempt at capturing finitude in First Order Logic; I say attempt, as I peeked in the textbook that this is firmly impossible without traversing to an infinite version of FOL, and I'm not going to refute the textbook itself on this subject.

≠

```
(exists (x)
        (and (and \phi_xOverline
                  \phi_xUnderline)
             (not (forall y
                          (and (= y x)
                               (not (and \phi_yOverline
                                         \phi_yUnderline)))))))
```

$\exists x: (\varphi_{xOverline} \land \varphi_{xUnderline}) \land \neg(\forall y: (y = x) \land \neg(\varphi_{yOverline} \land \varphi_{yUnderline}))$

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$\mathscr{L}_{\omega_1\omega}$ is coming . . .

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AI & The News, One Year Back …

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Sam Altman's Other Startup Is Building an App to Compete With Elon Musk's X

The CEO of OpenAI imagines a future where you'll need to constantly demonstrate that you're not a robot. His 'everything app' is the answer—but first, he needs to look deep into your eyes.

ILLUSTRATION: DANIEL HERTZBERG

By Christopher Mims Follow

Mar 07, 2025 09:00 p.m. ET

AI & The News, One Year Back …

Imagine a world full of basketball-sized "Orbs" that stare deep into our eyes, capturing the unique pattern of our irises.

These ubiquitous Orbs would allow us to do anything requiring identification, online or in real life, from buying bread to paying taxes. It's a vision reminiscent of other recent efforts—including Amazon's attempt to replace credit cards with our palms, and Ant Group's efforts in China to make it possible to pay with your face.

The big difference? The builders of an app called World—including Chief Executive Alex Blania and his co-founder Sam Altman of OpenAI fame—envision a time in the not-too-distant future when you can't do much *without* an ocular check-in. AI agents will be so prevalent, and so humanlike, that we'll need to repeatedly prove we're real to prevent those AIs from masquerading as humans on everything from payment platforms to social networks.

To accelerate adoption of what World calls its "anonymous proof-of-human" system, the company recently launched a mini app store inside its app, which is available for iPhones and Android devices.

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Discussion:
Literally Now a Sub-Discipline of AI …

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https://youtu.be/Umc9ezAyJv0

AI & The News, Now…

Sam Altman's 'Human Verification' Startup Leans on Consumer Brands

Tools for Humanity aims to promote World ID beyond the tech and crypto sets via partnerships with household names

Tools for Humanity Chief Business Officer Trevor Traina speaks at the opening of an ID verification location in San Francisco. NOAH BERGER/ASSOCIATED PRESS

By Patrick Coffee

Feb. 26, 2026 1:47 pm EST

AI & The News, Now…

Sam Altman's 'Human

Quick Summary

- Tools for Humanity, co-founded by Sam Altman, is partnering with brands like Gap, Visa, and Tinder to market its World ID human verification product.
- The company's World ID system uses Orbs to translate images of users' faces and irises into anonymized numbers stored on a user's device.
- Nearly 18 million people globally are now verified, but World ID has faced regulatory hurdles in some countries as well as New York state.

Tools for Humanity Chief Business Officer Trevor Traina speaks at the opening of an ID verification location in San Francisco. NOAH BERGER/ASSOCIATED PRESS

By Patrick Coffee

Feb. 26, 2026 1:47 pm EST

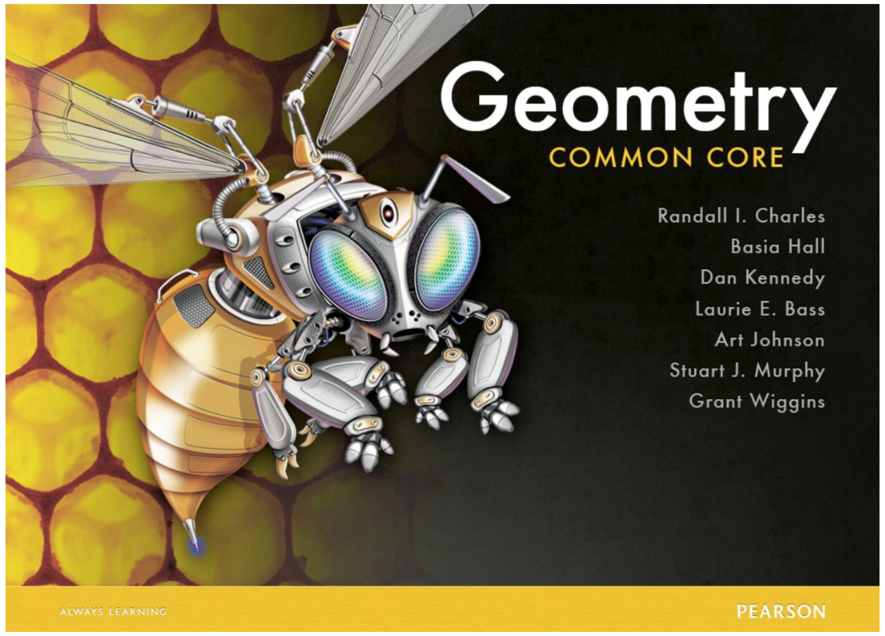
AI & The News, Now…

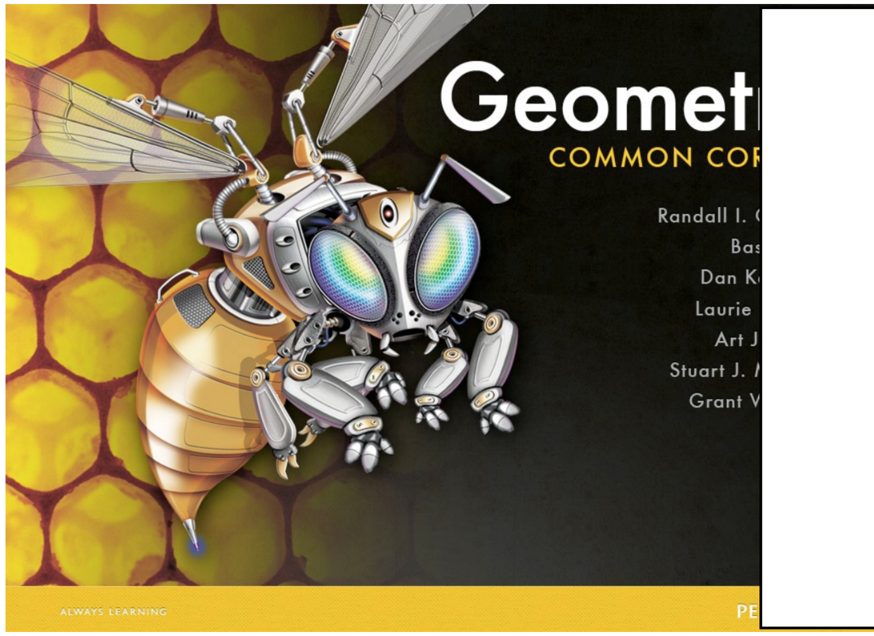
Sam Altman's project to help humans distinguish themselves from bots is increasingly banking on household names to sell its far-out concept.

A Gap store in San Francisco has begun helping visitors get World IDs, the "proof of human" product from the startup Tools for Humanity, by installing one of its signature volleyball-sized "Orb" devices to take images of faces and eyes.

A planned Visa V 0.98% ↑ payment card will let World ID holders spend digital assets including Worldcoin, the cryptocurrency that people receive in most markets as an incentive to sign up.

And the dating app Tinder is testing the ID in Japan to verify that users are human—and really the age they say they are.


Geometry
COMMON CORE
Randall I. Charles
Basia Hall
Dan Kennedy
Laurie E. Bass
Art Johnson
Stuart J. Murphy
Grant Wiggins
ALWAYS LEARNING
PEARSON



A proof involving indirect reasoning is an **indirect proof.** Often in an indirect proof, a statement and its negation are the only possibilities. When you see that one of these possibilities leads to a conclusion that contradicts a fact you know to be true, you can eliminate that possibility. For this reason, indirect proof is sometimes called *proof by contradiction*.

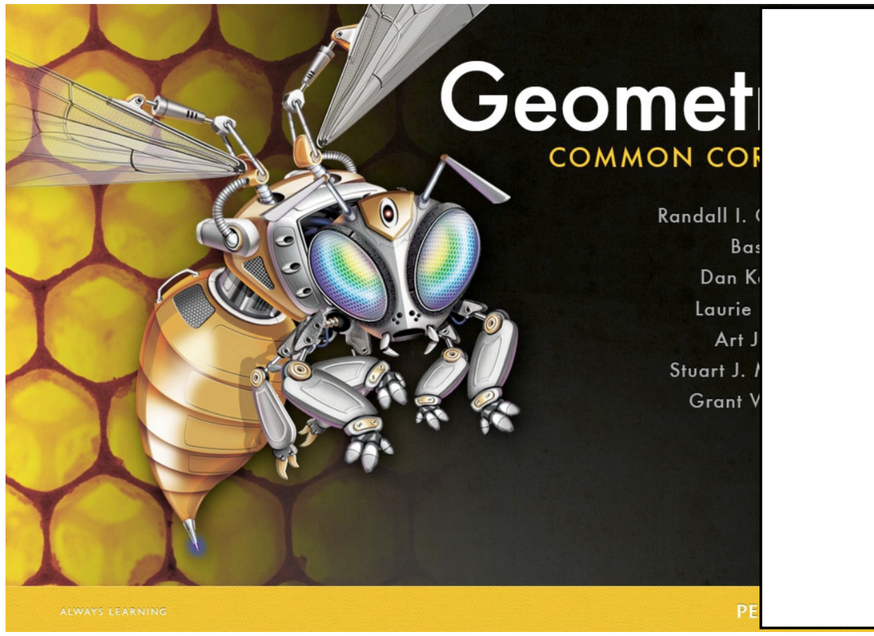
TAKE NOTE Key Concept

Writing an Indirect Proof

Step 1 State as a temporary assumption the opposite (negation) of what you want to prove.

Step 2 Show that this temporary assumption leads to a contradiction.

Step 3 Conclude that the temporary assumption must be false and that what you want to prove must be true.



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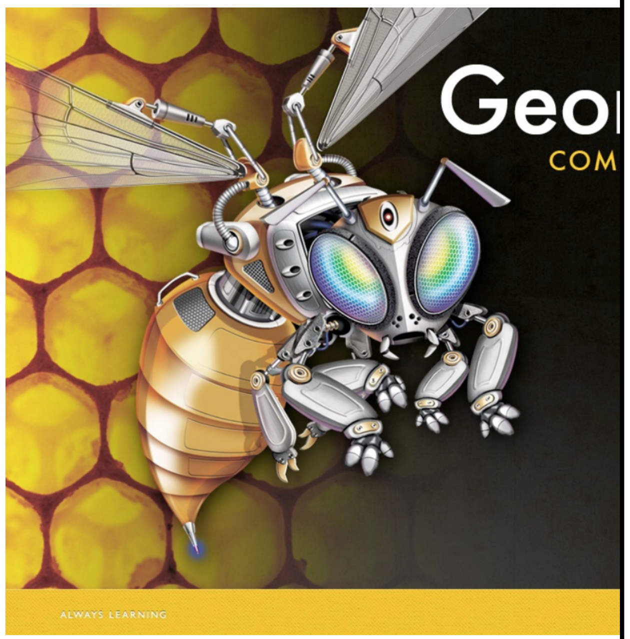
Problem 3 Writing an Indirect Proof

Proof

Given: $\triangle ABC$ is scalene.

Prove: $\angle A$, $\angle B$, and $\angle C$ all have different measures.

THINK	WRITE
Assume temporarily the opposite of what you want to prove.	Assume temporarily that two angles of $\triangle ABC$ have the same measure. Assume that $m\angle A = m\angle B$.
Show that this assumption leads to a contradiction.	By the Converse of the Isosceles Triangle Theorem, the sides opposite $\angle A$ and $\angle B$ are congruent. This contradicts the given information that $\triangle ABC$ is scalene.
Conclude that the temporary assumption must be false and that what you want to prove must be true.	The assumption that two angles of $\triangle ABC$ have the same measure must be false. Therefore, $\angle A$, $\angle B$, and $\angle C$ all have different measures.



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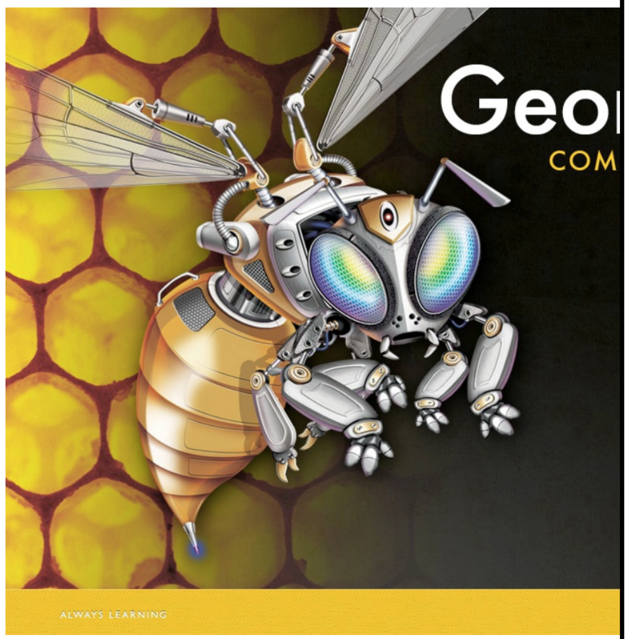

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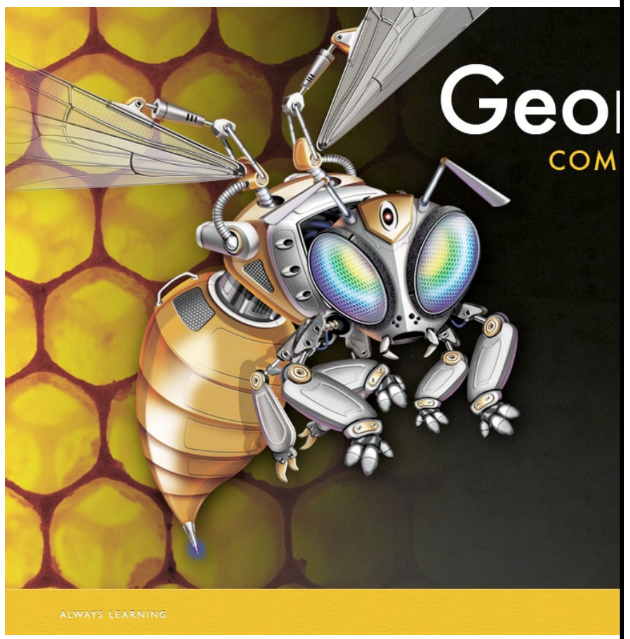

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Edit Problems

Leaderboard for Required

Required

Metrics for Required

TertiumNonDaturIFLAI1S24

The theorem to be proved here is *tertium non datur*, a.k.a. The Law of the Excluded Middle; you will need to prove this: $\vdash \phi \vee \neg\phi$. For some edifying supplementary reading, provided for the motivated, consult the SEP entry on Contradiction.

Deadline April 17, 2025 at 11:59 PM EDT

Reviewing the situation

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Types of Paradoxes

- Deductive Paradoxes
- Inductive Paradoxes — coming (e.g. The Lottery Paradox & The St Petersburg Paradox)

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https://www.megamillions.com

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1 in 302,575,350

Friday's Hill, Haslemere, 16 June 1902

Dear colleague,

For a year and a half I have been acquainted with your *Grundgesetze der Arithmetik*, but it is only now that I have been able to find the time for the thorough study I intended to make of your work. I find myself in complete agreement with you in all essentials, particularly when you reject any psychological element [[Moment]] in logic and when you place a high value upon an ideography [[Begriffsschrift]] for the foundations of mathematics and of formal logic, which, incidentally, can hardly be distinguished. With regard to many particular questions, I find in your work discussions, distinctions, and definitions that one seeks in vain in the works of other logicians. Especially so far as function is concerned (§ 9 of your *Begriffsschrift*), I have been led on my own to views that are the same even in the details. There is just one point where I have encountered a difficulty. You state (p. 17 [[p. 23 above]]) that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer its opposite follows. Therefore we must conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [[Menge]] does not form a totality.

I am on the point of finishing a book on the principles of mathematics and in it I should like to discuss your work very thoroughly.[1] I already have your books or shall buy them soon, but I would be very grateful to you if you could send me reprints of your articles in various periodicals. In case this should be impossible, however, I will obtain them from a library.

The exact treatment of logic in fundamental questions, where symbols fail, has remained very much behind; in your works I find the best I know of our time, and therefore I have permitted myself to express my deep respect to you. It is very regrettable that you have not come to publish the second volume of your *Grundgesetze*; I hope that this will still be done.

Very respectfully yours,

BERTRAND RUSSELL

The above contradiction, when expressed in Peano's ideography, reads as follows:

$$w = \mathrm{cls} \cap x \ni (x \sim\varepsilon\, x) .\supset: w\, \varepsilon\, w .=. w \sim\varepsilon\, w.$$

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http://plato.stanford.edu/entries/russell-paradox/#HOTP

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HyperGrader™ Problems HyperSlate My Submissions Leader Board Selmer.Bringsjord@gmail.com

FregTHEN2
KnightKnave_SmullyanKKProblem1.1
AthenCfromAthenBandBthenC
BiconditionalIntroByChaining
BogusBiconditional
CheatersNeverPropser
Contrapositive_NYS_2
Disj_Syll
GreenCheeseMoon2
HypSyll
LarryIsSomehowSmart
Modus_Tollens
RussellsLetter2Frege
ThxForThePCOracle
Explosion
OnlyMediumOrLargeLlamas
GreenCheeseMoon1
Disj_Elim
kok13_28
KingAce2
kok_13_31

☑ RussellsLetter2Frege

The challenge here is to prove that from Russell's instantiation of Frege's doomed Axiom V a contradiction can be promptly derived. The letter has of course been examined in some detail by S Bringsjord (in the Mar 16 2020 lecture in the 2020 lecture lineup); it, along with an astoundingly soft-spoken reply from Frege, can be found here. Put meta-logically, your task in the present problem is to build a proof that confirms this:

$$\{\exists x \forall y((y \in x) \rightarrow (y \notin y))\} \vdash \zeta \wedge \neg\zeta.$$

Make sure you understand that the given here is an instantiation of Frege's Axiom V; i.e. it's an instantiation of

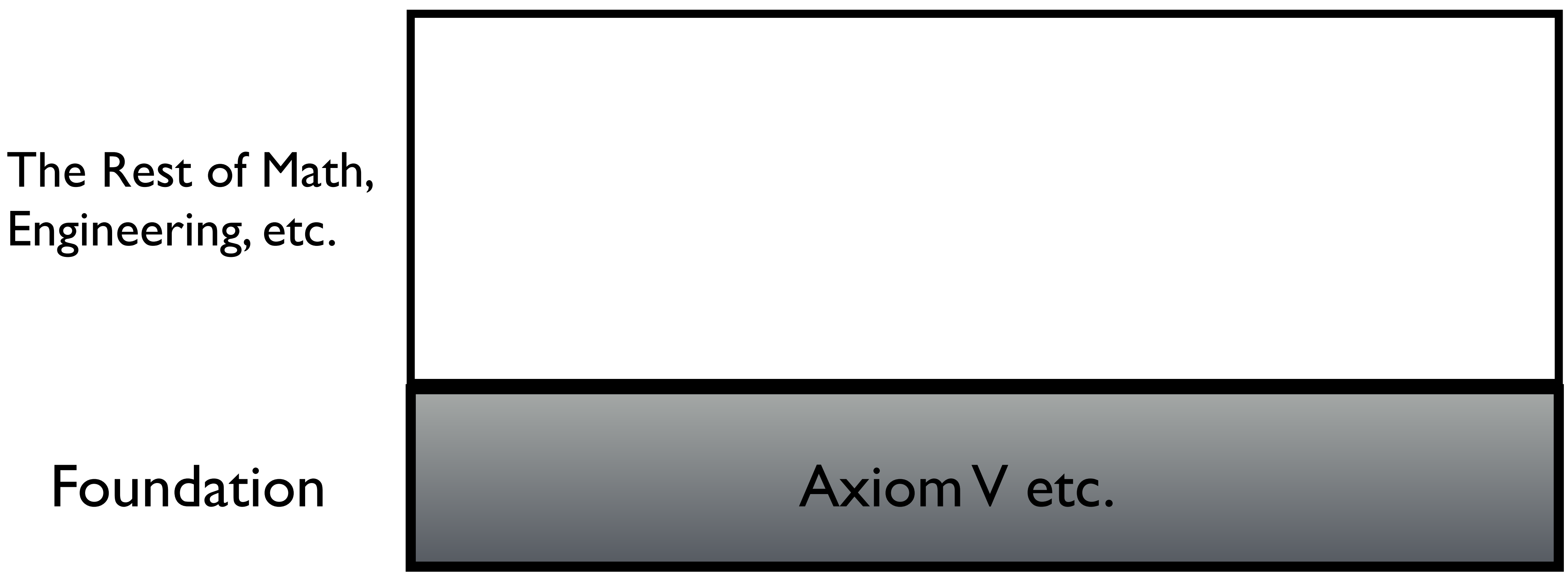
$$\exists x \forall y((y \in x) \rightarrow \phi(y)).$$

(The notation $\phi(y)$, recall, is the standard way in mathematical logic to say that y is free in ϕ.) **Note**: Your finished proof is allowed to make use the PC-provability oracle (but *only* that oracle).

(Now a brief remark on matters covered by in class by Bringsjord when second-order logic = $\mathscr{L}_2$ arrives on the scene: Longer term, and certainly constituting evidence of Frege's capacity for ingenius, intricate deduction, it has recently been realized that while Frege himself relied on Axiom V to obtain what is known as **Hume's Principle** (= HP), this reliance is avoidable. That from just HP we can deduce all of Peano Arithmetic (**PA**) (!) is a result Frege can be credited with showing; the result is known today as Frege's Theorem (= FT). Following the link just given will reward the reader with an understanding of HP, and how how to obtain **PA** from it.)

Solve

The Foundation Crumbles



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The Rest of Math,
Engineering, etc.

Foundation

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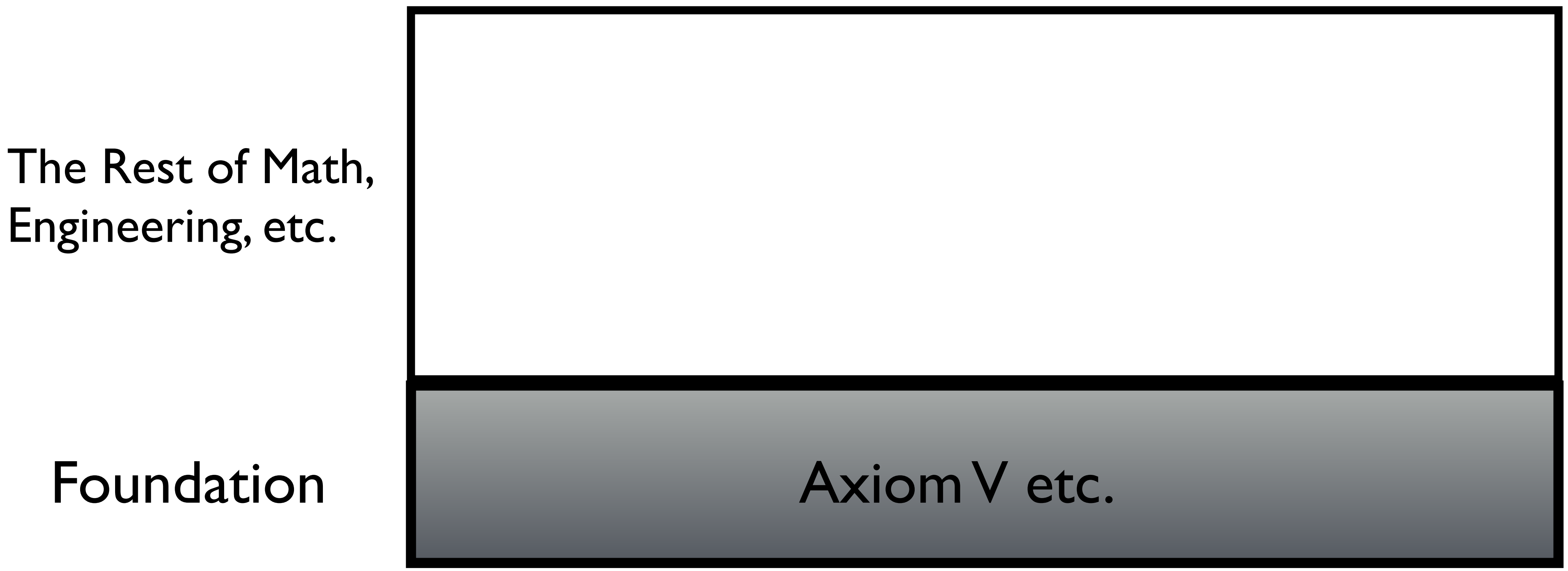

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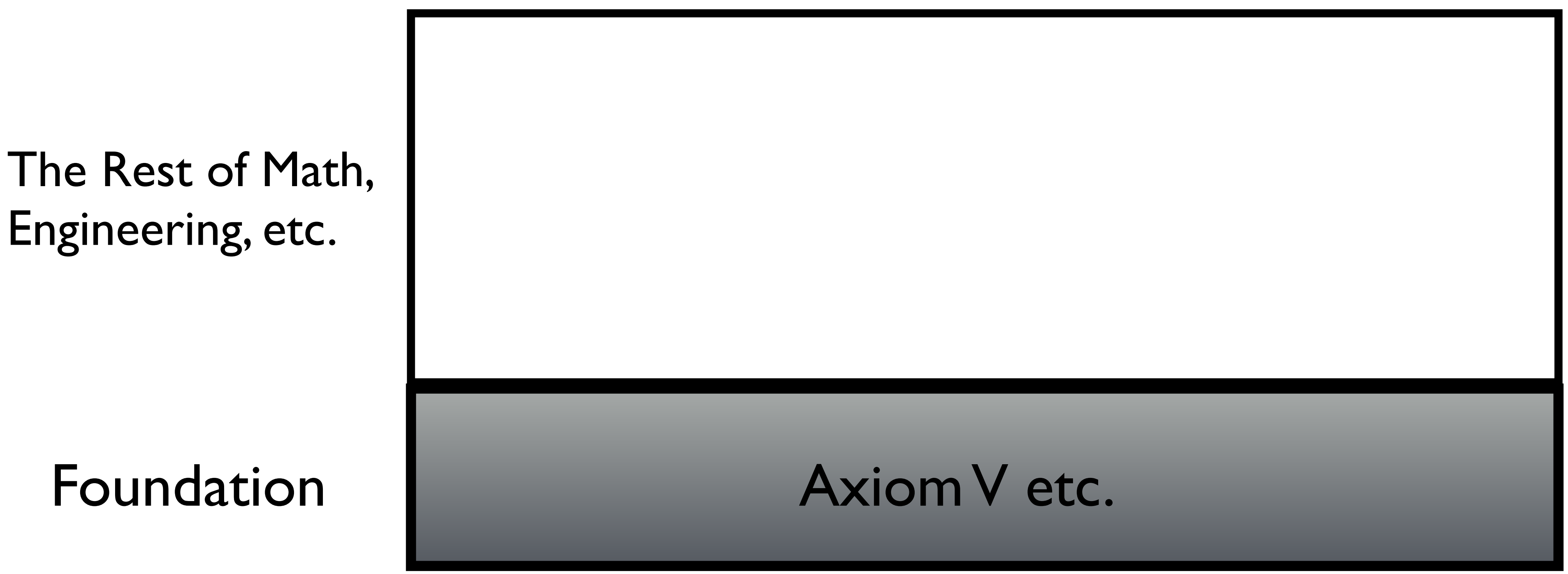

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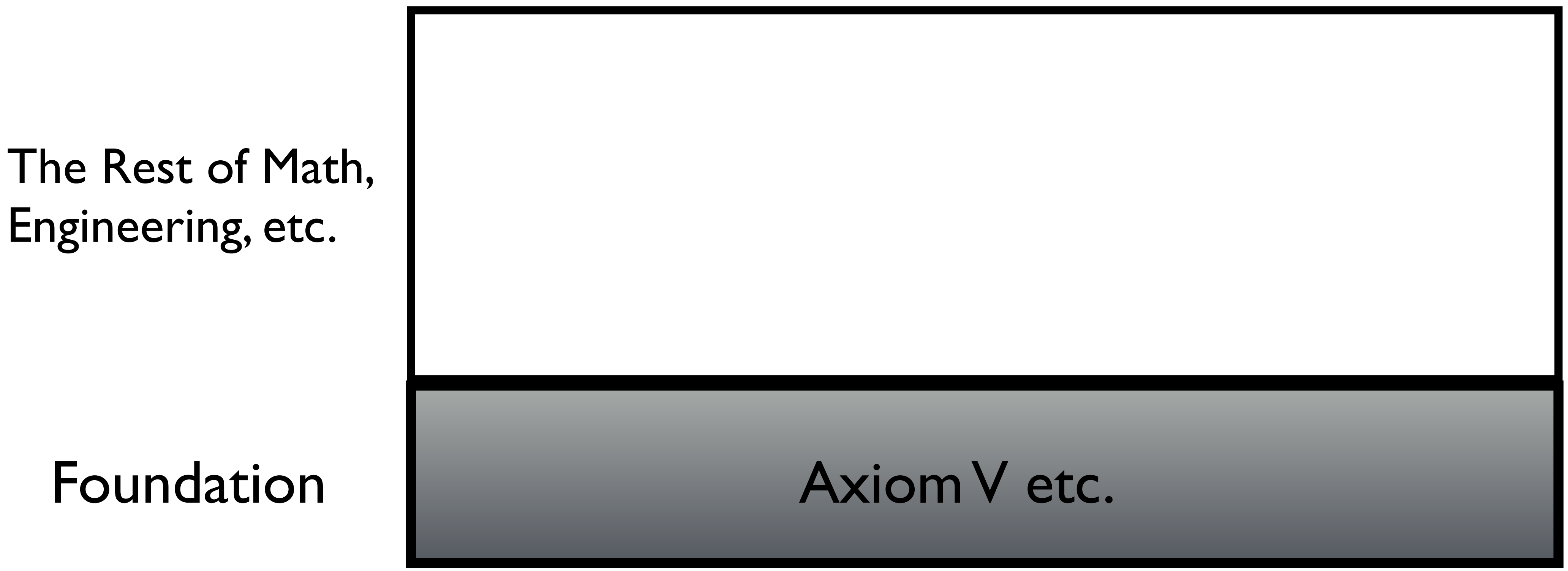

$$\text{Axiom V} \qquad \exists x \forall y [y \in x \leftrightarrow \phi(y)]$$

a formula of arbitrary size in which the variable *y* is free; this formula ascribes a property to *y*

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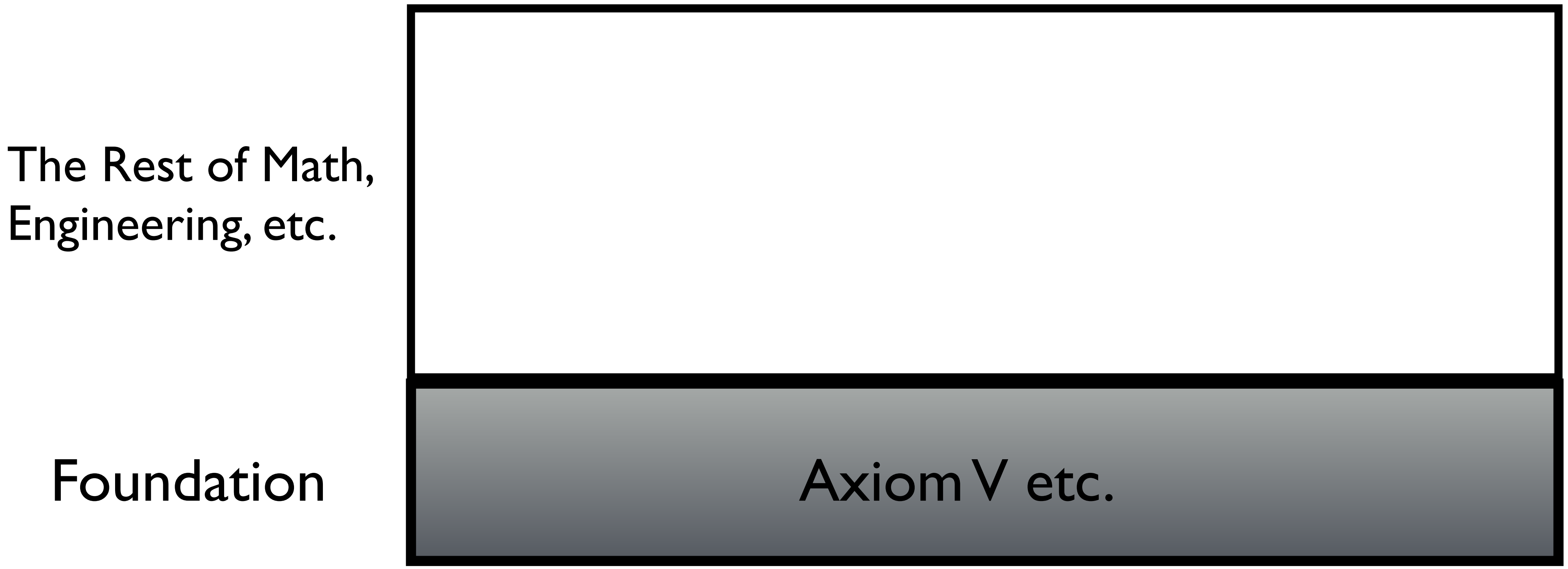

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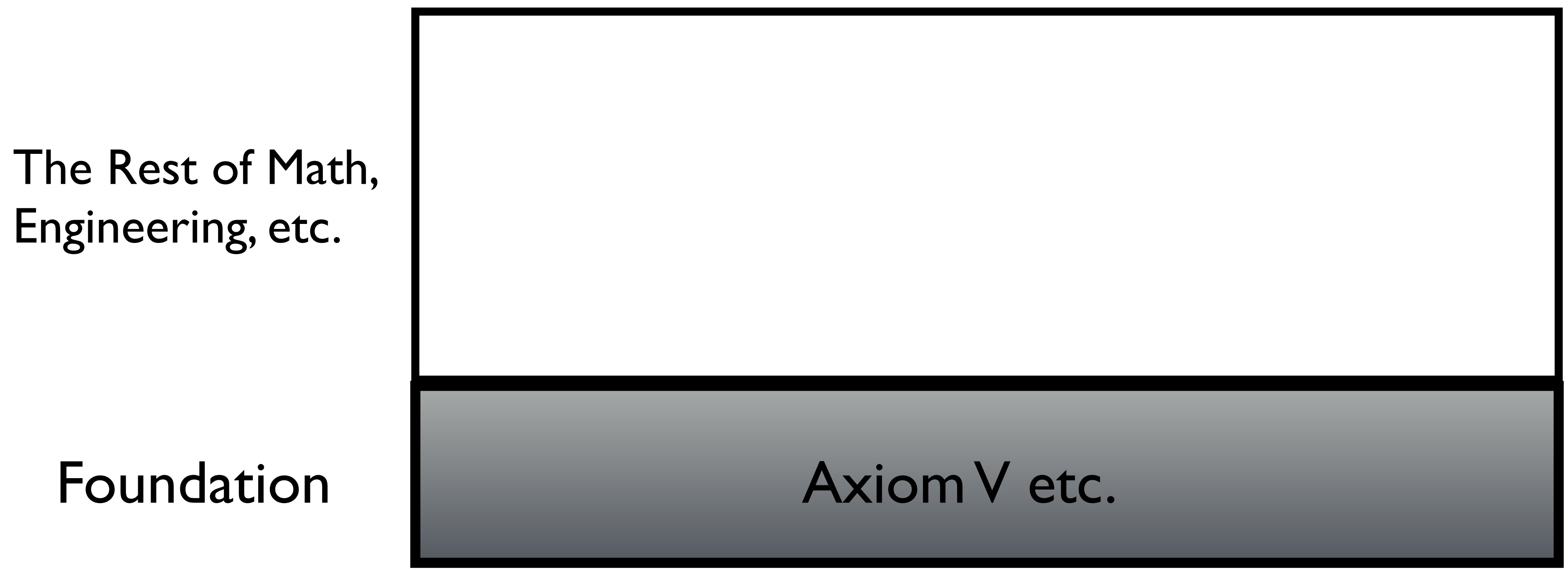

"For any property ascribed to all *y*'s, there's a set *x* composed of those *y*'s."

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The sub-formula ϕ of arbitrary size in which the variable y is free; this formula ascribes a property to y.

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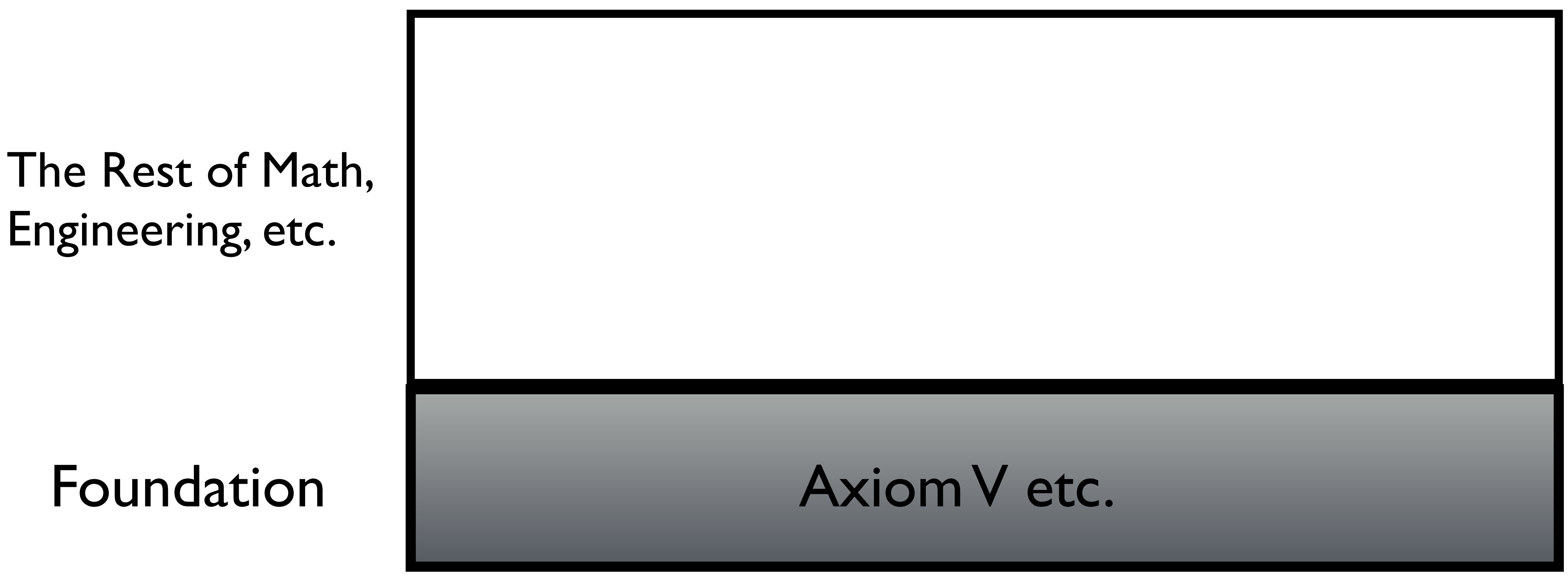

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The Rest of Math,
Engineering, etc.

Foundation

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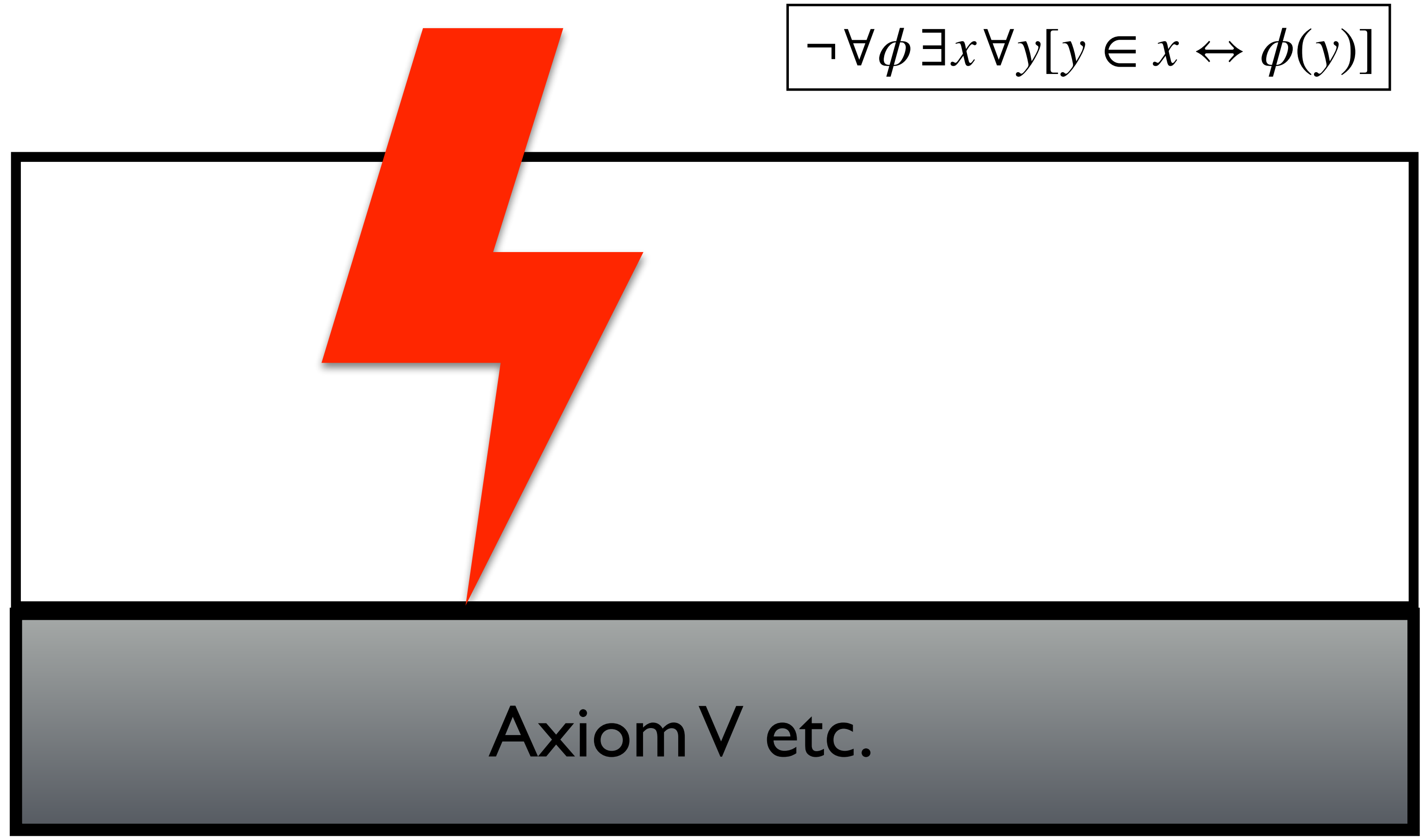
Foundation

Inconsistent!

Russell's Theorem:

$$\neg \forall \phi \, \exists x \forall y [y \in x \leftrightarrow \phi(y)]$$

Math, & all that rests upon it.



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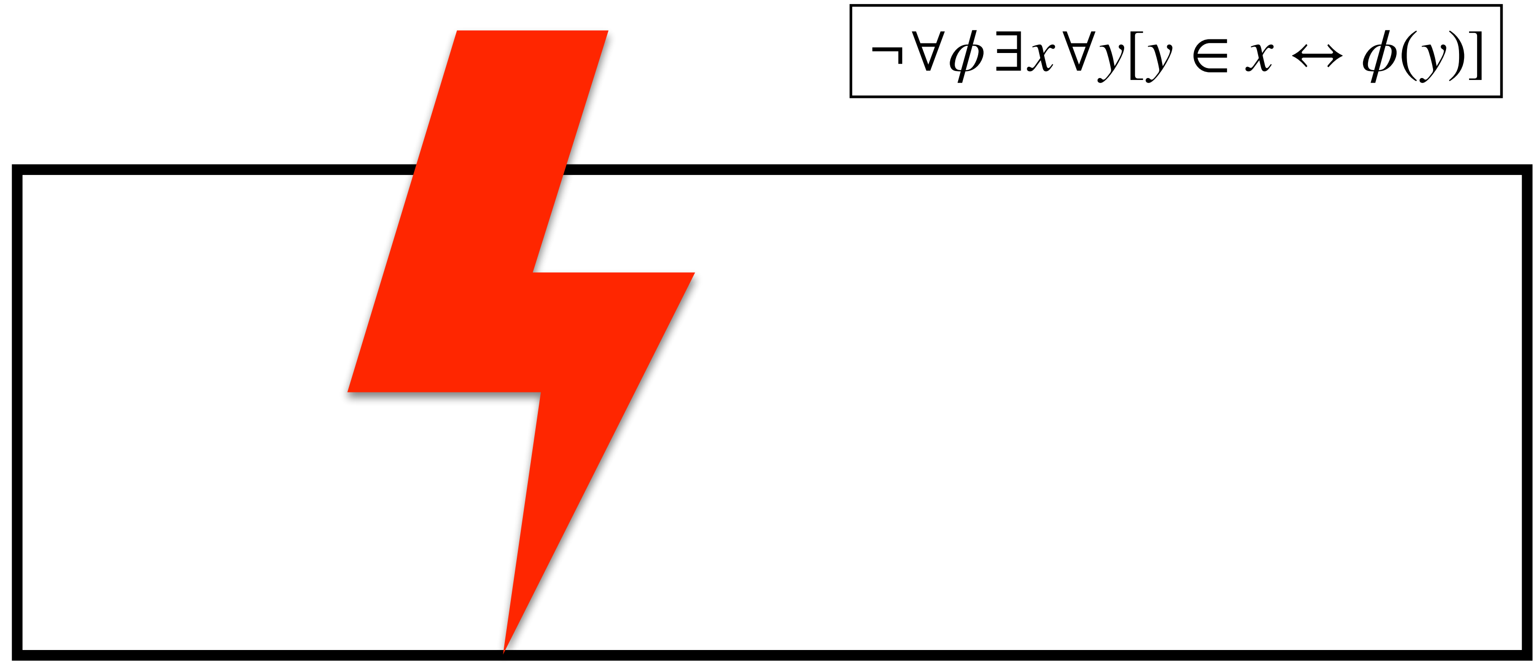

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The Foundation Rebuilt

The Rest of Math,
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New Foundation

The Foundation Rebuilt

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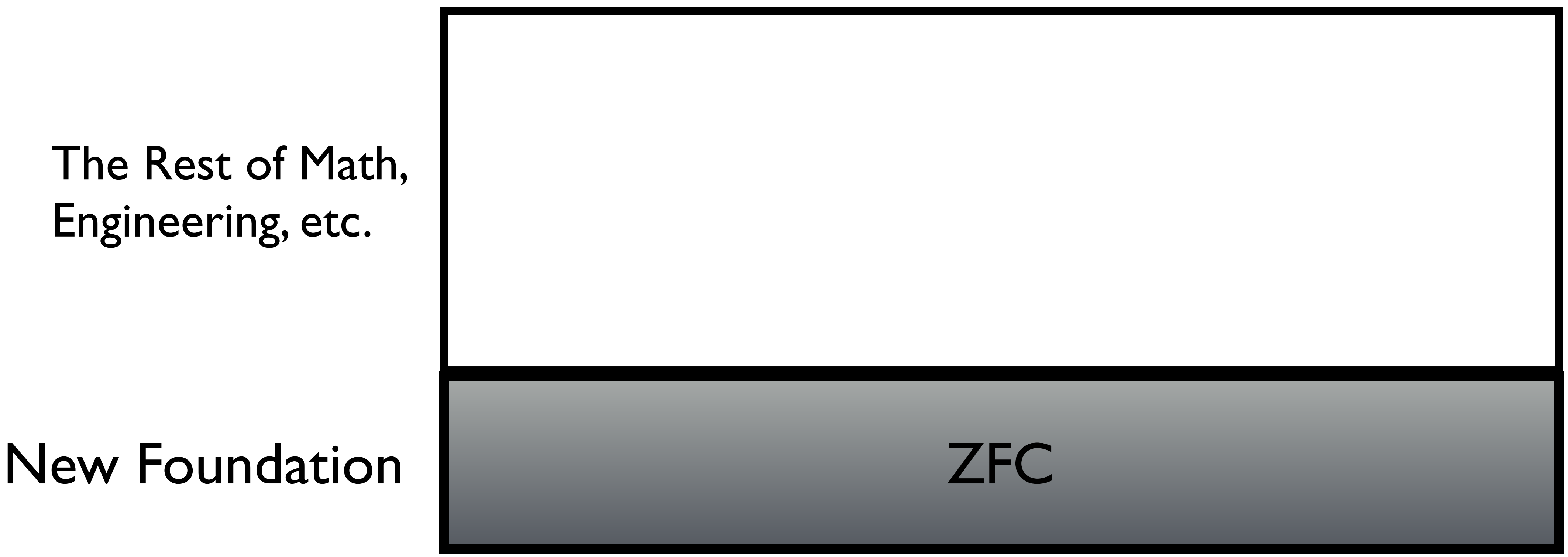

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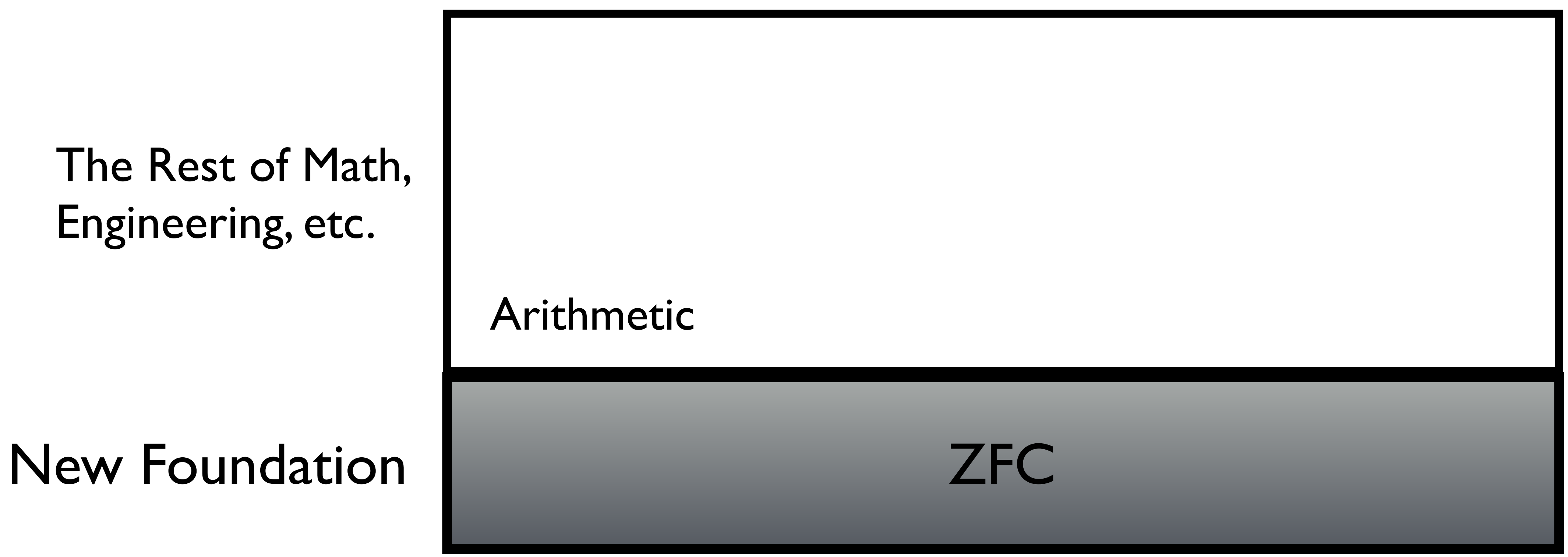

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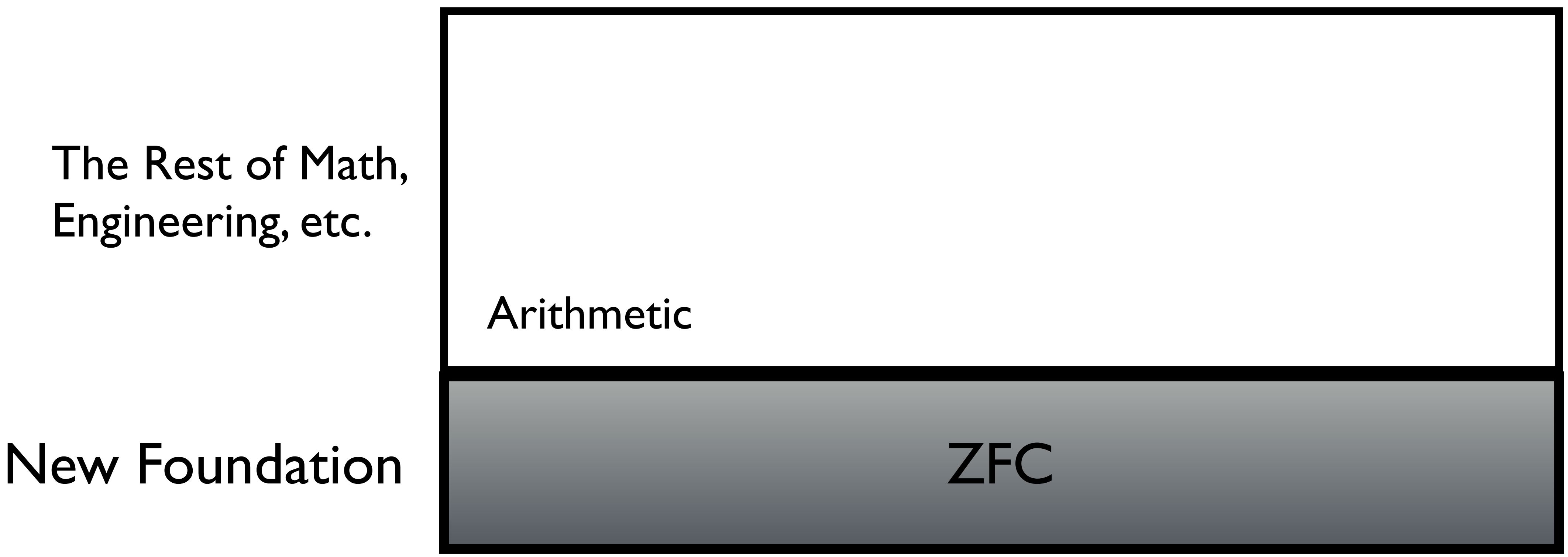

So what's an axiom system, anyway?

The Foundation Rebuilt

The Rest of Math,
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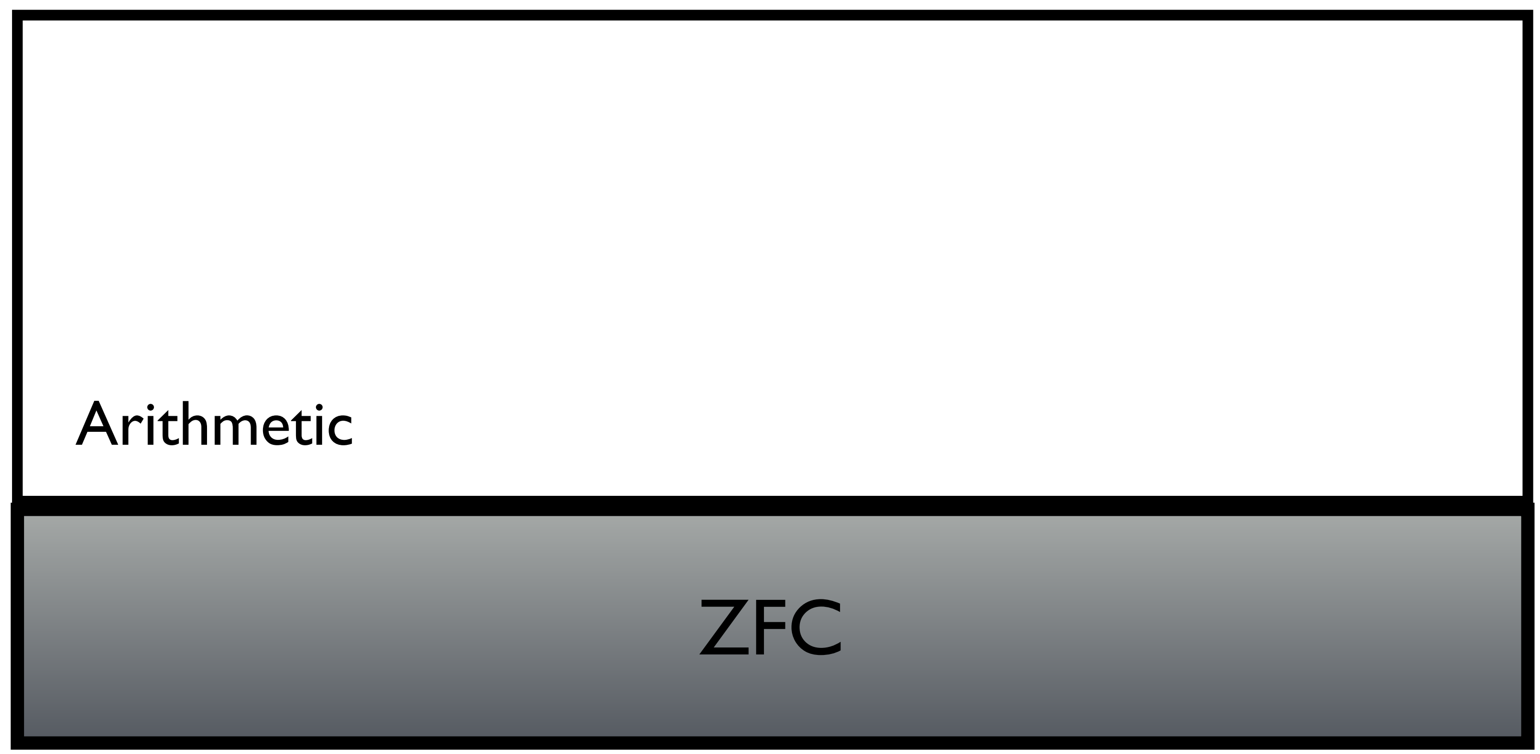
New Foundation



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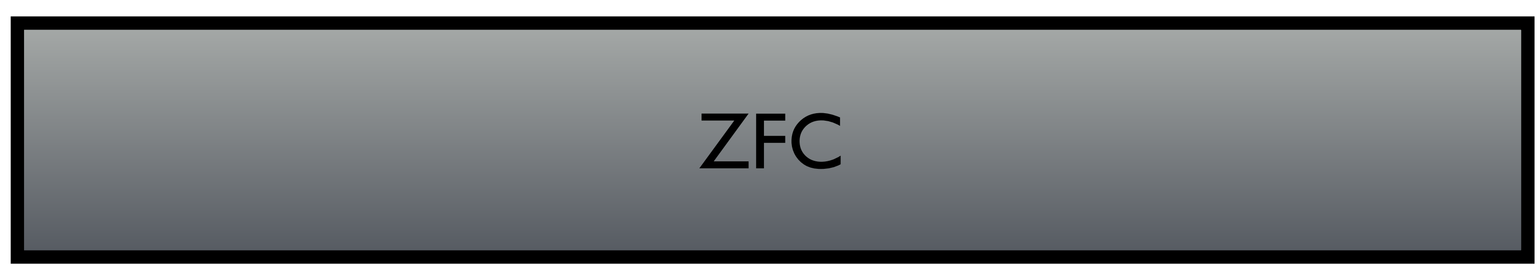
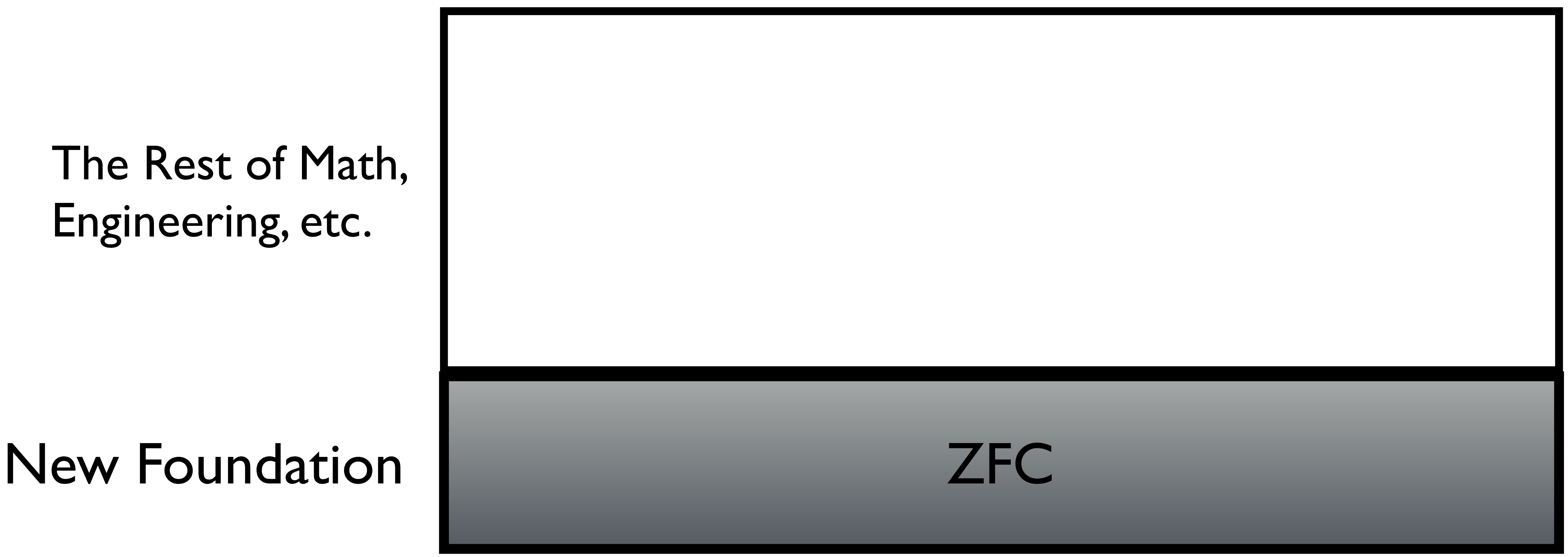
Engineering, etc.

New Foundation



The Foundation Rebuilt



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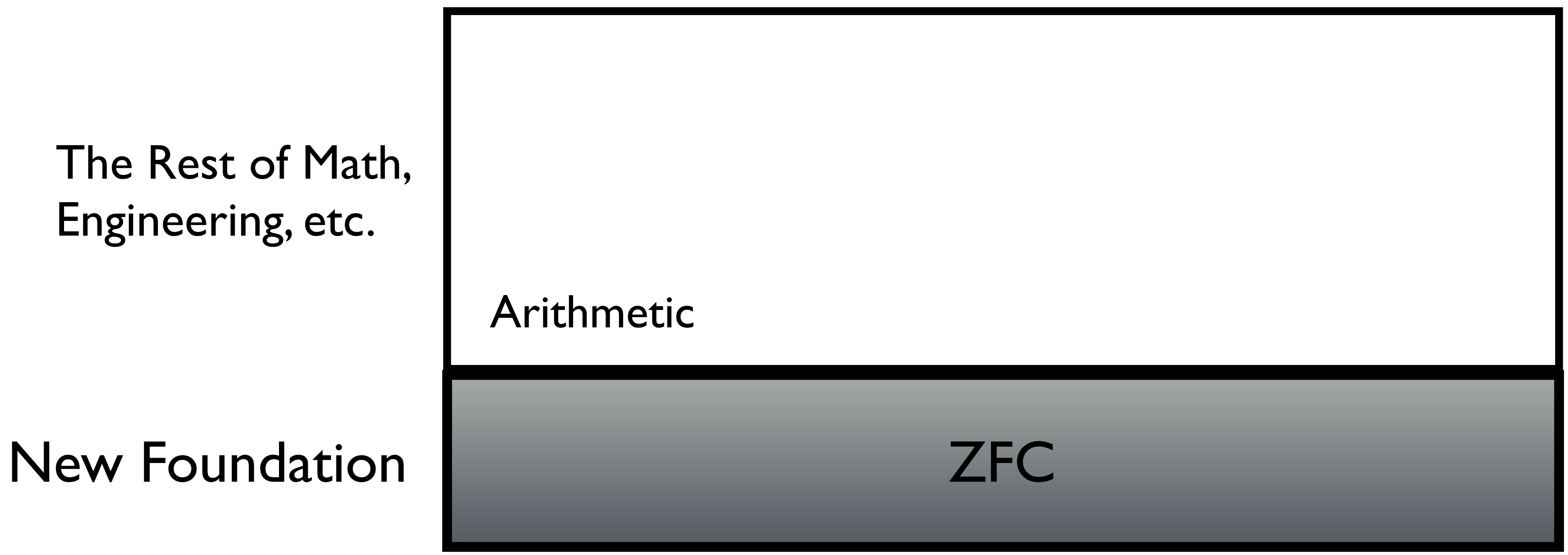

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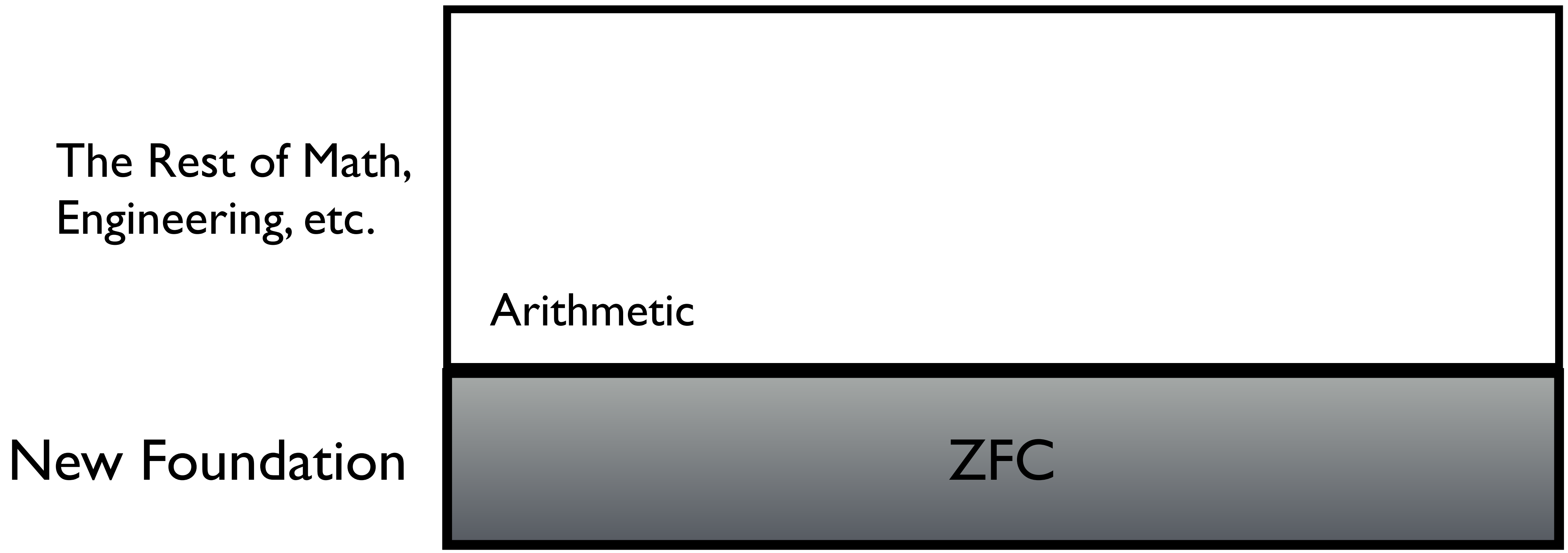
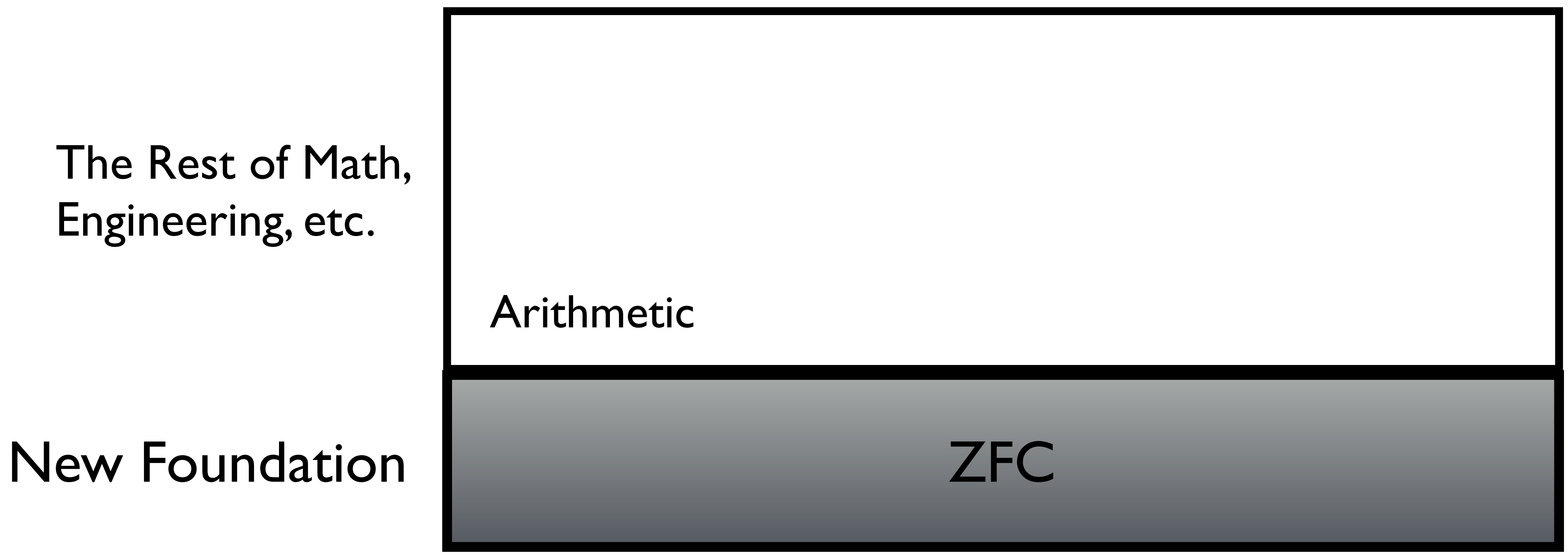


So what are the axioms in ZFC?

The Foundation Rebuilt



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Axiom *Schema* of Separation (SEP)

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SEP

$$\forall x_1 \ldots \forall x_k \forall x \exists y \forall z[z \in y \leftrightarrow (z \in x \wedge \phi(z, x_1, \ldots, x_k))]$$

where x and y are distinct, and are both distinct from z and the x_i; and, as usual for us now, ϕ expresses a property using $\in$.

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"Given *beforehand* a set *x* and property $\mathscr{P}$ captured by a formula ϕ that uses $\in$ for its relation and contains *z*, the set *y* composed of $\{z \in y : \mathscr{P}(z)\}$ exists."

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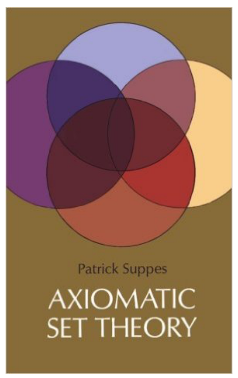
How does this neutralize Russell's letter to Frege?

Russell's Paradox … to ZFC

$$\vdash \neg\exists x \forall y (y \in x \leftrightarrow y \notin y)$$

(Russell's Theorem; poor Frege!)

http://plato.stanford.edu/entries/russell-paradox/#HOTP



Supplant Cantor's/Frege's Axiom V with the Axiom Schema of Separation (& put on our thinking caps …) and try to show Theorem 1 from Suppes:

$$\vdash \forall x (x \notin \emptyset)$$

Russell's Paradox … to ZFC

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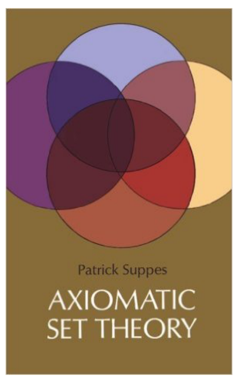

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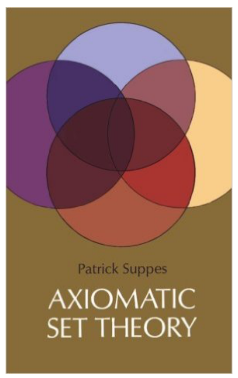

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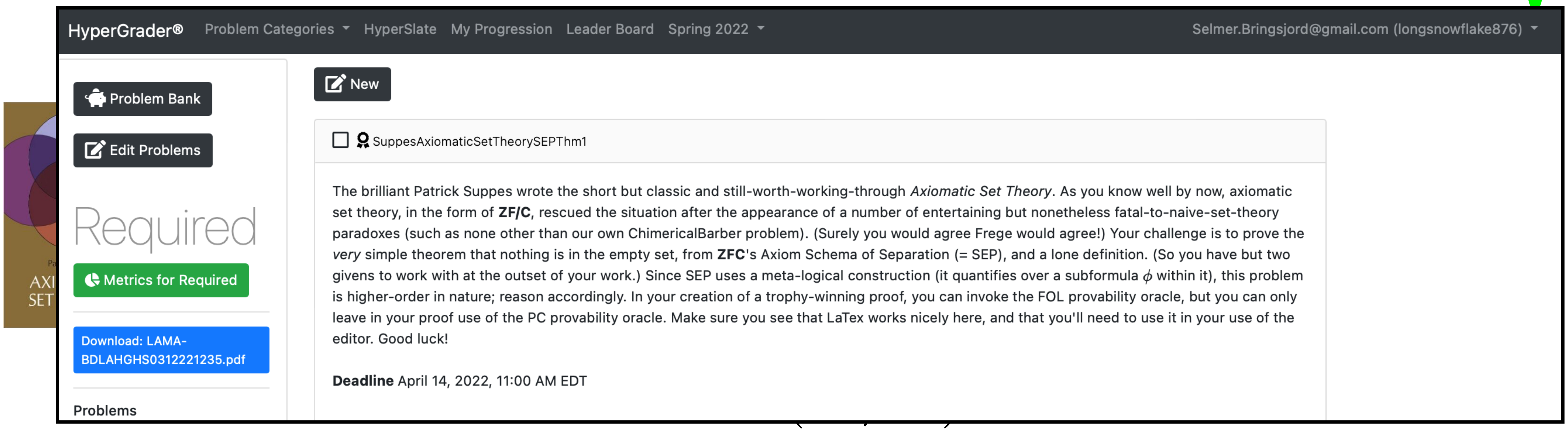

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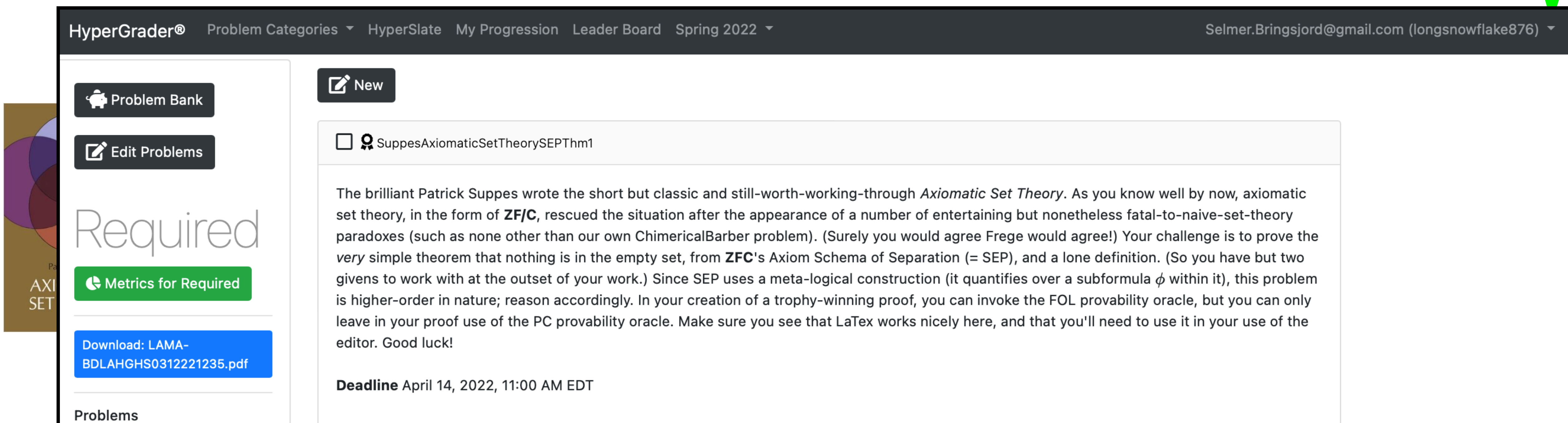

Try a second "Suppesian" theorem in ZFC:

Russell's Paradox … to ZFC

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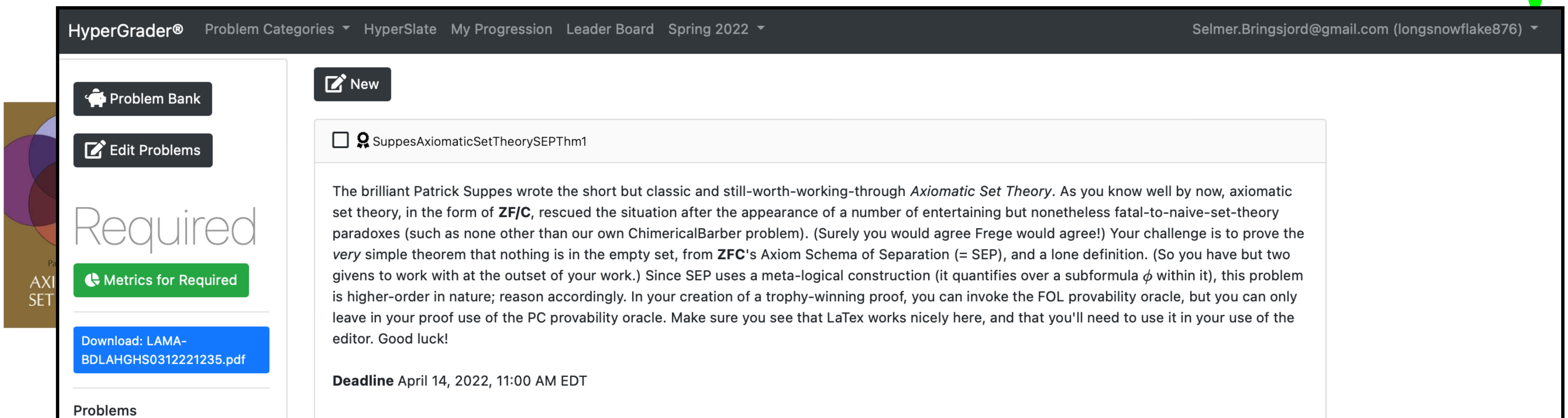

Try a second "Suppesian" theorem in ZFC:

$$\vdash \forall x[(\forall z(z \notin x)) \rightarrow x = \emptyset]$$

Russell's Paradox … to ZFC

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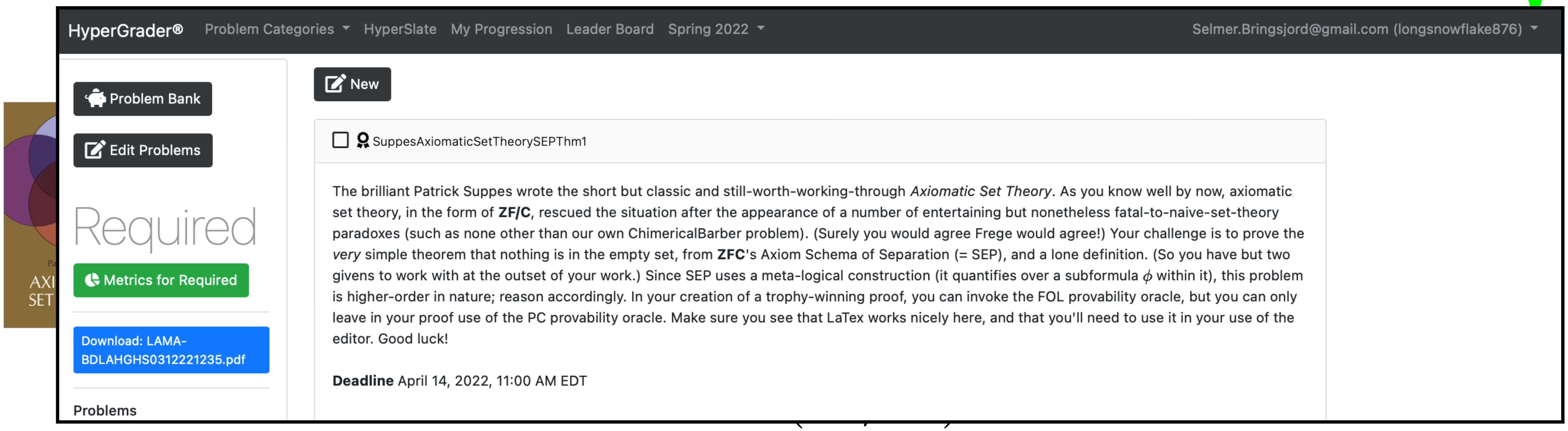

Try a second "Suppesian" theorem in ZFC:

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Now let's add the Definition of Subset to ZFC:

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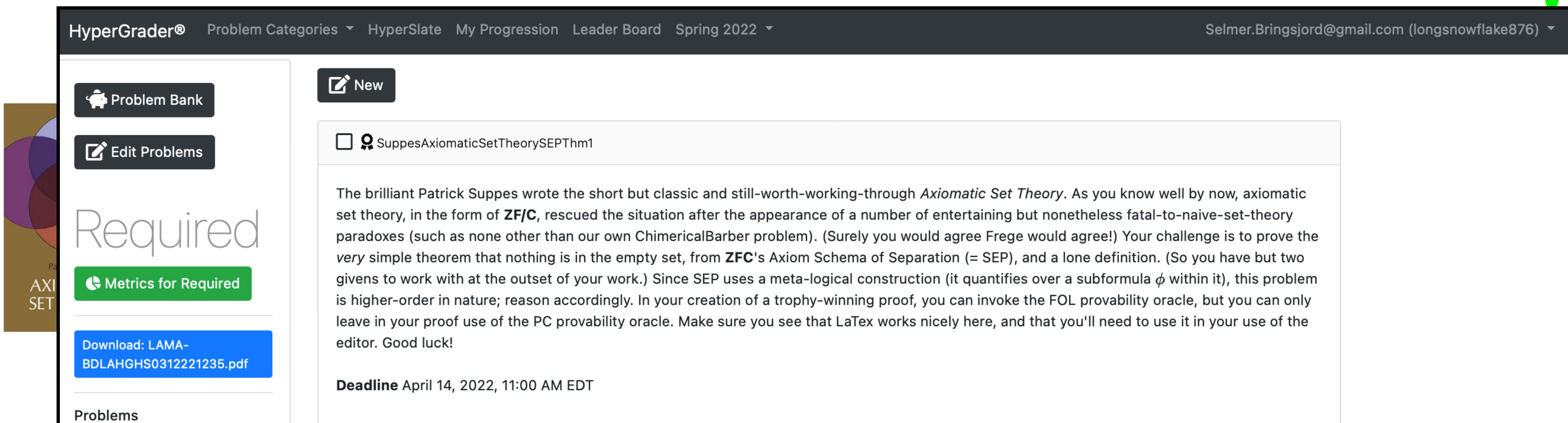

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$$\forall x \forall y[x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y)]$$

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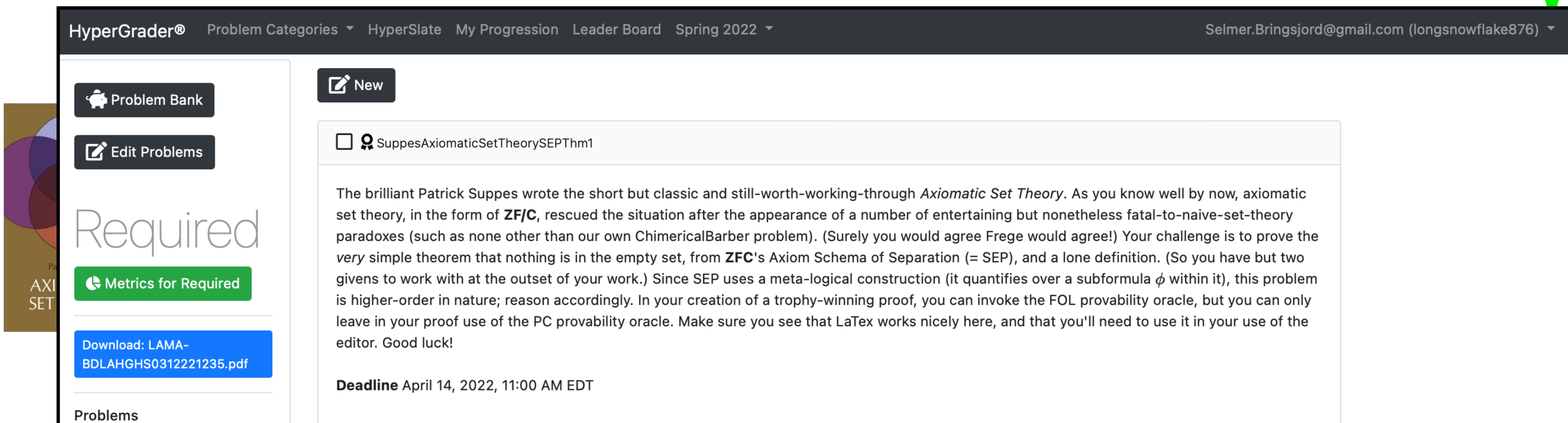

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With this definition, can you prove (Theorem 3) that every set is a subset of itself?

ZFC Completed

formulated with an eyes-wide-open understanding that paradoxes can rise up and threaten unreflective use of set-theoretic concepts. There are a number of different possibilities for specifying an axiomatic set theory. We turn now to the dominant one, known by the label 'ZFC.'

6.4.1 ZFC

The Zermelo-Fraenkel Axioms for Set Theory, or just 'ZFC' for short, include the following nine axioms.[34]

Axiom of Extensionality

$$\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$$

Axiom Schema of Separation

$$\forall x_0 \ldots \forall x_{n-1} \forall x \exists y \forall z(z \in y \leftrightarrow (z \in x \land \phi(z, x_0, \ldots, x_{n-1})))$$

Pair Set Axiom

$$\forall x \forall y \exists z \forall w(w \in z \leftrightarrow (w = x \lor w = y))$$

Sum Set Axiom

$$\forall x \exists y \forall z(z \in y \leftrightarrow \exists w(w \in x \land z \in w))$$

Power Set Axiom

$$\forall x \exists y \forall z(z \in y \leftrightarrow \forall w(w \in z \to w \in x))$$

Axiom of Infinity

$$\exists x(\emptyset \in x \land \forall y(y \in x \to y \cup \{y\} \in x))$$

Axiom Schema of Replacement

$$\forall x_0 \ldots \forall x_{n-1}(\forall x \exists^{=1} y\, \phi(x, y, x_0, \ldots, x_{n-1}) \to \forall u \exists v \forall y(y \in v \leftrightarrow \exists x(x \in u \land \phi(x, y, x_0, \ldots, x_{n-1}))))$$

Axiom of Choice

$$\forall x((\emptyset \notin x \land \forall u \forall v((u \in x \land v \in x \land u \neq v) \to u \cap v = \emptyset)) \to \exists y \forall w(w \in x \to \exists^{=1} z\, z \in w \cap y))$$

6.4.1.1 Exercises

1. The Axiom Schema of Separation was the replacement for Axiom V. Show that Russell's reasoning fails when the attempt is made to apply it to the Axiom Schema of Separation.

2. Provide for each axiom of ZFC one clear English sentence that expresses the axiom.

---

[3] While it's obvious what the 'Z' and 'F' abbreviate in the label 'ZFC,' what about 'C'? This letter refers to one of the axioms that follow: the Axiom of **C**hoice. 'ZF' refers then to the following list of axioms, *without* the Axiom of Choice.

[4] Note that when we write '$\phi(x)$' we are saying that variable x appears free in formula ϕ. In the Axiom Schema of Separation, y does not occur free in '$\phi(z, x_0, \ldots, x_{n-1})$.'

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Pair Set Axiom

Can then all of classical mathematics be derived deductively from a single HS workspace populated with these axioms?

Power Set Axiom

$$\forall x \exists y \forall z (z \in y \longleftrightarrow \forall w (w \in z \rightarrow w \in x))$$

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6.4.1.1 Exercises

Slutten